Intersecting Families of Graphs

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Often in extremal combinatorics, we're interested in determining the largest possible size of a family of subsets of a ground set, subject to a certain condition.
Often, the condition is that the intersection of any two members of the family contains a set of a certain kind.
The simplest example is when the ground-set is unstructured.

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The simplest example is when the ground-set is unstructured.

Let X be an n-element set.

Definition

We say that a family $\mathcal{F} \subset \mathcal{P}(X)$ is *t-intersecting* if $|A \cap B| \geq t$ for any $A, B \in \mathcal{F}$.

Question

What are the largest t-intersecting subsets of $\mathcal{P}([n])$?

Theorem (Katona, 1964)

Let $\mathcal{F} \subset \mathcal{P}([n])$ be a t-intersecting family. If $n + t = 2k$, then

$$|\mathcal{F}| \leq |[n]^{(\geq k)}| = \sum_{i=k}^{n} \binom{n}{i}.$$

If $n + t = 2k - 1$, then

$$|\mathcal{F}| \leq |[n]^{(\geq k)} \cup [n-1]^{(k-1)}| = \sum_{i=k}^{n} \binom{n}{i} + \binom{n-1}{k-1}.$$

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Let F be a fixed, unlabelled graph.

Definition

We say that a family $\mathcal{G}$ of graphs on $[n]$ is *F-intersecting* if the intersection of any two graphs in $\mathcal{G}$ contains a copy of F.

Question

What are the largest F-intersecting families of graphs on $[n]$?

Definition

$$m_n(F) = \max\{|\mathcal{G}| : \mathcal{G} \subset \mathcal{P}([n]^{(2)}),\ \mathcal{G} \text{ is } F\text{-intersecting}\}.$$

Similarly, let $\mathcal{F}$ be a fixed *family* of unlabelled graphs.

Definition

We say that a family $\mathcal{G}$ of graphs on $[n]$ is *$\mathcal{F}$-intersecting* if the intersection of any two graphs in $\mathcal{G}$ contains a copy of some graph in $\mathcal{F}$; we write

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- If F is a single edge,
  Can't have both $G, \overline{G} \in \mathcal{G}$
  $\Rightarrow |\mathcal{G}| \leq 2^{\binom{n}{2}-1}$.
- If $F = S_d$, the star with d rays,

$$m_n(S_d) = (1 - o(1))2^{\binom{n}{2}-1}.$$

Take

$$\mathcal{G} = \{G \in \mathcal{P}([n]^{(2)}) : \ \deg_G(1) \geq (n - 1 + d)/2\}.$$

Then

$$|\mathcal{G}| \geq (1 - o(1))\frac{1}{2}2^{\binom{n}{2}}.$$

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What happens if $G = \triangle$?

Conjecture (Simonovits-Sós, 1976)

Let $\mathcal{G}$ be a triangle-intersecting family of graphs on $[n]$. Then

$$|\mathcal{G}| \leq 2^{\binom{n}{2}-3}.$$

Equality holds if and only if $\mathcal{G}$ consists of all graphs containing a fixed triangle.

Theorem (Chung, Frankl, Graham, Shearer, 1984)

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Lemma (Shearer's Entropy/Projections Lemma)

Let X be a finite set, and let $\mathcal{A} \subset \mathcal{P}(X)$ be an r-cover of X, meaning every element of X is contained in at least r sets in $\mathcal{A}$. Let $\mathcal{F} \subset \mathcal{P}(X)$. For $A \subset S$, let

$$\mathcal{F}_A = \{F \cap A : \ F \in \mathcal{F}\}.$$

Then

$$|\mathcal{F}|^r \leq \prod_{A \in \mathcal{A}} |\mathcal{F}_A|.$$

Let $\mathcal{G} \subset \mathcal{P}([n]^{(2)})$ be triangle-intersecting.
Project onto the graph $A = K_S \sqcup K_{\bar{S}}$,
for each $S \subset [n] : \ |S| = \lfloor n/2 \rfloor$.
Any triangle contains at least one edge of A
$\therefore \mathcal{G}_A$ is 1-intersecting
$\therefore |\mathcal{G}_A| \leq 2^{e(A)-1} = 2^{\binom{\lfloor n/2 \rfloor}{2} + \binom{\lceil n/2 \rceil}{2} - 1}$.

