Exact Intersections

Algebraic Methods in Combinatorics, Lecture 6

We deduced from the Frankl-Wilson theorem that if $\mathcal{A} \subset [n]^{(r)}$ is such that $|x \cap y| \in L$ for any two distinct $x, y \in \mathcal{A}$, where L is a set of s non-negative integers, then

$$|\mathcal{A}| \leq \binom{n}{s}.$$

What happens for non-uniform families, i.e. subsets of $\mathcal{P}([n])$?

Definition. *If L is a finite set of integers, a family $\mathcal{A} \subset \mathcal{P}([n])$ is said to be* L-intersecting *if $|A \cap B| \in L$ for any two distinct $A, B \in \mathcal{A}$.*

We ask the following

Question. *What is the maximum possible size of an L-intersecting family of subsets of $\{1, 2, \ldots, n\}$?*

We write

$$M(n, L) = \max\{|\mathcal{A}| : \ \mathcal{A} \subset \mathcal{P}([n]), \ \mathcal{A} \text{ is } L\text{-intersecting}\}.$$

We will see that an L-intersecting family of subsets of $\{1, 2, \ldots, n\}$ has size at most

$$\sum_{i=0}^{s} \binom{n}{i},$$

where $s = |L|$. So for L fixed and $n \to \infty$, $M(n, L)$ grows no faster than a polynomial of degree $|L|$. It is a longstanding open problem to determine, in terms of L, the rate of growth of $M(n, L)$ as $n \to \infty$ (for each fixed L).

Remark. *Similarly, we define*

$$M(n, r, L) = \max\{|\mathcal{A}| : \ \mathcal{A} \subset [n]^{(r)}, \ \mathcal{A} \textit{ is } L\textit{-intersecting}\}.$$

Again, it would be interesting to determine, in terms of L and r, the rate of growth of $M(n, r, L)$ as $n \to \infty$ (for each fixed L and r). It is not even known exactly which L and r force $M(n, r, L)$ to be linear in n. For a survey of partial results, see the book by Babai and Frankl [1].

We start off with the simplest case, $|L| = 1$:

Theorem 1 (Fisher)**.** *If $l \in \mathbb{N}$, and $\mathcal{A} \subset \mathcal{P}([n])$ is $\{l\}$-intersecting, i.e. $|A \cap B| = l$ for any two distinct $A, B \in \mathcal{A}$, then*

$$|\mathcal{A}| \leq n.$$

Proof. Let $\mathcal{A} \subset \mathcal{P}([n])$ be $\{l\}$-intersecting. First, observe that if there exists $A_0 \in \mathcal{A}$ with $|A_0| = l$, then we are done. Indeed, every other set in $\mathcal{A}$ then contains A_0, so the sets $B \setminus A_0$ (for $B \in \mathcal{A} \setminus \{A_0\}\}$) must be pairwise disjoint, so $|\mathcal{A}| \leq n$, as required.

Assume from now on that $|A| > l$ for all $A \in \mathcal{A}$. We claim that the characteristic vectors

$$\{\chi_A : \ A \in \mathcal{A}\}$$

form a linearly independent set in $\mathbb{R}^n$. (In fact, this is true in the first case as well, but it is easier to use the above argument.)

To see this, suppose that

$$\sum_{A \in \mathcal{A}} c_A \chi_A = 0$$

for some real numbers $\{c_A : \ A \in \mathcal{A}\}$. Taking the (standard Euclidean) inner product with χ_B gives

$$c_B|B| + \sum_{A \in \mathcal{A} \setminus \{B\}} c_A |A \cap B| = 0,$$

and therefore

$$c_B(|B| - l) = -l \sum_{A \in \mathcal{A}} c_A \quad \forall B \in \mathcal{A}.$$

If $\sum_{A \in \mathcal{A}} c_A \neq 0$, then for each $B \in \mathcal{A}$, c_B is non-zero and has the opposite sign to $\sum_{A \in \mathcal{A}} c_A$, a contradiction. Hence, $\sum_{A \in \mathcal{A}} c_A = 0$, so $c_B = 0$ for all $B \in \mathcal{A}$, proving the claim.

Since $\mathbb{R}^n$ has dimension n, it follows that $|\mathcal{A}| \leq n$, proving the theorem. □

Remark. *We have equality if* $l = 1$ *and* $\mathcal{A} = \{\{1\}\} \cup \{\{1, i\} : \ 2 \leq i \leq n\}$, *or if* $l = n - 2$ *and* $\mathcal{A} = [n]^{(n-1)}$.

