Stability for t-intersecting families of permutations

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June 2009

Abstract

A family of permutations $\mathcal{A} \subset S_n$ is said to be *t-intersecting* if any two permutations in $\mathcal{A}$ agree on at least t points, i.e. for any $\sigma, \pi \in \mathcal{A}$, $|\{i \in [n] : \sigma(i) = \pi(i)\}| \geq t$. It was proved by Friedgut, Pilpel and the author in [6] that for n sufficiently large depending on t, a t-intersecting family $\mathcal{A} \subset S_n$ has size at most $(n-t)!$, with equality only if $\mathcal{A}$ is a coset of the stabilizer of t points (or 't-coset' for short), proving a conjecture of Deza and Frankl. Here, we first obtain a rough stability result for t-intersecting families of permutations, namely that for any $t \in \mathbb{N}$ and any positive constant c, if $\mathcal{A} \subset S_n$ is a t-intersecting family of permutations of size at least $c(n-t)!$, then there exists a t-coset containing all but at most a $O(1/n)$-fraction of $\mathcal{A}$. We use this to prove an exact stability result: for n sufficiently large depending on t, if $\mathcal{A} \subset S_n$ is a t-intersecting family which is not contained within a t-coset, then $\mathcal{A}$ is at most as large as the family

$$\begin{aligned} \mathcal{D} &= \{\sigma \in S_n : \sigma(i) = i \ \forall i \leq t, \ \sigma(j) = j \text{ for some } j > t+1\} \\ &\quad \cup \{(1\ t+1), (2\ t+1), \ldots, (t\ t+1)\}, \end{aligned}$$

which has size $(1 - 1/e + o(1))(n-t)!$. Moreover, if $\mathcal{A}$ is the same size as $\mathcal{D}$ then it must be a 'double translate' of $\mathcal{D}$, meaning that there exist $\pi, \tau \in S_n$ such that $\mathcal{A} = \pi\mathcal{D}\tau$. The $t = 1$ case of this was a

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conjecture of Cameron and Ku and was proved by the author in [5]. We build on the methods of [5], but the representation theory of S_n and the combinatorial arguments are more involved. We also obtain an analogous result for t-intersecting families in the alternating group A_n.

1 Introduction

We work first on the symmetric group S_n, the group of all permutations of $\{1, 2, \ldots, n\} = [n]$. A family of permutations $\mathcal{A} \subset S_n$ is said to be *t-intersecting* if any two permutations in $\mathcal{A}$ agree on at least t points, i.e. for any $\sigma, \pi \in \mathcal{A}$, $|\{i \in [n] : \sigma(i) = \pi(i)\}| \geq t$. Deza and Frankl [4] conjectured that for n sufficiently large depending on t, a t-intersecting family $\mathcal{A} \subset S_n$ has size at most $(n-t)!$; this became known as the Deza-Frankl conjecture. It was proved in 2008 by the author and independently by Friedgut and Pilpel, using the same methods: eigenvalue techniques, together with the representation theory of the symmetric group; we have submitted a joint paper [6]. Friedgut and Pilpel gave a very elegant proof that equality holds only if $\mathcal{A}$ is a coset of the stabilizer of t points (or 't-coset' for short); this is also contained in [6]. In the following paper, we will first prove a rough stability result for t-intersecting families of permutations. Namely, we show that for any fixed $t \in \mathbb{N}$ and $c > 0$, if $\mathcal{A} \subset S_n$ is a t-intersecting family of size at least $c(n-t)!$, then there exists a t-coset $\mathcal{C}$ such that $|\mathcal{A} \setminus \mathcal{C}| \leq \Theta((n-t-1)!)$, i.e. $\mathcal{C}$ contains all but at most a $O(1/n)$-fraction of $\mathcal{A}$.

We will then use some additional combinatorial arguments to prove an exact stability result: for n sufficiently large depending on t, if $\mathcal{A} \subset S_n$ is a t-intersecting family which is not contained within a t-coset, then $\mathcal{A}$ is at most as large as the family

$$\begin{aligned}\mathcal{D} \;=\; & \{\sigma \in S_n : \ \sigma(i) = i \ \forall i \leq t, \ \sigma(j) = j \text{ for some } j > t+1\} \\ & \cup \{(1\ t+1), (2\ t+1), \ldots, (t\ t+1)\},\end{aligned}$$

which has size $(1 - 1/e + o(1))(n-t)!$. Moreover, if $\mathcal{A}$ is the same size as $\mathcal{D}$, then it must be a 'double translate' of $\mathcal{D}$, meaning that there exist $\pi, \tau \in S_n$ such that $\mathcal{A} = \pi \mathcal{D} \tau$. Note that if $\mathcal{F} \subset S_n$, any double translate of $\mathcal{F}$ has the same size as $\mathcal{F}$, is t-intersecting iff $\mathcal{F}$ is and is contained within a t-coset of S_n iff $\mathcal{F}$ is; this will be our notion of 'isomorphism'.

In other words, if we demand that our t-intersecting family $\mathcal{A} \subset S_n$ is not contained within a t-coset of S_n, then it is best to take $\mathcal{A}$ such that all but t of its permutations are contained within some t-coset.

One may compare this with the situation for t-intersecting families of r-sets. We say a family $\mathcal{A} \subset [n]^{(r)}$ of r-element subsets of $[n]$ is *t-intersecting* if any two of its sets contain at least t elements in common, i.e. $|x \cap y| \geq t$ for any $x, y \in \mathcal{A}$. Wilson [11] proved using an eigenvalue technique that provided $n \geq (t+1)(r-t+1)$, a t-intersecting family $\mathcal{A} \subset [n]^{(r)}$ has size at most $\binom{n-t}{r-t}$, and that for $n > (t+1)(r-t+1)$, equality holds only if $\mathcal{A}$ consists of all r-sets containing some fixed t-set. Later, Ahlswede and Khachatrian [1] characterized the t-intersecting families of maximum size in $[n]^{(r)}$ for all values of t, r and n using entirely combinatorial methods based on left-compression. They also proved, in [2], that for $n > (t+1)(r-t+1)$, if $\mathcal{A} \subset [n]^{(r)}$ is t-intersecting and *non-trivial*, meaning that there is no t-set contained in all of its members, then $\mathcal{A}$ is at most as large as the family

$$\{x \in [n]^{(r)} : \ [t] \subset x, \ x \cap \{t+1, \ldots, r+1\} \neq \emptyset\} \cup \{[r+1] \setminus \{i\} : \ i \in [t]\}$$

if $r > 2t+1$, and at most as large as the family

$$\{x \in [n]^{(r)} : |x \cap [t+2]| \geq t+1\}$$

if $r \leq 2t+1$. This had been proved under the assumption $n \geq n_1(r,t)$ by Frankl [7] in 1978. Note that the first family above is 'almost trivial', and is the natural analogue of our family $\mathcal{D}$.

The $t = 1$ case of our result was a conjecture of Cameron and Ku and was proved by the author in [5]. We build on the methods of [5], but the representation theory of S_n and the combinatorial arguments required are more involved.

We also obtain analogous results for t-intersecting families of permutations in the alternating group A_n. We use the methods of [6] to show that for n sufficiently large depending on t, if $\mathcal{A} \subset A_n$ is t-intersecting, then $|\mathcal{A}| \leq (n-t)!/2$. Interestingly, it does not seem possible to use the method in [6] to show that equality holds only if $\mathcal{A}$ is a coset of the stabilizer of t points. Instead, we deduce this from a stability result. Using the same techniques as for S_n, we prove that if $\mathcal{A} \subset A_n$ is t-intersecting but not contained within a t-coset, then it is at most as large as the family

$$\begin{aligned}\mathcal{E} &= \{\sigma \in A_n : \sigma(i) = i \ \forall i \leq t, \ \sigma(j) = (n-1 \ n)(j) \text{ for some } j > t+1\} \\ &\quad \cup \{(1 \ t+1)(n-1 \ n), (2 \ t+1)(n-1 \ n), \ldots, (t \ t+1)(n-1 \ n)\},\end{aligned}$$

which has size $(1-1/e+o(1))(n-t)!/2$; if $\mathcal{A}$ is the same size as $\mathcal{E}$, then it must be a double translate of $\mathcal{E}$, meaning that $\mathcal{A}=\pi\mathcal{E}\tau$ for some $\pi,\tau\in A_n$.

2 Background

In [6], in order to prove the Deza-Frankl conjecture, we constructed (for n sufficiently large depending on t) a weighted graph Y which was a real linear combination of Cayley graphs on S_n generated by conjugacy-classes of permutations with less than t fixed points, such that the matrix A of weights of Y had maximum eigenvalue 1 and minimum eigenvalue

$$\omega_{n,t} = -\frac{1}{n(n-1)\dots(n-t+1)-1}.$$

The 1-eigenspace was the subspace of $\mathbb{C}[S_n]$ consisting of the constant functions. The direct sum of the 1-eigenspace and the $\omega_{n,t}$-eigenspace was the subspace V_t of $\mathbb{C}[S_n]$ spanned by the characteristic vectors of the t-cosets of S_n. All other eigenvalues were $O(|\omega_{n,t}|/n^{1/6})$; this can in fact be improved to $O(|\omega_{n,t}|/n)$, but any bound of the form $o(|\omega_{n,t}|)$ will suffice for our purposes. We then appealed to a weighted version of Hoffman's bound (Theorem 11 in [6]):

Theorem 1. *Let A be a real, symmetric, $N\times N$ matrix with real eigenvalues $\lambda_1\geq\lambda_2\geq\ldots\geq\lambda_N$ (where $\lambda_1>0$), such that the all-1's vector $\mathbf{f}$ is an eigenvector of A with eigenvalue λ_1, i.e. all row and column sums of A equal λ_1. Let $X\subset[N]$ such that $A_{x,y}=0$ for any $x,y\in X$. Let U be the direct sum of the subspace of constant vectors and the λ_N-eigenspace. Then*

$$|X| \leq \frac{|\lambda_N|}{\lambda_1+|\lambda_N|}N,$$

and equality holds only if the characteristic vector v_X lies in the subspace U.