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Let $\mathcal{G}$ be an odd-cycle-intersecting family of graphs on $[n]$. Then

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Definition

If G and H are graphs on $[n]$, we write

$$G \nabla H = (G \cap H) + (\overline{G} \cap \overline{H}) = \overline{G \Delta H}$$

for the set of edges on which they 'agree'.

Let $\mathcal{F}$ be any family of graphs on $[n]$.

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Lemma (Chung, Frankl, Graham, Shearer, 1984)

$$\bar{m}_n(\mathcal{F}) = m_n(\mathcal{F}).$$

So the theorem we wish to prove is equivalent to:

Theorem
Let $\mathcal{G}$ be an odd-cycle-agreeing *family of graphs on $[n]$. Then*

$$|\mathcal{G}| \leq 2^{\binom{n}{2}-3}.$$

Equality holds iff $\mathcal{G}$ is a triangle-junta, *meaning it consists of all graphs with prescribed intersection with a fixed triangle.*

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If Z is an Abelian group, and $Y \subset Z$ is symmetric ($Y = -Y$), the *Cayley graph* $\mathrm{Cay}(Z, Y)$ is the graph on Z with edge-set $\{\{z, z+y\} : z \in Z, y \in Y\}$.

Let

$$\mathcal{C} = \mathrm{Cay}(\mathbb{Z}^{[n]^{(2)}}, \{\bar{B} : B \text{ is bipartite}\}).$$

Observe:

$$\mathcal{G} \text{ is odd-cycle-agreeing } \Leftrightarrow \mathcal{G} \text{ is an independent set in } \mathcal{C} :$$

$$\begin{aligned}
& \mathcal{G} \text{ odd-cycle-agreeing} \\
\Leftrightarrow \quad & G \nabla H \text{ contains an odd cycle } \forall G, H \in \mathcal{G} \\
\Leftrightarrow \quad & G \nabla H \text{ is non-bipartite } \forall G, H \in \mathcal{G} \\
\Leftrightarrow \quad & G \Delta H \text{ cannot be the complement of a bipartite graph} \\
\Leftrightarrow \quad & GH \notin E(\mathcal{C}) \ \forall G, H \in \mathcal{G} \\
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Task:

Show that the largest independent sets in $\mathcal{C}$ are the triangle-juntas.

Theorem (Hoffman)

Let $G = (V, E)$ be a d-regular graph, and let A be the adjacency matrix of G, so that the all-1's vector $\mathbf{f}$ is an eigenvector of A with eigenvalue d. Let $\lambda_{\min}$ denote the least eigenvalue of A. If $S \subset V$ is an independent set in G, then

$$\frac{|S|}{|V|} \leq \frac{-\lambda_{\min}}{d - \lambda_{\min}}.$$

If equality holds, then the characterstic vector f_S of S satisfies:

$$f_S - \tfrac{|S|}{|V|}\mathbf{f} \in Ker(A - \lambda_{\min} I).$$

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Let $G = (V, E)$ be a graph. A *pseudo-adjacency matrix* for G is a matrix $A \in \mathbb{R}[V \times V]$ such that $A_{v,w} = 0$ whenenver $vw \notin E(G)$.

Observation (Delsarte)

Hoffman's bound holds if A is a pseudo-adjacency matrix for G.

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Recall: if Z is a finite Abelian group, the group of characters $\hat{Z}$ forms an orthonormal basis of eigenvectors for the adjacency matrix of any Cayley graph on Z.
In our case, the characters of $\mathbb{Z}_2^X$ are the *Fourier-Walsh functions*

$$\begin{aligned} u_R : \{0,1\}^X &\rightarrow \{-1,1\}; \\ u_R(S) &= (-1)^{|R \cap S|}. \end{aligned}$$