We now turn to the case of general L.

Theorem 2 (Frankl, Wilson, 1981)**.** *Let* $\mathcal{A} \subset \mathcal{P}[n]$ *be* L*-intersecting, where* $|L| = s$. *Then*

$$|\mathcal{A}| \leq \sum_{i=0}^{s} \binom{n}{i}.$$

Remark. *This is sharp when* $L = \{0, 1, \ldots, s-1\}$*: take* $\mathcal{A} = [n]^{(\leq s)}$.

The proof we give is due to Babai (1988).

Proof. Write

$$\mathcal{A} = \{A_1, \ldots, A_m\},$$

where $|A_1| \geq |A_2| \geq \ldots \geq |A_m|$. We will find m linearly independent objects in a vector-space of dimension $\sum_{i=0}^{s} \binom{n}{i}$. Our objects will be real-valued functions on $\{0, 1\}^n$. For each set A_j, we will choose a function $f_j : \{0, 1\}^n \to \mathbb{R}$ such that

$$f_j(\chi_{A_j}) \neq 0 \ \forall j \in [m],$$

but

$$f_j(\chi_{A_k}) = 0 \ \forall k > j.$$

This 'upper-triangular' condition immediately implies that the f_j's are linearly independent.

Claim. *Let U be a set, and let $f_1, \ldots, f_m$ be real-valued functions on U. If there exist points $u_1, \ldots, u_m$ such that*

$$f_j(u_j) \neq 0 \ \forall j \in [m],$$

but

$$f_j(u_k) = 0 \ \forall k > j,$$

then the functions $f_1, \ldots, f_m$ are linearly independent.

Proof of Claim: The $m \times m$ matrix $(f_j(u_k))_{1 \leq j \leq m,\ 1 \leq k \leq m}$ is *strictly upper-triangular* (upper triangular with all its diagonal entries non-zero). Such a matrix clearly has non-zero determinant, so its rows are linearly independent. In other words, the restrictions $f_j|_{\{u_1,\ldots,u_m\}}$ are linearly independent (as real-valued functions), so certainly the f_j's are. □

For each j, define $f_j : \{0,1\}^n \to \mathbb{R}$ by

$$f(z_1, \ldots, z_n) = \prod_{l \in L:\ l < |A_j|} \left(\sum_{i \in A_j} z_i - l \right).$$

We then have

$$f_j(\chi_{A_k}) = \prod_{l \in L:\ l < |A_j|} (|A_j \cap A_k| - l) \quad \forall j, k \in [m].$$

Hence, $f_j(\chi_{A_j}) \neq 0$ for each $j \in [m]$, but $f_k(\chi_{A_j}) = 0$ whenever $k > j$, as desired. It follows from the claim above that the f_j's are linearly independent as real-valued functions on $\{0,1\}^n$.

Observe that each function f_j can be considered as a polynomial in $\mathbb{R}[X_1, \ldots, X_n]$ with total degree at most s. We now perform a process known as *multilinearization*: the idea is to find a *multilinear* polynomial $\tilde{f}_j$ with total degree at most s, such that $\tilde{f}_j$ agrees with f_j on all of $\{0,1\}^n$. (A multivariate polynomial is *multilinear* iff it has degree at most 1 in each variable.)

For each j, express f_j as a real linear combination of monomials of the form

$$\prod_{i \in T} X_i^{a_i},$$

where $a_i \geq 1$ for each i, and $|T| \leq s$. For each of these monomials, replace it by

$$\prod_{i \in T} X_i.$$

The resulting polynomial $\tilde{f}_j$ clearly agrees with f_j on all of $\{0,1\}^n$, and is a linear combination of monomials which are products of at most s distinct X_i's, i.e. it is a multilinear polynomial of total degree at most s. The vector-space of multilinear polynomials of total degree at most s has dimension

$$\sum_{i=0}^{s} \binom{n}{i};$$

indeed, the monomials of the form

$$\prod_{i\in T} X_i \quad (T \in [n]^{(\leq s)})$$

are a basis. Hence, the f_j's live in a vector space of dimension at most

$$\sum_{i=0}^{s} \binom{n}{i},$$

proving the theorem. □

References

[1] L. Babai, P. Frankl, *Linear Algebra Methods in Combinatorics with Applications to Geometry and Computer Science*, Department of Computer Science, University of Chicago, preliminary version, 1992.