Applying this to our weighted graph Y proved the Deza-Frankl conjecture:

Theorem 2. *For n sufficiently large depending on t, a t-intersecting family $\mathcal{A}\subset S_n$ has size $|\mathcal{A}|\leq(n-t)!$.*

Note that equality holds only if the characteristic vector $v_{\mathcal{A}}$ of $\mathcal{A}$ lies in the subspace V_t spanned by the characteristic vectors of the t-cosets of S_n. It was proved in [6] that the Boolean functions in V_t are precisely the disjoint unions of t-cosets of S_n, implying that equality holds only if $\mathcal{A}$ is a t-coset of S_n.

We also appealed to the following 'cross-independent' weighted version of Hoffman's bound:

Theorem 3. *Let A be as in Theorem 1, and let $\nu = \max(|\lambda_2|, |\lambda_N|)$. Let $X, Y \subset [N]$ such that $A_{x,y} = 0$ for any $x \in X$ and $y \in Y$. Let U be the direct sum of the subspace of constant vectors and the $\pm\nu$-eigenspaces. Then*

$$|X||Y| \leq \left(\frac{\nu}{\lambda_1 + \nu} N\right)^2,$$

and equality holds only if $|X| = |Y|$ and the characteristic vectors v_X and v_Y lie in the subspace U.

Applying this to our weighted graph Y yielded the following:

Theorem 4. *For n sufficiently large depending on t, if $\mathcal{A}, \mathcal{B} \subset S_n$ are t-cross-intersecting, then $|\mathcal{A}||\mathcal{B}| \leq ((n-t)!)^2$.*

This will be a crucial tool in our stability analysis. Note that if equality holds in Theorem 4, then the characteristic vectors $v_{\mathcal{A}}$ and $v_{\mathcal{B}}$ lie in the subspace V_t spanned by the characteristic vectors of the t-cosets of S_n, so by the same argument as before, $\mathcal{A}$ and $\mathcal{B}$ must both be equal to the same t-coset of S_n.

We will need the following 'stability' version of Theorem 1:

Lemma 5. *Let A, X and U be as in Theorem 1. Let $\alpha = |X|/N$. Let λ_M be the negative eigenvalue of second largest modulus. Equip $\mathbb{C}^N$ with the inner product:*

$$\langle x, y \rangle = \frac{1}{N} \sum_{i=1}^{N} \bar{x}_i y_i,$$

and let

$$||x|| = \sqrt{\frac{1}{N} \sum_{i=1}^{N} |x_i|^2}$$

be the induced norm. Let D be the Euclidean distance from the characteristic vector v_X of X to the subspace U, i.e. the norm $||P_{U^\perp}(v_X)||$ of the projection *of* v_X *onto* $U^\perp$*. Then*

$$D^2 \leq \frac{(1-\alpha)|\lambda_N| - \lambda_1\alpha}{|\lambda_N| - |\lambda_M|}\alpha.$$

For completeness, we include a proof:

Proof. Choose real eigenvectors $u_1, u_2, \ldots, u_N$ of A corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_N$, such that $\{u_1, u_2, \ldots, u_N\}$ is an orthonormal basis of $\mathbb{C}^N$ with respect to the inner product above. We may choose $u_1 = \mathbf{f}$, the all-1's vector. Write

$$v_X = \sum_{i=1}^{N} \xi_i u_i$$

as a linear combination of the eigenvectors of A; we have $\xi_1 = \alpha$ and

$$\sum_{i=1}^{N} \xi_i^2 = ||v_X||^2 = |X|/N = \alpha.$$

Then we have the crucial property:

$$0 = \sum_{x,y \in X} A_{x,y} = v_X^\top A v_X = \sum_{i=1}^{N} \lambda_i \xi_i^2 \geq \lambda_1 \xi_1^2 + \lambda_N \sum_{i:\lambda_i=\lambda_N} \xi_i^2 + \lambda_M \sum_{i>1:\lambda_i \neq \lambda_N} \xi_i^2.$$

Note that

$$\sum_{i>1:\lambda_i \neq \lambda_N} \xi_i^2 = D^2$$

and

$$\sum_{i:\lambda_i=\lambda_N} \xi_i^2 = \alpha - \alpha^2 - D^2,$$

so we have

$$0 \geq \lambda_1\alpha^2 + \lambda_N(\alpha - \alpha^2 - D^2) + \lambda_M D^2.$$

Rearranging, we obtain:

$$D^2 \leq \frac{(1-\alpha)|\lambda_N| - \lambda_1\alpha}{|\lambda_N| - |\lambda_M|}\alpha$$

as required. □

Our weighted graph Y has $\lambda_N = \omega_{n,t}$ and $|\lambda_M| = O(|\omega_{n,t}|/n^{1/6})$, so applying the above result to a t-intersecting family $\mathcal{A} \subset S_n$ gives:

$$||P_{V_t^{\perp}}(v_{\mathcal{A}})||^2 \leq (1 - |\mathcal{A}|/(n-t)!)(1 + O(n^{1/6}))|\mathcal{A}|/n!. \quad (1)$$

Next, we find a formula for the projection $P_{V_t}(v_{\mathcal{A}})$ of the characteristic vector of $\mathcal{A}$ onto the subspace V_t spanned by the characteristic vectors of the t-cosets of S_n. But first, we need some background on non-Abelian Fourier analysis and the representation theory of the symmetric group.

Background from non-Abelian Fourier analysis

We now recall some information we need from [6]. [Notes for algebraists are included in square brackets and may be ignored without prejudicing the reader's understanding.]

If G is a finite group, a *representation* of G is a vector space W together with a group homomorphism $\rho : G \to \mathrm{GL}(W)$ from G to the group of all automorphisms of W, or equivalently a linear action of G on W. If $W = \mathbb{C}^m$, then $\mathrm{GL}(W)$ can be identified with the group of all complex invertible $m \times m$ matrices; we call ρ a *complex matrix representation* of degree (or dimension) m. [Note that ρ makes $\mathbb{C}^m$ into a $\mathbb{C}G$-module of dimension m.]

We say a representation (ρ, W) is *irreducible* if it has no proper subrepresentation, i.e. no proper subspace of W is fixed by $\rho(g)$ for every $g \in G$. We say that two (complex) representations (ρ, W) and (ρ', W') are *equivalent* if there exists a linear isomorphism $\phi : W \to W'$ such that $\rho'(g) \circ \phi = \phi \circ \rho(g)\ \forall g \in G$.

For any finite group G, there are only finitely many equivalence classes of irreducible complex representations of G. Let $(\rho_1, \rho_2, \ldots, \rho_k)$ be a complete set of pairwise non-equivalent complex irreducible matrix representations of G (i.e. containing one from each equivalence class of complex irreducible representations).

Definition 1. *The* (non-Abelian) Fourier transform *of a function $f : G \to \mathbb{C}$ at the irreducible representation ρ_i is the matrix*

$$\hat{f}(\rho_i) = \frac{1}{|G|} \sum_{g \in G} f(g)\rho_i(g).$$

Let V_{ρ_i} be the subspace of functions whose Fourier transform is concentrated on ρ_i, i.e. with $\hat{f}(\rho_j) = 0$ for each $j \neq i$. [Identifying the space $\mathbb{C}[G]$ of all complex-valued functions on G with the *group module* $\mathbb{C}G$, V_{ρ_i} is the sum of all submodules of the group module isomorphic to the module defined by ρ_i; it has dimension $\dim(V_{\rho_i}) = (\dim(\rho_i))^2$. The group module decomposes as

$$\mathbb{C}G = \bigoplus_{i=1}^{k} V_{\rho_i}.$$

Write $\mathrm{Id} = \sum_{i=1}^{k} e_i$, where $e_i \in V_{\rho_i}$ for each $i \in [k]$. The e_i's are called the *primitive central idempotents* of $\mathbb{C}G$; they are given by the following formula:

$$e_i = \frac{\dim(\rho_i)}{|G|} \sum_{g \in G} \chi_i(g^{-1}) g.$$

They are in the *centre* $Z(\mathbb{C}G)$ of the group module, and satisfy $e_i e_j = \delta_{i,j}$. Note that V_{ρ_i} is the two-sided ideal of $\mathbb{C}G$ generated by e_i. For any $x \in \mathbb{C}G$, the unique decomposition of x into elements of the V_{ρ_i}'s is given by $x = \sum_{i=1}^{k} e_i x$.]

A function $f : G \to \mathbb{C}$ may be recovered from its Fourier transform using the Fourier Inversion Formula:

$$f(g) = \sum_{i=1}^{k} \dim(\rho_i) \mathrm{Tr}\left(\hat{f}(\rho_i)\rho_i(g^{-1})\right)$$

where $\mathrm{Tr}(M)$ denotes the trace of the matrix M. It follows from this that the projection of f onto V_{ρ_i} has g-coordinate

$$P_{V_{\rho_i}}(f)_g = \frac{\dim(\rho_i)}{|G|} \sum_{h \in G} f(h) \mathrm{Tr}(\rho_i(hg^{-1})) = \frac{\dim(\rho_i)}{|G|} \sum_{h \in G} f(h) \chi_{\rho_i}(hg^{-1})$$

where $\chi_{\rho_i}(g) = \mathrm{Tr}(\rho_i(g))$ denotes the character of the representation ρ_i.

Background on the representation theory of S_n

A *partition* of n is a non-increasing sequence of positive integers summing to n, i.e. a sequence $\alpha = (\alpha_1, \ldots, \alpha_k)$ with $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_l \geq 1$ and $\sum_{i=1}^{l} \alpha_i = n$; we write $\alpha \vdash n$. For example, $(3, 2, 2) \vdash 7$; we sometimes use the shorthand $(3, 2, 2) = (3, 2^2)$.