Easy check: these form an orthonormal basis for $\mathbb{R}[\mathbb{Z}_2^X]$ with respect to the inner product

$$\langle f, g \rangle = \frac{1}{2^{|X|}} \sum_{S \subset X} f(S) g(S).$$

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$$\begin{aligned} u_R : \{0,1\}^X &\rightarrow \{-1,1\}; \\ u_R(S) &= (-1)^{|R \cap S|}. \end{aligned}$$

Easy check: these form an orthonormal basis for $\mathbb{R}[\mathbb{Z}_2^X]$ with respect to the inner product

$$\langle f, g \rangle = \frac{1}{2^{|X|}} \sum_{S \subset X} f(S) g(S).$$

If $f : \mathbb{Z}_2^X \to \mathbb{R}$, we may write

$$f = \sum_{R \subset X} \hat{f}(R) u_R,$$

where $\hat{f}(R) = \langle f, u_R \rangle$.
The function $\hat{f} : \mathbb{Z}_2^X \to \mathbb{R}$ is called the *Fourier transform* of f.
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In particular: let B be a bipartite graph on $[n]$;
let A_B be the adjacency matrix of $\mathrm{Cay}(\mathbb{Z}^{[n]^{(2)}}, \{\overline{B}\})$.

$$A_B f(H) = f(H \oplus \overline{B}) \quad (H \in \mathbb{Z}_2^{[n]^{(2)}}).$$

For each graph G on $[n]$,

$$A_B u_G(H) = u_G(H \oplus \overline{B}) = u_G(\overline{B}) u_G(H),$$

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$$A = \sum_B c_B A_B,$$

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Definition

We call a function $\lambda : \mathbb{Z}_2^{[n]^{(2)}} \longrightarrow \mathbb{R}$ of the form

$$G \mapsto (-1)^{e(G)} \sum_B c_B u_B(G)$$

an *admissible spectrum*.

We'll construct one such that $\frac{-\lambda_{\min}}{\lambda_\emptyset - \lambda_{\min}} = 1/8$, and $\lambda_H = \lambda_{\min}$ if and only if H is a nonempty subgraph of a triangle.

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We'll then apply the following pseudo-Hoffman bound:

Theorem

Let $\mathcal{C}$ be a Cayley graph on $\mathbb{Z}_2^X$, and let A be a pseudo-adjacency matrix for $\mathcal{C}$ which is a linear combination of Cayley subgraphs of $\mathcal{C}$. Then the characters $\{u_S : S \subset X\}$ are all eigenvectors of A. Let λ_R be the eigenvalue corresponding to u_R. If $\mathcal{G} \subset \mathbb{Z}_2^X$ is an independent set in $\mathcal{C}$, then

$$\frac{|\mathcal{G}|}{2^{|X|}} \leq \frac{-\lambda_{\min}}{\lambda_{\emptyset} - \lambda_{\min}}.$$

If equality holds, then the Fourier transform of $\mathcal{G}$ is totally supported on

$$\{R : \lambda_R = \lambda_{\min}\} \cup \{\emptyset\}.$$

This will prove that if $\mathcal{G} \subset \mathcal{P}([n]^{(2)})$ is odd-cycle-agreeing, then $|\mathcal{G}| \leq 2^{\binom{n}{2}-3}$, with equality only if the characteristic function of $\mathcal{G}$ has its Fourier transform totally concentrated on subgraphs of triangles.

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Let

$$\mathcal{U} = \mathrm{Span}\{u_B : \ B \subset K_n, \ B \text{ is bipartite}\};$$

we call the functions in $\mathcal{U}$ '*legal*'.
Note that any function of the form

$$G \mapsto (-1)^{e(G)} p(H) \quad (p \in \mathcal{U})$$

is legal.
Let B_0 be fixed bipartite graph on $[n]$.
Let f be any function on subgraphs of B_0.
Then

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Hence, if R is a fixed *unlabelled* bipartite graph,

$$G \mapsto \mathbf{1}\{G \cap B_0 \cong R\}$$

is legal.

$$\begin{aligned} q_R : G &\mapsto \frac{1}{2^n} \sum_{S \subset [n]} \mathbf{1}\{G \cap K_{S,\bar{S}} \cong R\} \\ &= \text{Prob}\{\text{a random cut in } G \cong R\} \end{aligned}$$

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$$\begin{aligned} q_i : G &\mapsto \frac{1}{2^n} \sum_{S \subset [n]} \mathbf{1}\{e(G \cap K_{S,\bar{S}}) = i\} \\ &= \text{Prob}\{\text{a random cut in } G \text{ has exactly } i \text{ edges}\} \end{aligned}$$

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Our first admissible spectrum will be of the form

$$\lambda_G = (-1)^{e(G)} \sum_{i \geq 0} a_i q_i(G).$$

Recall that we must have

$$\frac{-\lambda_{\mathsf{min}}}{\lambda_\emptyset - \lambda_{\mathsf{min}}} = \tfrac{1}{8},$$

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From $\lambda_\emptyset = 1$, $\lambda_- = \lambda_< = \lambda_\triangle = -\frac{1}{7}$,
we get the equations

$$\begin{aligned} a_0 &= 1 \\ \tfrac{1}{2}a_0 + \tfrac{1}{2}a_1 &= \tfrac{1}{7} \\ \tfrac{1}{4}a_0 + \tfrac{1}{2}a_1 + \tfrac{1}{4}a_2 &= -\tfrac{1}{7} \\ \tfrac{1}{4}a_0 + \tfrac{3}{4}a_2 &= \tfrac{1}{7} \end{aligned}$$

We have 4 equations in 3 unknowns, but they are (luckily!) solvable, with:

$$a_0 = 1,\ a_1 = -\tfrac{5}{7},\ a_2 = -\tfrac{1}{7}.$$

The equation $\lambda_{F_4} \geq -\frac{1}{7}$ gives $a_3 + 4a_4 \geq \frac{3}{28}$, whereas
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Fortunately, setting $a_3 = \frac{3}{28}$ and $a_i = 0$ for all $i \geq 4$
works for all graphs!
—one can check that

$$(-1)^{e(G)}(q_0(G) - \tfrac{5}{7}q_1(G) - \tfrac{1}{7}q_2(G) + \tfrac{3}{28}q_3(G)) \geq -\tfrac{1}{7}$$

for all G.
Since for fixed i, $q_i(G) \to 0$ as $e(G) \to \infty$,
this is essentially a finite task, and can be easily done with the use of a computer.
To avoid the need for this, we prove several upper bounds on the 'random cut statistics' q_i.
This proves that an odd-cycle-agreeing family of graphs on $[n]$
has size at most $2^{\binom{n}{2}-3}$.

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Unfortunately, we have $\lambda_G = \lambda_{\min} = -\frac{1}{7}$ when G is a 2-edge matching, a 4-edge forest or a K_4^-.
To fix this problem, we add to our admissible spectrum a very small multiple of the following admissible spectrum:

$$G \mapsto (-1)^{e(G)} \left(q_{=}(G) + \sum_{F \in \mathcal{F}_4} q_F(G) - q_{\square}(G) \right).$$

The resulting admissible spectrum still has
$\lambda_{\emptyset} = 1$ and $\lambda_{\min} = -\frac{1}{7}$,
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Hence, we may deduce from the pseudo-Hoffman theorem that if $\mathcal{G}$ is an odd-cycle-agreeing family of size $2^{\binom{n}{2}-3}$,
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If we also know that $\mathcal{G}$ is monotone-increasing, we can use:

Theorem (Friedgut, 2004)

Let $N \in \mathbb{N}$. Suppose $\mathcal{A} \subset \{0,1\}^N$ is monotone-increasing, with $|\mathcal{A}| = 2^{N-t}$, and with characteristic function f satisfying

$$\hat{f}(S) = 0 \textit{ whenever } |S| > t.$$

Then f depends on at most t coordinates.

If $\mathcal{G}$ is odd-cycle-*intersecting* with $|\mathcal{G}| = 2^{\binom{n}{2}-3}$, then it is monotone increasing, since it is maximal.
Applying Friedgut's theorem with $t = 3$ shows that $\mathcal{G}$ depends on at most 3 edges, which must form a triangle.
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Applying Friedgut's theorem with $t = 3$ shows that $\mathcal{G}$ depends on at most 3 edges, which must form a triangle.
Hence, $\mathcal{G}$ consists of all graphs on $[n]$ containing a fixed triangle.