The *cycle-type* of a permutation $\sigma \in S_n$ is the partition of n obtained by expressing σ as a product of disjoint cycles and listing its cycle-lengths in non-increasing order. The conjugacy-classes of S_n are precisely

$$\{\sigma \in S_n : \text{ cycle-type}(\sigma) = \alpha\}_{\alpha \vdash n}.$$

Moreover, there is an explicit 1-1 correspondence between irreducible representations of S_n (up to isomorphism) and partitions of n, which we now describe.

Let $\alpha = (\alpha_1, \ldots, \alpha_l)$ be a partiton of n. The *Young diagram* of α is an array of n cells, having l left-justified rows where row i contains α_i cells. For example, the Young diagram of the partition $(3, 2^2)$ is

If the array contains the numbers $\{1, 2, \ldots, n\}$ in some order in place of the dots, we call it an α-*tableau*; for example,

6	1	7
5	4	
3	2	

is a $(3, 2^2)$-tableau. Two α-tableaux are said to be *row-equivalent* if for each row, they have the same numbers in that row. If an α-tableau s has rows $R_1, \ldots, R_l \subset [n]$ and columns $C_1, \ldots, C_k \subset [n]$, we let $R_s = S_{R_1} \times S_{R_2} \times \ldots \times S_{R_l}$ be the row-stablizer of s and $C_s = S_{C_1} \times S_{C_2} \times \ldots \times S_{C_k}$ be the column-stabilizer.

An α-*tabloid* is an α-tableau with unordered row entries (or formally, a row-equivalence class of α-tableaux); given a tableau s, we write $[s]$ for the tabloid it produces. For example, the $(3, 2^2)$-tableau above produces the following $(3, 2^2)$-tabloid:

$$\begin{array}{lll} \{1 & 6 & 7\} \\ \{4 & 5\} & \\ \{2 & 3\} & \end{array}$$

Consider the natural left action of S_n on the set X^α of all α-tabloids; let $M^\alpha = \mathbb{C}[X^\alpha]$ be the corresponding permutation module, i.e. the complex vector space with basis X^α and S_n action given by extending this action linearly. Given an α-tableau s, we define the corresponding *α-polytabloid*

$$e_s := \sum_{\pi \in C_s} \epsilon(\pi)\pi[s].$$

We define the *Specht module* S^α to be the submodule of M^α spanned by the α-polytabloids:

$$S^\alpha = \mathrm{Span}\{e_s : \ s \text{ is an } \alpha\text{-tableau}\}.$$

A central observation in the representation theory of S_n is that *the Specht modules are a complete set of pairwise non-isomorphic, irreducible representations of S_n*. Hence, any irreducible representation ρ of S_n is isomorphic to some S^α. For example, $S^{(n)} = M^{(n)}$ is the trivial representation; $M^{(1^n)}$ is the left-regular representation, and $S^{(1^n)}$ is the sign representation S.

We say that a tableau is *standard* if the numbers strictly increase along each row and down each column. It turns out that for any partition α of n,

$$\{e_t : t \text{ is a standard } \alpha\text{-tableau}\}$$

is a basis for the Specht module S^α.

Given a partition α of n, for each cell (i, j) in its Young diagram, we define the 'hook-length' ($h^\alpha_{i,j}$) to be the number of cells in its 'hook' (the set of cells in the same row to the right of it or in the same column below it, including itself) — for example, the hook-lengths of $(3, 2^2)$ are as follows:

5	4	1
3	2	
2	1	

The dimension f^α of the Specht module S^α is given by the following formula

$$f^\alpha = n!/\prod(\text{hook lengths of } [\alpha]). \tag{2}$$

From now on we will write $[\alpha]$ for the equivalence class of the irreducible representation S^α, χ_α for the irreducible character χ_{S^α}, and ξ_α for the character of the permutation representation M^α. Notice that the set of α-tabloids form a basis for M^α, and therefore $\xi_\alpha(\sigma)$, the trace of the corresponding permutation representation at σ, is precisely the number of α-tabloids fixed by σ.

We now explain how the permutation modules M^β decompose into irreducibles.

Definition 2. *Let* α, β *be partitions of* n*. A* generalized α-tableau *is produced by replacing each dot in the Young diagram of* α *with a number between 1 and* n*; if a generalized* α*-tableau has* β_i i*'s* $(1 \leq i \leq n)$ *it is said to have* content β*. A generalized* α*-tableau is said to be* semistandard *if the numbers are non-decreasing along each row and strictly increasing down each column.*

Definition 3. *Let* α, β *be partitions of* n*. The* Kostka number $K_{\alpha,\beta}$ *is the number of semistandard generalized* α*-tableaux with content* β*.*

Young's Rule states that for any partition β of n, the permutation module M^{β} decomposes into irreducibles as follows:

$$M^{\beta} \cong \oplus_{\alpha \vdash n} K_{\alpha,\beta} S^{\alpha}.$$

For example, $M^{(n-1,1)}$, which corresponds to the natural permutation action of S_n on $[n]$, decomposes as

$$M^{(n-1,1)} \cong S^{(n-1,1)} \oplus S^{(n)},$$

and therefore

$$\xi_{(n-1,1)} = \chi_{(n-1,1)} + 1.$$

Let V_{α} be the subspace of $\mathbb{C}[S_n]$ consisting of functions whose Fourier transform is concentrated on $[\alpha]$; equivalently, V_{α} is the sum of all submodules of $\mathbb{C}S_n$ isomorphic to the Specht module S^{α}.

We call a partition of n (or an irreducible representation of S_n) 'fat' if its Young diagram has first row of length at least $n-t$. Let $\mathcal{F}_{n,t}$ denote the set of all fat partitions of n; note that for $n \geq 2t$,

$$|\mathcal{F}_{n,t}| = \sum_{s=0}^{t} p(s),$$

where $p(s)$ denotes the number of partitions of s. This grows very rapidly with t, but (as will be crucial for our stability analysis) it is independent of n for $n \geq 2t$. Note that $\{[\alpha] : \alpha \text{ is fat}\}$ are precisely the irreducible constituents of the permutation module $M^{(n-t,1^t)}$ corresponding to the action of S_n on t-tuples of distinct numbers, since $K_{\alpha,(n-t,1^t)} \geq 1$ iff there exists a semistandard generalized α-tableau of content $(n-t,1^t)$, i.e. iff $\alpha_1 \geq n-t$.

Recall from [6] that V_t is the subspace of functions whose Fourier transform is concentrated on the 'fat' irreducible representations of S_n; equivalently,

$$V_t = \bigoplus_{\text{fat } \alpha} V_\alpha. \tag{3}$$

The projection of $u \in \mathbb{C}[S_n]$ onto V_α has σ-coordinate

$$P_{V_\alpha}(u)_\sigma = \frac{f^\alpha}{n!} \sum_{\pi \in S_n} u(\pi)\chi_\alpha(\pi\sigma^{-1}),$$

and therefore the projection of u onto V_t has σ-coordinate

$$P_{V_t}(u)_\sigma = \frac{1}{n!} \sum_{\text{fat } \alpha} f^\alpha \sum_{\pi \in S_n} u(\pi)\chi_\alpha(\pi\sigma^{-1}). \tag{4}$$

3 Stability

We are now in a position to prove our rough stability result:

Theorem 6. *Let $t \in \mathbb{N}, c > 0$ be fixed. If $\mathcal{A} \subset S_n$ is a t-intersecting family with $|\mathcal{A}| \geq c(n-t)!$, then there exists a t-coset $\mathcal{C}$ such that $|\mathcal{A} \setminus \mathcal{C}| \leq O((n-t-1)!)$.*

In other words, if $\mathcal{A} \subset S_n$ is a t-intersecting family of size at least a constant proportion of the maximum possible size $(n-t)!$, then there is some t-coset containing all but at most a $O(1/n)$-fraction of $\mathcal{A}$.

To prove this, we will first prove the following weaker statement:

Lemma 7. *Let $t \in \mathbb{N}, c > 0$ be fixed. If $\mathcal{A} \subset S_n$ is a t-intersecting family of size at least $c(n-t)!$, then there exist i and j such that all but at most $O((n-t-1)!)$ permutations in $\mathcal{A}$ map i to j.*

In other words, a large t-intersecting family is almost contained within a 1-coset. Theorem 6 will follow easily from this by an inductive argument.

Given distinct $i_1, \ldots, i_l$ and distinct $j_1, \ldots, j_l$, we will write

$$\mathcal{A}_{i_1 \mapsto j_1, i_2 \mapsto j_2, \ldots, i_l \mapsto j_l} := \{\sigma \in \mathcal{A} : \sigma(i_k) = j_k \ \forall k \in [l]\}$$

To prove Lemma 7, we will first observe from (1) that if $\mathcal{A} \subset S_n$ is a t-intersecting family of size at least $c(n-t)!$ then the characteristic vector $v_\mathcal{A}$

of $\mathcal{A}$ is close to the subspace V_t spanned by the characteristic vectors of the t-cosets. We will use this, combined with representation-theoretic arguments, to show that there exists some t-coset $\mathcal{C}_0$ such that

$$|\mathcal{A} \cap \mathcal{C}_0| \geq \omega((n-2t)!)$$

—without loss of generality, $\mathcal{C}_0 = \{\sigma \in S_n : \ \sigma(1) = 1, \ldots, \sigma(t) = t\}$, so

$$|\mathcal{A}_{1\mapsto 1, 2\mapsto 2, \ldots, t\mapsto t}| \geq \omega((n-2t)!).$$

Note that the average size of the intersection of $\mathcal{A}$ with a t-coset is

$$|\mathcal{A}|/n(n-1)\ldots(n-t+1) = \Theta((n-2t)!).$$

We only know that $\mathcal{A} \cap \mathcal{C}_0$ has size ω of the average size. This statement would at first seem too weak to help us. However, for any distinct $j_1 \neq 1, j_2 \neq 2, \ldots,$ and $j_t \neq t$, the pair of families

$$\mathcal{A}_{1\mapsto 1, 2\mapsto 2, \ldots, t\mapsto t}, \quad \mathcal{A}_{1\mapsto j_1, 2\mapsto j_2, \ldots, t\mapsto j_t}$$

is t-cross-intersecting, so we may compare their sizes. In detail, we will deduce from Theorem 4 that

$$|\mathcal{A}_{1\mapsto 1, 2\mapsto 2, \ldots, t\mapsto t}||\mathcal{A}_{1\mapsto j_1, 2\mapsto j_2, \ldots, t\mapsto j_t}| \leq ((n-2t)!)^2,$$

giving $|\mathcal{A}_{1\mapsto j_1, \ldots, t\mapsto j_t}| \leq o((n-2t)!)$. Summing over all choices of $j_1, \ldots, j_t$ will show that all but at most $o((n-t)!)$ permutations in $\mathcal{A}$ fix some point of $[t]$, enabling us to complete the proof.

Proof of Lemma 7:
Let $\mathcal{A} \subset S_n$ be a t-intersecting family of size at least $c(n-t)!$; write $\delta = 1 - c < 1$. From (1), we know that the Euclidean distance from $v_{\mathcal{A}}$ to V_t is small:

$$||P_{V_t^{\perp}}(v_{\mathcal{A}})||^2 \leq \delta(1 + O(n^{1/6}))|\mathcal{A}|/n!.$$

From (4), the projection of $v_{\mathcal{A}}$ onto V_t has σ-coordinate:

$$P_{V_t}(v_{\mathcal{A}})_\sigma = \frac{1}{n!} \sum_{\text{fat } \alpha} f^\alpha \sum_{\pi \in \mathcal{A}} \chi_\alpha(\pi\sigma^{-1}).$$

Write $P_\sigma = P_{V_t}(v_{\mathcal{A}})_\sigma$; then

$$\frac{1}{n!}\left(\sum_{\sigma\in\mathcal{A}}(1-P_\sigma)^2+\sum_{\sigma\notin\mathcal{A}}P_\sigma^2\right)\leq \delta(1+O(1/n^{1/6}))|\mathcal{A}|/n!,$$

i.e.

$$\sum_{\sigma\in\mathcal{A}}(1-P_\sigma)^2+\sum_{\sigma\notin\mathcal{A}}P_\sigma^2\leq \delta(1+O(1/n^{1/6}))|\mathcal{A}|.$$

Choose $C>0$ such that $|\mathcal{A}|(1-1/n)\delta(1+C/n^{1/6})$ is at least the right-hand-side; then the subset

$$\mathcal{S} := \{\sigma\in\mathcal{A} : (1-P_\sigma)^2 < \delta(1+C/n^{1/6})\}$$

has size at least $|\mathcal{A}|/n$. Similarly, $P_\sigma^2 < 2\delta/n$ for all but at most

$$n|\mathcal{A}|(1+O(1/n))/2$$

permutations $\sigma\notin\mathcal{A}$. Provided n is sufficiently large, $|\mathcal{A}|\leq (n-t)!$, and therefore the subset $\mathcal{T}=\{\sigma\notin\mathcal{A} : P_\sigma^2 < 2\delta/n\}$ has size

$$|\mathcal{T}|\geq n!-(n-t)!-n(n-t)!(1+O(1/n))/2.$$

The permutations $\sigma\in\mathcal{S}$ have P_σ close to 1; the permutations $\pi\in\mathcal{T}$ have P_π close to 0. Using only our lower bounds on the sizes of $\mathcal{S}$ and $\mathcal{T}$, we may prove the following:

Claim: There exist permutations $\sigma\in\mathcal{S},\ \pi\in\mathcal{T}$ such that $\sigma^{-1}\pi$ is a product of at most $h=h(n)$ transpositions, where $h=\sqrt{2(t+2)(n-1)\log n}$.

Proof of Claim: Define the *transposition graph* H to be the Cayley graph on S_n generated by the transpositions, i.e. $V(H)=S_n$ and $\sigma\pi\in E(H)$ iff $\sigma^{-1}\pi$ is a transposition. We use the following isoperimetric inequality for H, essentially the martingale inequality of Maurey:

Theorem 8. *Let $X\subset V(H)$ with $|X|\geq\gamma n!$ where $0<\gamma<1$. Then for any $h\geq h_0 := \sqrt{\frac{1}{2}(n-1)\log\frac{1}{\gamma}}$,*

$$|N_h(X)|\geq\left(1-e^{-\frac{2(h-h_0)^2}{n-1}}\right)n!.$$

$\square$

For a proof, see for example [10]. Applying this to the set $\mathcal{S}$, which has $|\mathcal{S}| \geq (1-\delta)(n-t)!/n \geq \frac{n!}{n^{t+2}}$ (provided n is sufficiently large), with $\gamma = 1/n^{t+2}$, $h = 2h_0$, gives $|N_h(\mathcal{S})| \geq (1-n^{-(t+2)})n!$, so certainly $N_h(\mathcal{S}) \cap \mathcal{T} \neq \emptyset$, proving the claim.

We now have two permutations $\sigma \in \mathcal{A}$, $\pi \notin \mathcal{A}$ which are 'close' to one another in H (differing in only $O(\sqrt{n \log n})$ transpositions) such that

$$P_\sigma > 1 - \sqrt{\delta(1 + C/n^{1/6})}, \quad P_\pi < \sqrt{2\delta/n},$$

and therefore

$$P_\sigma - P_\pi > 1 - \sqrt{\delta} - O(1/n^{1/12}).$$

Hence, by averaging, there exist two permutations ρ, τ that differ by just one transposition and satisfy

$$P_\rho - P_\tau > (1 - \sqrt{\delta} - O(1/n^{1/12}))/h \geq \frac{1 - \sqrt{\delta} - O(1/n^{1/12})}{\sqrt{2(t+2)n \log n}},$$

i.e.

$$\sum_{\alpha \in \mathcal{F}_{n,t}} \frac{f^\alpha}{n!} \left(\sum_{\pi \in \mathcal{A}} \chi_\alpha(\pi\rho^{-1}) - \sum_{\pi \in \mathcal{A}} \chi_\alpha(\pi\tau^{-1}) \right) \geq \frac{1 - \sqrt{\delta} - O(1/n^{1/12})}{\sqrt{2(t+2)n \log n}}.$$

By double translation, we may assume without loss of generality that $\rho = \mathrm{Id}$, $\tau = (1\ 2)$. So we have:

$$\sum_{\alpha \in \mathcal{F}_{n,t}} \frac{f^\alpha}{n!} \left(\sum_{\pi \in \mathcal{A}} \chi_\alpha(\pi) - \sum_{\pi \in \mathcal{A}} \chi_\alpha(\pi(1\ 2)) \right) \geq \frac{1 - \sqrt{\delta} - O(1/n^{1/12})}{\sqrt{2(t+2)n \log n}}.$$

The above sum is over $|\mathcal{F}_{n,t}| = \sum_{s=0}^{t} p(s)$ partitions α of n; this grows very rapidly with t, but is independent of n for $n \geq 2t$. By averaging, there exists some $\alpha \in \mathcal{F}_{n,t}$ such that

$$\begin{aligned} \frac{f^\alpha}{n!} \left(\sum_{\pi \in \mathcal{A}} \chi_\alpha(\pi) - \sum_{\pi \in \mathcal{A}} \chi_\alpha(\pi(1\ 2)) \right) &\geq \frac{1 - \sqrt{\delta} - O(1/n^{1/12})}{\sqrt{2(t+2)n \log n}\sum_{s=0}^{t} p(s)} \\ &= \Omega(1/\sqrt{n \log n}). \end{aligned}$$

Recall that the 'fat' irreducible representations $\{[\alpha] : \alpha \in \mathcal{F}_{n,t}\}$ are precisely the irreducible constituents of $M^{(n-t,1^t)}$, so very crudely, for each fat α,

$$f^\alpha \leq \dim(M^{(n-t,1^t)}) = n(n-1)\dots(n-t+1).$$

Hence,

$$\sum_{\pi \in \mathcal{A}} \chi_\alpha(\pi) - \sum_{\pi \in \mathcal{A}} \chi_\alpha(\pi(1\ 2)) \geq \Omega(1/\sqrt{n \log n})(n-t)!.$$

But for any $\alpha \in \mathcal{F}_{n,t}$, we may express the irreducible character χ_α as a linear combination of permutation characters $\xi_\beta : \beta \in \mathcal{F}_{n,t}$ using the following 'determinantal formula' (see [8]). For any partition α of n,

$$\chi_\alpha = \sum_{\pi \in S_n} \epsilon(\pi)\xi_{\alpha-\mathrm{id}+\pi}.$$

Here, for $\alpha = (\alpha_1, \dots, \alpha_l) \vdash n$, we set $\alpha_i = 0$ $(l < i \leq n)$, we think of α, id and π as sequences of length n, and we define addition and subtraction of these sequences pointwise. In general,

$$\alpha - \mathrm{id} + \pi = (\alpha_1 - 1 + \pi(1), \alpha_2 - 2 + \pi(2), \dots, \alpha_n - n + \pi(n))$$

will be a sequence of n integers with sum n, i.e. a *composition* of n. If λ is a composition of n with all its terms non-negative, then let $\bar{\lambda}$ be the partition of n produced by ordering the terms of λ in non-increasing order, and define $\xi_\lambda = \xi_{\bar{\lambda}}$; if λ has a negative term, we define $\xi_\lambda = 0$. If $\alpha \in \mathcal{F}_{n,t}$, then as $\alpha_1 \geq n-t$, any composition occurring in the above sum has first term at least $n-t$, and therefore ξ_β can only occur in the above sum if $\beta \in \mathcal{F}_{n,t}$. Observe further that since α has at most $t+1$ non-zero parts, $\alpha_i = 0$ for every $i > t+1$, and therefore any permutation $\pi \in S_n$ with $\xi_{\alpha-\mathrm{id}+\pi} \neq 0$ must have $\pi(i) \geq i$ for every $i > t+1$, so must fix $t+2, t+3, \dots,$ and n. Therefore, the above sum is only over $\pi \in S_{\{1,\dots,t+1\}}$, i.e.