If $\mathcal{G}$ is an odd-cycle-*agreeing* family of size $2^{\binom{n}{2}-3}$, then we may turn it into an odd-cycle-intersecting family $\mathcal{G}'$ of the same size, using a sequence of e-monotononization operations.
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Stability of the extremal families

It turns out that if $\mathcal{G} \subset \mathcal{P}([n]^{(2)})$ is odd-cycle-agreeing with

$$|\mathcal{G}| \geq (1-\epsilon)2^{\binom{n}{2}-3},$$

then there exists a triangle-junta $\mathcal{T}$ such that

$$|\mathcal{G}\Delta\mathcal{T}| \leq K\epsilon 2^{\binom{n}{2}},$$

where $K > 0$ is an absolute constant.

Our admissible spectrum also satisfies:

$$\lambda_G \geq -\tfrac{1}{7} + \tfrac{1}{100}$$

whenever G is not a subgraph of a triangle.

Using a stability version of our pseudo-Hoffman theorem, it follows that if $\mathcal{G}$ is an odd-cycle-agreeing family of graphs on $[n]$ with

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Theorem (Kindler, Safra, 2004)

For every $t \in \mathbb{N}$, there exist $\epsilon_t > 0$, $c_t > 0$ and $T_t \in \mathbb{N}$ such that the following holds. Let $N \in \mathbb{N}$, and let $f : \{0,1\}^N \to \{0,1\}$ be a Boolean function such that

$$\sum_{|S|>t} \widehat{f}(S)^2 = \varepsilon < \epsilon_t.$$

Then there exists a Boolean function $g : \{0,1\}^N \to \{0,1\}$, depending on at most T_t coordinates, such that

$$|\{R \subset [N] : \; f(R) \neq g(R)\}| \leq c_t \epsilon 2^N.$$

It follows that there exists a family $\mathcal{T}$ depending on at most T_3 edges such that

$$|\mathcal{G} \Delta \mathcal{T}| \leq K \epsilon 2^{\binom{n}{2}}.$$

Provided ϵ is sufficiently small depending on K and T_3, $\mathcal{T}$ must be a triangle-junta.

Our method generalizes to prove an analogous result about families of non-uniform hypergraphs...

Definition

We say that a family $\mathcal{F}$ of hypergraphs on $[n]$ is *odd-linear-dependency-intersecting* if for any $G, H \in \mathcal{F}$ there exist $l \in \mathbb{N}$ and nonempty sets $A_1, A_2, \ldots, A_{2l+1} \in G \cap H$ such that

$$A_1 \triangle A_2 \triangle \ldots \triangle A_{2l+1} = \emptyset.$$

Equivalently...

Definition

A family $\mathcal{F}$ of subsets of $\mathbb{Z}_2^n$ is *odd-linear-dependency-intersecting* if for any two subsets $S, T \in \mathcal{F}$, there exist $l \in \mathbb{N}$ and non-zero vectors $v_1, v_2, \ldots, v_{2l+1} \in S \cap T$ such that

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A family $\mathcal{T}$ of subsets of $\mathbb{Z}_2^n$ is a *Schur junta* if there exists a Schur-triple $\{x, y, x+y\}$ such that $\mathcal{T}$ consists of all subsets of $\mathbb{Z}_2^n$ with prescribed intersection with $\{x, y, x+y\}$.

Theorem (E., Filmus, Friedgut)

Let $\mathcal{F}$ be an odd-linear-dependency-agreeing family of subsets of $\mathbb{Z}_2^n$. Then

$$|\mathcal{F}| \leq 2^{2^n-3}.$$

Equality holds if and only if $\mathcal{F}$ is a Schur junta.

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Open Problems

Let F be a fixed, unlabelled graph.

$$m_n(F) = \max\{|\mathcal{G}| : \mathcal{G} \subset \mathcal{P}([n]^{(2)}),\ \mathcal{G} \text{ is } F\text{-intersecting}\}.$$

Determine $m_n(F)$.

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