$$\chi_\alpha = \sum_{\pi \in S_{t+1}} \epsilon(\pi)\xi_{\alpha-\mathrm{id}+\pi}\ \forall \alpha \in \mathcal{F}_{n,t}.$$

Therefore, χ_α is a (± 1)-linear combination of at most $(t+1)!$ permutation characters ξ_β $(\beta \in \mathcal{F}_{n,t})$, possibly with repeats. Hence, by averaging, there exists some $\beta \in \mathcal{F}_{n,t}$ such that

$$\begin{aligned} \left| \sum_{\pi \in \mathcal{A}} \xi_\beta(\pi) - \sum_{\pi \in \mathcal{A}} \xi_\beta(\pi(1\ 2)) \right| &\geq \Omega(1/\sqrt{n \log n}) \frac{(n-t)!}{(t+1)!} \\ &= \Omega(1/\sqrt{n \log n})(n-t)!. \end{aligned}$$

Without loss of generality, we may assume that the above quantity is positive, i.e.

$$\sum_{\pi \in \mathcal{A}} \xi_\beta(\pi) - \sum_{\pi \in \mathcal{A}} \xi_\beta(\pi(1\ 2)) \geq \Omega(1/\sqrt{n \log n})(n-t)!.$$

Let $\mathbb{T}_\beta$ be the set of β-tabloids; the left-hand-side is then

$$\begin{aligned} & \#\{(T,\pi) : T \in \mathbb{T}_\beta, \pi \in \mathcal{A}, \pi(T) = T\} \\ - \ & \#\{(T,\pi) : T \in \mathbb{T}_\beta, \pi \in \mathcal{A}, \pi(1\ 2)(T) = T.\} \end{aligned}$$

Interchanging the order of summation, this equals

$$\sum_{T \in \mathbb{T}_\beta} \left(\#\{\pi \in \mathcal{A} : \pi(T) = T\} - \#\{\pi \in \mathcal{A} : \pi(1\ 2)(T) = T\}\right).$$

The above summand is zero for all β-tabloids T with 1 and 2 in the first row of T (as then $(1\ 2)T = T$). Write $\beta = (n-s, \beta_2, \ldots, \beta_l)$, where $0 \leq s \leq t$. The number of β-tabloids with 1 not in the first row is

$$s(n-1)(n-2)\ldots(n-s+1)/\prod_{i=2}^{l} \beta_i!,$$

and therefore the number of β-tabloids with 1 or 2 below the first row is at most

$$\begin{aligned} 2s(n-1)(n-2)\ldots(n-s+1)/\prod_{i=2}^{l}\beta_i! &\leq 2t(n-1)(n-2)\ldots(n-s+1) \\ &= \frac{2t(n-1)!}{(n-s)!}. \end{aligned}$$

Hence by averaging, for one such β-tabloid T,

$$\begin{aligned} &\#\{\pi \in \mathcal{A} : \pi(T) = T\} - \#\{\pi \in \mathcal{A} : \pi(1\ 2)(T) = T\} \\ &\qquad \geq \Omega(1/\sqrt{n \log n})\frac{(n-s)!}{2t(n-1)!}(n-t)!, \end{aligned}$$

and therefore the number of permutations in $\mathcal{A}$ fixing T satisfies

$$\#\{\pi \in \mathcal{A} : \pi(T) = T\} \geq \Omega(1/\sqrt{n \log n})\frac{(n-s)!}{2t(n-1)!}(n-t)!.$$

Without loss of generality, we may assume that the first row of T consists of the numbers $\{s+1,\dots,n\}$. There are $\beta_2!\beta_3!\dots\beta_l! \leq s! \leq t!$ permutations of $[s]$ fixing the 2^{nd},3^{rd},..., and l^{th} rows of T; any permutation fixing T must agree with one of these permutations on $[s]$. Hence, there exists a permutation ρ of $[s]$ such that at least

$$\Omega(1/\sqrt{n\log n})\frac{(n-s)!(n-t)!}{2t(n-1)!t!}$$

permutations in $\mathcal{A}$ agree with ρ on $[s]$. Without loss of generality, we may assume that $\rho = \mathrm{Id}_{[s]}$, so the number of permutations in $\mathcal{A}$ fixing $[s]$ pointwise satisfies

$$\begin{aligned} |\mathcal{A}_{1\mapsto 1,\dots,s\mapsto s}| &\geq \Omega(1/\sqrt{n\log n})\frac{(n-s)!(n-t)!}{2t(n-1)!t!} \\ &= \Omega(1/\sqrt{n\log n})\frac{(n-s)!(n-t)!}{(n-1)!}. \end{aligned}$$

We may write $\mathcal{A}_{1\mapsto 1,\dots,s\mapsto s}$ as a disjoint union

$$\mathcal{A}_{1\mapsto 1,\dots,s\mapsto s} = \bigcup_{j_{s+1},\dots,j_t>s \text{ distinct}} \mathcal{A}_{1\mapsto 1\dots,s\mapsto s,s+1\mapsto j_{s+1},\dots,t\mapsto j_t},$$

and there are $(n-s)(n-s-1)\dots(n-t+1)$ choices of $j_{s+1},\dots,j_t$, so by averaging, there exists a choice such that

$$|\mathcal{A}_{1\mapsto 1\dots,s\mapsto s,s+1\mapsto j_{s+1},\dots,t\mapsto j_t}| \geq \Omega(1/\sqrt{n\log n})\frac{((n-t)!)^2}{(n-1)!}.$$

By translation, we may assume without loss of generality that $j_k = k$ for each k, so

$$\begin{aligned} |\mathcal{A}_{1\mapsto 1,2\mapsto 2,\dots,t\mapsto t}| &\geq \Omega(1/\sqrt{n\log n})\frac{((n-t)!)^2}{(n-1)!} \\ &= \Omega(\sqrt{n/\log n})(n-2t)! \\ &= \omega((n-2t)!). \end{aligned}$$

We will use this to show that the number of permutations in $\mathcal{A}$ with no fixed point in $[t]$ is small. We may write

$$\mathcal{A} \setminus (\mathcal{A}_{1\mapsto 1} \cup \dots \cup \mathcal{A}_{t\mapsto t}) = \bigcup_{j_1,\dots,j_t \text{ distinct } : j_k\neq k \ \forall k\in[t]} \mathcal{A}_{1\mapsto j_1,\dots,t\mapsto j_t}.$$

We now show that each $\mathcal{A}_{1\mapsto j_1,\ldots,t\mapsto j_t}$ is small using Theorem 4. Let $J = \{j_1,\ldots,j_t\}$. Notice that $\mathcal{E} := \mathcal{A}_{1\mapsto 1,\ldots,t\mapsto t}$, $\mathcal{F} := \mathcal{A}_{1\mapsto j_1,\ldots,t\mapsto j_t}$ is a t-cross-intersecting pair of families, so for any $\sigma \in \mathcal{E}$ and $\pi \in \mathcal{F}$, there are t distinct points $i_1, i_2 \ldots, i_t > t$ such that $\sigma(i_k) = \pi(i_k) \notin [t] \cup J$ for each $k \in [t]$. But then

$$(1\ j_1)(2\ j_2)\ldots(t\ j_t)\pi(i_k) = \sigma(i_k) \quad \text{for each } k \in [t],$$

so letting $\mathcal{G} := (1\ j_1)(2\ j_2)\ldots(t\ j_t)\mathcal{F}$, the pair of families $\mathcal{E}, \mathcal{G}$ fix $[t]$ pointwise and t-cross-intersect on $\{t+1, t+2, \ldots, n\}$. Deleting $1, \ldots, t$ we obtain a t-cross-intersecting pair $\mathcal{E}', \mathcal{G}'$ of subsets of $S_{\{t+1,\ldots,n\}}$. By Theorem 4,

$$|\mathcal{A}_{1\mapsto 1,\ldots,t\mapsto t}||\mathcal{A}_{1\mapsto j_1,\ldots,t\mapsto j_t}| = |\mathcal{E}||\mathcal{G}| = |\mathcal{E}'||\mathcal{G}'| \leq ((n-2t)!)^2.$$

Since

$$|\mathcal{A}_{1\mapsto 1,\ldots,t\mapsto t}| \geq \omega((n-2t)!),$$

we have

$$|\mathcal{A}_{1\mapsto j_1,\ldots,t\mapsto j_t}| \leq o((n-2t)!).$$

There are $\leq n(n-1)(n-2)\ldots(n-t+1)$ possible choices of $j_1,\ldots,j_t$, and therefore the number of permutations in $\mathcal{A}$ with no fixed point in $[t]$ satisfies

$$\begin{aligned} |\mathcal{A} \setminus (\mathcal{A}_{1\mapsto 1} \cup \mathcal{A}_{2\mapsto 2} \cup \ldots \cup \mathcal{A}_{t\mapsto t})| &\leq o((n-2t)!)n(n-1)\ldots(n-t+1) \\ &= o((n-t)!). \end{aligned}$$

Since $|\mathcal{A}| \geq c(n-t)!$, we have

$$|\mathcal{A}_{1\mapsto 1} \cup \mathcal{A}_{2\mapsto 2} \cup \ldots \cup \mathcal{A}_{t\mapsto t}| \geq (c - o(1))(n-t)!.$$

By averaging, there exists some $i \in [t]$ such that

$$|\mathcal{A}_{i\mapsto i}| \geq (c - o(1))(n-t)!/t.$$

We may assume that $i = 1$, so $|\mathcal{A}_{1\mapsto 1}| \geq (c-o(1))(n-t)!/t$. Now, using the same trick as before, we may use Theorem 4 to show that $|\mathcal{A} \setminus \mathcal{A}_{1\mapsto 1}| \leq O((n-t-1)!)$. Indeed, write $\mathcal{A} \setminus \mathcal{A}_{1\mapsto 1}$ as a disjoint union

$$\mathcal{A} \setminus \mathcal{A}_{1\mapsto 1} = \bigcup_{j\neq 1} \mathcal{A}_{1\mapsto j}.$$

We will show that each $\mathcal{A}_{1\mapsto j}$ is small. Notice as before that the pair of families $\mathcal{A}_{1\mapsto 1}$, $(1\ j)\mathcal{A}_{1\mapsto j}$ fixes 1 and t-cross-intersects on the domain $\{2,\ldots,n\}$, so Theorem 4 gives

$$|\mathcal{A}_{1\mapsto 1}||\mathcal{A}_{1\mapsto j}| \leq ((n-t-1)!)^2.$$

Since $|\mathcal{A}_{1\mapsto 1}| \geq \Omega((n-t)!)$, we obtain $|\mathcal{A}_{1\mapsto j}| \leq O((n-t-2)!)$, and therefore

$$|\mathcal{A} \setminus \mathcal{A}_{1\mapsto 1}| = \sum_{j\neq 1} |\mathcal{A}_{1\mapsto j}| \leq O((n-t-1)!).$$

proving Lemma 7.

□

Proof of Theorem 6:
By induction on t. The $t = 1$ case is the same as that of Lemma 7. Assume the theorem is true for $t-1$; we will prove it for t. Let $\mathcal{A} \subset S_n$ be a t-intersecting family of size at least $c(n-t)!$. By Lemma 7, there exist i and j such that $|\mathcal{A} \setminus \mathcal{A}_{i\mapsto j}| \leq O((n-t-1)!)$. Without loss of generality we may assume that $i = j = 1$, so $|\mathcal{A} \setminus \mathcal{A}_{1\mapsto 1}| \leq O((n-t-1)!)$. Hence, $|\mathcal{A}_{1\mapsto 1}| \geq |\mathcal{A}| - O((n-t-1)!)$. Deleting 1 from each permutation in $\mathcal{A}_{1\mapsto 1}$, we obtain a $(t-1)$-intersecting family $\mathcal{A}' \subset S_{\{2,3,\ldots,n\}}$ of size $\geq (c - O(1/n))(n-t)!$. Choose any positive constant $c' < c$; then provided n is sufficiently large, we have $|\mathcal{A}'| \geq c'(n-t)!$. By the induction hypothesis, there exists a $(t-1)$-coset $\mathcal{C}'$ of $S_{2,\ldots,n}$ such that $|\mathcal{A}' \setminus \mathcal{C}'| \leq O((n-t-1)!)$. Then if $\mathcal{C}$ is the t-coset obtained from $\mathcal{C}'$ by adjoining $1 \mapsto 1$, we have $|\mathcal{A} \setminus \mathcal{C}| \leq O((n-t-1)!)$. This completes the induction and proves Theorem 6.

□

We now use our rough stability result to prove an exact stability result. First, we need some more definitions.

Let d_n be the number of *derangements* of $[n]$ (permutations of $[n]$ without fixed points). It is a well-known consequence of the inclusion-exclusion formula that $d_n = (1/e + o(1))n!$.

Following Cameron and Ku [3], given a permutation $\rho \in S_n$ and $i \in [n]$, we define the *i-fix* of ρ to be the permutation ρ_i which fixes i, maps the preimage of i to the image of i, and agrees with ρ at all other points of $[n]$, i.e.

$$\rho_i(i) = i;\ \rho_i(\rho^{-1}(i)) = \rho(i);\ \rho_i(k) = \rho(k)\ \forall k \neq i, \rho^{-1}(i).$$

In other words, $\rho_i = \rho(\rho^{-1}(i)\ i)$. We inductively define

$$\rho_{i_1,\ldots,i_l} = (\rho_{i_1,\ldots,i_{l-1}})_{i_l}.$$

Notice that if σ fixes j, then σ agrees with ρ_j wherever it agrees with ρ.

Theorem 9. *For n sufficiently large depending on t, if $\mathcal{A} \subset S_n$ is a t-intersecting family which is not contained within a t-coset, then $\mathcal{A}$ is no larger than the family*

$$\begin{aligned}\mathcal{D} &= \{\sigma \in S_n : \ \sigma(i) = i \ \forall i \leq t, \ \sigma(j) = j \textit{ for some } j > t+1\} \\ &\quad \cup \{(1\ t+1), (2\ t+1), \ldots, (t\ t+1)\},\end{aligned}$$

which has size $(n-t)! - d_{n-t} - d_{n-t-1} + t = (1 - 1/e + o(1))(n-t)!$. If $\mathcal{A}$ is the same size as $\mathcal{D}$, then $\mathcal{A}$ is a double translate of $\mathcal{D}$, i.e. $\mathcal{A} = \pi \mathcal{D} \tau$ for some $\pi, \tau \in S_n$.

Proof. Suppose $\mathcal{A} \subset S_n$ is a t-intersecting family which is not contained within a t-coset, and has size

$$|\mathcal{A}| \geq (n-t)! - d_{n-t} - d_{n-t-1} + t = (1 - 1/e + o(1))(n-t)!.$$

Applying Theorem 6 with any constant c such that $0 < c < 1 - 1/e$, we see that (provided n is sufficiently large) there exists a t-coset $\mathcal{C}$ such that

$$|\mathcal{A} \setminus \mathcal{C}| \leq O(1/n)(n-t)!.$$

By double translation, without loss of generality we may assume that $\mathcal{C} = \{\sigma \in S_n : \sigma(1) = 1, \ldots, \sigma(t) = t\}$. We have:

$$\begin{aligned}|\mathcal{A} \cap \mathcal{C}| &\geq (n-t)! - d_{n-t} - d_{n-t-1} + t - O(1/n)(n-t)! \\ &= (1 - 1/e + o(1))(n-t)!. \qquad (5)\end{aligned}$$

We now claim that every permutation in $\mathcal{A} \setminus \mathcal{C}$ fixes exactly $t-1$ points of $[t]$. Suppose for a contradiction that $\mathcal{A}$ contains a permutation τ fixing at most $t-2$ points of $[t]$. Then every permutation in $\mathcal{A} \cap \mathcal{C}$ must agree with τ on at least 2 points of $\{t+1, \ldots, n\}$, so

$$|\mathcal{A} \cap \mathcal{C}| \leq \binom{n-t}{2}(n-t-2)! = \tfrac{1}{2}(n-t)!$$

contradicting (5), provided n is sufficiently large.

Since we are assuming that $\mathcal{A}$ is not contained within a t-coset, $\mathcal{A} \setminus \mathcal{C}$ contains some permutation τ; τ must fix all points of $[t]$ except for one. By double translation, we may assume that $\tau = (1\ t+1)$. We will show that under these hypotheses, $\mathcal{A} = \mathcal{D}$.

Every permutation in $\mathcal{A} \cap \mathcal{C}$ must t-intersect $(1\ t+1)$ and must therefore have at least one fixed point $> t+1$, i.e. $\mathcal{A} \cap \mathcal{C}$ is a subset of the family

$$\mathcal{E} := \{\sigma \in S_n : \sigma(i) = i\ \forall i \in [t],\ \sigma(j) = j \text{ for some } j > t+1\},$$

which has size

$$(n-t)! - d_{n-t} - d_{n-t-1}.$$

We now make the following observation:

Claim: $\mathcal{A} \setminus \mathcal{C}$ may only contain the transpositions $\{(i\ t+1)\ : i \in [t]\}$.

Proof of Claim:
Suppose for a contradiction that $\mathcal{A} \setminus \mathcal{C}$ contains a permutation ρ not of this form. Then $\rho(j) \neq j$ for some $j \geq t+2$. We will show that there are at least d_{n-t-1} permutations in $\mathcal{E}$ which fix j and disagree with ρ at every point of $\{t+1, t+2, \ldots, n\}$, and therefore cannot t-intersect ρ. Let l be the unique point of $[t]$ not fixed by ρ. If σ fixes both l and j, then σ agrees with $\rho_{j,l} = (\rho_j)_l$ wherever it agrees with ρ. Notice that $\rho_{j,l}$ fixes $1, 2, \ldots, t$ and j. There are exactly d_{n-t-1} permutations in $\mathcal{E}$ which fix j and disagree with $\rho_{j,l}$ at every point of $\{t+1, t+2, \ldots, n\} \setminus \{j\}$; each disagrees with ρ at every point of $\{t+1, t+2, \ldots, n\}$. So none t-intersect ρ, so none are in $\mathcal{A}$, and therefore

$$|\mathcal{A} \cap \mathcal{C}| \leq |\mathcal{E}| - d_{n-t-1} = (n-t)! - d_{n-t} - 2d_{n-t-1}.$$

Since we are assuming that $|\mathcal{A}| \geq (n-t)! - d_{n-t} - d_{n-t-1} + t$, this means that

$$|\mathcal{A} \setminus \mathcal{C}| \geq d_{n-t-1} + t = (1/e + o(1))(n-t-1)!.$$

Notice that for any $m \leq n$ we have the following trivial upper bound on the size of an m-intersecting family $\mathcal{H} \subset S_n$:

$$|\mathcal{H}| \leq \binom{n}{m}(n-m)! = n!/m!$$

since every permutation in $\mathcal{H}$ must agree with a fixed permutation in $\mathcal{H}$ in at least m places.

Hence, $\mathcal{A} \setminus \mathcal{C}$ cannot be $(\log n)$-intersecting and therefore contains two permutations π, τ agreeing on at most $\log n$ points. The number of permutations fixing $[t]$ pointwise and agreeing with both π and τ at one of these

$\log n$ points is therefore at most $(\log n)(n-t-1)!$. All other permutations in $\mathcal{A} \cap \mathcal{C}$ agree with π and τ at two separate points of $\{t+1,\ldots,n\}$, and by the above argument, the same holds for π_p and τ_q, where p and q are the points of $[t]$ shifted by π and τ respectively. The number of permutations in $\mathcal{C}$ that agree with π_p and τ_q at two separate points of $\{t+1,\ldots,n\}$ is at most $((1-1/e)^2+o(1))(n-t)!$ (it is easily checked that given two fixed permutations, the probability that a uniform random permutation agrees with them at separate points is at most $(1-1/e)^2+o(1)$), which implies that

$$\begin{aligned} |\mathcal{A} \cap \mathcal{C}| &\leq ((1-1/e)^2+o(1))(n-t)! + (\log n)(n-t-1)! \\ &= ((1-1/e)^2+o(1))(n-t)! \end{aligned}$$

contradicting (5), provided n is sufficiently large. This proves the claim.

Since we are assuming $|\mathcal{A}| \geq |\mathcal{E}| + t$, we must have equality, so $\mathcal{A} = \mathcal{D}$, proving Theorem 9. □

Similar arguments give the following stability results for t-cross-intersecting families. Say two pairs of families $(\mathcal{A},\mathcal{B})$, $(\mathcal{C},\mathcal{D})$ in S_n are *isomorphic* if there exist permutations $\pi,\rho \in S_n$ such that $\mathcal{A} = \pi\mathcal{C}\rho$ and $\mathcal{B} = \pi\mathcal{D}\rho$. We have:

Theorem 10. *For n sufficiently large depending on t, if $\mathcal{A},\mathcal{B} \subset S_n$ are t-cross-intersecting but not both contained within the same t-coset, then*

$$\min(|\mathcal{A}|,|\mathcal{B}|) \leq (n-t)! - d_{n-t} - d_{n-t-1} + t.$$

Equality holds if and only if $(\mathcal{A},\mathcal{B})$ is isomorphic to the pair of families

$$\{\sigma : \sigma(i) = i\ \forall i \leq t,\ \sigma(j) = \tau(j) \text{ for some } j > t+1\} \cup \{(i\ t+1) :\ i \in [t]\}$$

$$\{\sigma : \sigma(i) = i\ \forall i \leq t,\ \sigma(j) = j \text{ for some } j > t+1\} \cup \{(1i)\tau(1i) : i \in [t]\}$$

where $\tau(1) \neq 1$ and if $t \geq 2$, τ fixes $2,3,\ldots,t$ and at least two points $> t+1$, whereas if $t = 1$, τ intersects $(1\ 2)$.

Theorem 11. *For n sufficiently large depending on t, if $\mathcal{A},\mathcal{B} \subset S_n$ are t-cross-intersecting but not both contained within the same t-coset, then*

$$|\mathcal{A}||\mathcal{B}| \leq ((n-t)! - d_{n-t} - d_{n-t-1})((n-t)! + t).$$

Equality holds if and only if $(\mathcal{A},\mathcal{B})$ is isomorphic to the pair of families

$$\{\sigma \in S_n : \sigma(i) = i\ \forall i \leq t,\ \sigma(j) = j \text{ for some } j > t+1\}$$

$$\{\sigma \in S_n : \sigma(i) = i\ \forall i \leq t\} \cup \{(1\ t+1),(2\ t+1),\ldots,(t\ t+1)\}$$

The proofs are very similar to the proof of Theorem 9, and we omit them.

4 The Alternating Group

We now turn our attention to the alternating group A_n, the index-2 subgroup of S_n consisting of the even permutations of $\{1, 2, \ldots, n\}$. The following may be deduced from the proof of the Deza-Frankl conjecture in [6]:

Theorem 12. *For n sufficiently large depending on t, if $\mathcal{A} \subset A_n$ is t-intersecting, then $|\mathcal{A}| \leq (n-t)!/2$.*

Remark: This implies the Deza-Frankl conjecture. To see this, let $\mathcal{A} \subset S_n$ be t-intersecting; then $\mathcal{A} \cap A_n$ and $(\mathcal{A} \setminus A_n)(1\ 2)$ are both t-intersecting families of permutations in A_n, so by Theorem 12, both have size at most $(n-t)!/2$. Hence,

$$|\mathcal{A}| = |\mathcal{A} \cap A_n| + |\mathcal{A} \setminus A_n| \leq (n-t)!.$$

Proof. Recall that in [6], we constructed a weighted graph Y_{even} which was a real linear combination of Cayley graphs on S_n generated by conjugacy-classes of *even* permutations with less than t fixed points, and whose matrix of weights had maximum eigenvalue 1 and minimum eigenvalue

$$\omega_{n,t} = -\frac{1}{n(n-1)\ldots(n-t+1)-1}.$$

Clearly, Y_{even} has no (non-zero) edges between A_n and $S_n \setminus A_n$. Let Y_1 be the weighted subgraph of Y_{even} induced on A_n, and Y_2 the weighted subgraph induced on $S_n \setminus A_n$. Notice that the map

$$\begin{aligned} \phi : A_n &\rightarrow S_n \setminus A_n; \\ \sigma &\mapsto (1\ 2)\sigma \end{aligned}$$

is a graph isomorphism from Y_1 to Y_2. To see this, note that

$$\phi(\sigma)(\phi(\pi))^{-1} = ((1\ 2)\sigma)((1\ 2)\pi)^{-1} = (1\ 2)\sigma\pi^{-1}(1\ 2),$$

which is conjugate to $\sigma\pi^{-1}$. Since Y_{even} is a linear combination of Cayley graphs generated by conjugacy-classes of S_n, the edge $\phi(\sigma)\phi(\pi)$ has the same weight in Y_{even} as the edge $\sigma\pi$. Hence, Y_{even} is a disjoint union of the two isomorphic subgraphs Y_1 and Y_2, so the eigenvalues of Y_{even} are the same as those of Y_1 (with double the multiplicities). Applying Theorem 1 to Y_1 proves Theorem 12. □

Our next aim is to show that equality holds in Theorem 12 only if $\mathcal{A}$ is a coset of the stabilizer of t points. As for S_n, we will call these families the 't-cosets of A_n'.

Let W_t be the subspace of $\mathbb{C}[A_n]$ spanned by the characteristic vectors of the t-cosets of A_n. It is easily checked that W_t is the direct sum of the 1- and $\omega_{n,t}$-eigenspaces of Y_1. Hence, by Theorem 1, if equality holds in Theorem 12, then the characteristic vector $v_{\mathcal{A}}$ of $\mathcal{A}$ lies in the subspace W_t.

We would like to show that the Boolean functions which are linear combinations of the characteristic functions of the t-cosets of A_n are precisely the characteristic functions of the disjoint unions of t-cosets of A_n. To do this for S_n in [6], it was first proved that if a non-negative function $f : S_n \to \mathbb{R}_{\geq 0}$ is a linear combination of the characteristic functions of the t-cosets of S_n, then it can be expressed as a linear combination of them with non-negative coefficients. However, this is not true in the case of A_n, even for $t = 1$:

Claim: There exists a non-negative function in W_1 which cannot be written as a non-negative linear combination of the characteristic functions of the 1-cosets of A_n.

Proof of Claim: Let $w_{i \mapsto j}$ be the characteristic function of the 1-coset $\{\sigma \in A_n : \sigma(i) = j\}$. We say a real $n \times n$ matrix B *represents* a function $f \in W_1$ if f can be written as a linear combination of $w_{i \mapsto j}$'s with coefficients given by the matrix B, i.e.

$$f = \sum_{i,j=1}^{n} b_{i,j} w_{i \mapsto j},$$

or equivalently,

$$f(\sigma) = \sum_{i=1}^{n} b_{i,\sigma(i)} \quad \forall \sigma \in A_n.$$

It is easy to see that, provided $n \geq 4$, any function $f \in W_1$ has a unique extension to a function $\tilde{f} \in V_1$. Hence, if B and C are two matrices both representing f, they must both represent the same function $\tilde{f} : S_n \to \mathbb{R}$, and therefore

$$\sum_{i=1}^{n} b_{i,\sigma(i)} = \sum_{i=1}^{n} c_{i,\sigma(i)} \quad \forall \sigma \in S_n.$$

Now let f be the function represented by the matrix

$$B = \begin{pmatrix} 1 & -1/2 & 1 & 1 & \dots & 1 \\ -1/2 & 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 0 & 1 & \dots & 1 \\ \vdots & \vdots & & \ddots & & \vdots \\ 1 & 1 & \dots & & & 0 \end{pmatrix}.$$

This takes only non-negative values on A_n, since

$$\sum_{i=1}^{n} b_{i,\sigma(i)} \geq 0 \quad \forall \sigma \in A_n,$$

but if τ is the transposition (1 2), then

$$\sum_{i=1}^{n} b_{i,\tau(i)} = -1.$$

Hence, any matrix C representing the same function as B must also have

$$\sum_{i=1}^{n} c_{i,\tau(i)} = -1,$$

and therefore cannot have non-negative entries. Therefore, f is a non-negative function in W_1 that cannot be written as a non-negative linear combination of the $w_{i\mapsto j}$'s, proving the claim.

Instead, we obtain our desired characterization of equality in Theorem 12 from a stability result for t-intersecting families in A_n.

Let e_n, o_n denote the number of respectively even/odd derangements of $[n]$. It is well-known (and easy to prove, by induction on n) that $e_n - o_n = (-1)^{n-1}(n-1)\ \forall n \in \mathbb{N}$; combining this with the fact that $d_n = (1/e + o(1))n!$ gives $e_n = (1/(2e) + o(1))n!,\ o_n = (1/(2e) + o(1))n!$.

We now prove the following analogue of Theorem 9:

Theorem 13. *For n sufficiently large depending on t, if $\mathcal{A} \subset A_n$ is a t-intersecting family which is not contained within a t-coset of A_n, then $\mathcal{A}$ cannot be larger than the family*

$$\begin{aligned} \mathcal{B} &= \{\sigma \in A_n : \sigma(i) = i\ \forall i \leq t,\ \sigma(j) = (n-1\ n)(j) \text{ for some } j > t+1\} \\ &\quad \cup \{(1\ t+1)(n-1\ n), (2\ t+1)(n-1\ n), \dots, (t\ t+1)(n-1\ n)\}, \end{aligned}$$

which has size $(n-t)!/2 - o_{n-t} - o_{n-t-1} + t = (1-1/e+o(1))(n-t)!/2$. *If* $\mathcal{A}$ *is the same size as* $\mathcal{B}$*, then* $\mathcal{A}$ *is a double translate of* $\mathcal{B}$*, meaning that* $\mathcal{A} = \pi\mathcal{B}\tau$ *for some* $\pi, \tau \in A_n$*.*

Proof. Let $\mathcal{A} \subset A_n$ be a t-intersecting family which is not contained within a t-coset of A_n and has size

$$|\mathcal{A}| \geq (n-t)!/2 - o_{n-t} - o_{n-t-1} + t = (1-1/e+o(1))(n-t)!/2.$$

Applying Theorem 6 with any constant c such that $0 < c < (1-1/e)/2$, we see that (provided n is sufficiently large) there exists a t-coset $\mathcal{C}$ such that

$$|\mathcal{A} \setminus \mathcal{C}| \leq O(1/n)(n-t)!.$$

By double translation, without loss of generality we may assume that $\mathcal{C} = \{\sigma \in A_n : \sigma(1) = 1, \ldots, \sigma(t) = t\}$. We have:

$$\begin{aligned} |\mathcal{A} \cap \mathcal{C}| &\geq (n-t)!/2 - o_{n-t} - o_{n-t-1} + t - O(1/n)(n-t)! \\ &= (1-1/e+o(1))(n-t)!/2. \end{aligned} \tag{6}$$

We now claim that every permutation in $\mathcal{A} \setminus \mathcal{C}$ fixes exactly $t-1$ points of $[t]$. Suppose for a contradiction that $\mathcal{A}$ contains a permutation τ fixing at most $t-2$ points of $[t]$. Then every permutation in $\mathcal{A} \cap \mathcal{C}$ must agree with τ on at least 2 points of $\{t+1, \ldots, n\}$, so

$$|\mathcal{A} \cap \mathcal{C}| \leq \binom{n-t}{2}(n-t-2)!/2 = \tfrac{1}{2}(n-t)!/2,$$

contradicting (6), provided n is sufficiently large.

Since we are assuming that $\mathcal{A}$ is not contained within a t-coset, $\mathcal{A} \setminus \mathcal{C}$ contains some permutation τ; τ must fix all points of $[t]$ except for one. By double translation, we may assume that $\tau = (1\ t+1)(n-1\ n)$. We will show that under these hypotheses, $\mathcal{A} = \mathcal{B}$. Every permutation in $\mathcal{A} \cap \mathcal{C}$ must agree with $(n-1\ n)$ at some point $\geq t+2$, i.e. $\mathcal{A} \cap \mathcal{C}$ is a subset of the family

$$\mathcal{E} := \{\sigma \in A_n : \sigma(i) = i\ \forall i \in [t],\ \sigma(j) = (n-1\ n)(j) \text{ for some } j \geq t+2\},$$

which has size

$$(n-t)!/2 - o_{n-t} - o_{n-t-1}.$$

We now make the following observation:

Claim: $\mathcal{A}\backslash\mathcal{C}$ may only contain the permutations $\{(i\,t{+}1)(n{-}1\;n) : i \in [t]\}$.

Proof of Claim:
Suppose for a contradiction that $\mathcal{A} \setminus \mathcal{C}$ contains a permutation ρ not of this form. Then $\rho(j) \neq (n-1\;n)(j)$ for some $j \geq t+2$, so by a very similar argument to in the proof of Theorem 9, there are at least $\min(e_{n-t-1}, o_{n-t-1})$ even permutations which fix $1, 2, \ldots, t$ and agree with $(n-1\;n)$ at j (and are therefore in $\mathcal{E}$) and also disagree with ρ at all points of $\{t{+}1, t{+}2, \ldots, n\}\backslash\{j\}$. Since ρ has exactly $t-1$ fixed points in $[t]$, none of these permutations can t-intersect ρ, and therefore

$$\begin{aligned}|\mathcal{A} \cap \mathcal{C}| &\leq |\mathcal{E}| - \min(e_{n-t-1}, o_{n-t-1}) \\ &= (n-t)! - o_{n-t} - o_{n-t-1} - \min(e_{n-t-1}, o_{n-t-1}).\end{aligned}$$

Since we are assuming that $|\mathcal{A}| \geq (n-t)! - o_{n-t} - o_{n-t-1} + t$, this means that

$$|\mathcal{A} \setminus \mathcal{C}| \geq \min(e_{n-t-1}, o_{n-t-1}) + t = (1/e + o(1))(n-t-1)!/2.$$

Notice that for any $m < n$ we have the following trivial upper bound on the size of an m-intersecting family $\mathcal{H} \subset A_n$:

$$|\mathcal{H}| \leq \binom{n}{m}(n-m)!/2 = n!/(2m!)$$

since every permutation in $\mathcal{H}$ must agree with a fixed permutation in $\mathcal{H}$ in at least m places.

Hence, $\mathcal{A} \setminus \mathcal{C}$ cannot be $(\log n)$-intersecting and therefore contains two permutations π, τ agreeing on at most $\log n$ points. The number of permutations in $\mathcal{C}$ which agree with π and τ at one of these $\log n$ points is clearly at most $(\log n)(n-t-1)!/2$. All other permutations in $\mathcal{A} \cap \mathcal{C}$ agree with π and τ at two separate points of $\{t+1, \ldots, n\}$, and therefore the same holds for π_p and τ_q, where p and q are the unique points of $[t]$ shifted by π and τ respectively. The number of permutations in $\mathcal{C}$ that agree with π_p and τ_q at two separate points of $\{t+1, \ldots, n\}$ is at most $((1-1/e)^2 + o(1))(n-t)!/2$ (it is easily checked that given two fixed permutations, the probability that a uniform random even permutation agrees with them at separate points is at most $(1-1/e)^2 + o(1)$), which implies that

$$\begin{aligned}|\mathcal{A} \cap \mathcal{C}| &\leq ((1-1/e)^2 + o(1))(n-t)!/2 + (\log n)(n-t-1)!/2 \\ &= ((1-1/e)^2 + o(1))(n-t)!/2,\end{aligned}$$

contradicting (6), provided n is sufficiently large. This proves the claim.

Since we are assuming $|\mathcal{A}| \geq |\mathcal{E}| + t$, we must have equality, so $\mathcal{A} = \mathcal{B}$, proving Theorem 13. $\square$

5 Open Problems

One of the most important open problems in the area is to obtain an analogue of the Ahlswede-Khachatrian Theorem [1] for t-intersecting families in S_n, i.e. to determine the maximum-sized t-intersecting families in S_n for *every* value of n and t. We make the following conjecture:

Conjecture 14. *A t-intersecting family in S_n of maximum size must be a double translate of one of the families*

$$\mathcal{F}_i = \{\sigma \in S_n : \sigma \text{ has at least } t{+}i \text{ fixed points in } [t{+}2i]\} \ (0 \leq i \leq (n{-}t)/2).$$

This would imply that the maximum size is $(n-t)!$ for $n \geq 2t+2$. It is natural to ask how large n must be for our method in [6] to give this bound, i.e. when there exists a weighted graph Y which is a real linear combination of Cayley graphs on S_n generated by conjugacy-classes of permutations with less than t fixed points, such that the matrix A of weights of Y has maximum eigenvalue 1 and minimum eigenvalue

$$\omega_{n,t} = -\frac{1}{n(n-1)\dots(n-t+1)-1}.$$

It turns out that this need not be the case when $n = 2t+2$; indeed, it fails for $t = 2$ and $n = 6$. We believe that new techniques will be required to prove the above conjecture.

We now turn to the question of improving Theorem 6, our rough stability result. We conjecture that the hypothesis $|\mathcal{A}| \geq c(n-t)!$ is unnecessary, i.e. that for *any* t-intersecting family $\mathcal{A} \subset S_n$, there exists a t-coset containing all but at most $O((n-t-1)!)$ of the permutations in $\mathcal{A}$. In fact, we conjecture that for n sufficiently large depending on t, the t-intersecting families which are furthest from being contained within a t-coset are precisely the double translates of $\mathcal{F}_1$:

Conjecture 15. *For n sufficiently large depending on t, if $\mathcal{A} \subset S_n$ is t-intersecting, then there exists a t-coset containing all but at most*

$$t((n-t-1)! - (n-t-2)!)$$

of the permutations in $\mathcal{A}$. This is sharp precisely when $\mathcal{A}$ is a double translate of $\mathcal{F}_1$.

It would also be interesting to obtain an analogue of the complete non-trivial t-intersection theorem of Ahlswede and Khachatrian in [1]. We make the following conjecture:

Conjecture 16. *For any n and t, if $\mathcal{A} \subset S_n$ is a t-intersecting family which is not contained within a t-coset, and has the maximum size subject to these conditions, then it must be a double translate of the family $\mathcal{D}$ in Theorem 9, or of one of the $\mathcal{F}_i$'s.*

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