Intersecting families of permutations and other problems in extremal combinatorics

David Ellis
Trinity College, Cambridge

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Introduction

In this dissertation, we examine questions of the following kind: given a set of objects, S, what is the maximum size of a family of objects in S with a given property, P? What are the families which have this maximum size (the 'extremal' families with property P)? If a family has size close to the maximum, must it be 'close' to an extremal family in some appropriate sense?

First, we consider some extremal problems on the symmetric group S_n. A family of permutations $\mathcal{A} \subset S_n$ is said to be *intersecting* if any two permutations in $\mathcal{A}$ agree in at least one point, i.e. for any $\sigma, \pi \in \mathcal{A}$, there is some $i \in [n]$ such that $\sigma(i) = \pi(i)$. In 1977, Deza and Frankl [8] proved that an intersecting family of permutations in S_n has size at most $(n-1)!$, which is best possible because of intersecting families of the form $\mathcal{A} = \{\sigma \in S_n : \sigma(i) = j\}$ for fixed $i, j \in [n]$, i.e. cosets of point-stabilizers. It turned out to be surprisingly hard to show that equality holds only for these families, but this was eventually done by Cameron and Ku [6]; Larose and Malvenuto [23] independently found a different proof.

Cameron and Ku [6] then conjectured a 'stability' version of this result, namely that there exists a constant $c > 0$ such that any intersecting family $\mathcal{A} \subset S_n$ of size at least $(1-c)(n-1)!$ is 'centred', meaning that every permutation in it maps i to j for some fixed $i, j \in [n]$. They also made the stronger 'Hilton-Milner' type conjecture that for $n \geq 6$, if $\mathcal{A} \subset S_n$ is a non-centred intersecting family, then $\mathcal{A}$ cannot be larger than the family $\mathcal{B} = \{\sigma \in S_n : \sigma(1) = 1, \sigma(i) = i \text{ for some } i > 2\} \cup \{(12)\}$, which has size $(1 - 1/e + o(1))(n-1)!$, and that the extremal families are precisely the 'double translates' $\{\pi \mathcal{B} \tau : \pi, \tau \in S_n\}$ of this family.

In Chapter 1, we prove this stability conjecture, and also the Hilton-Milner type conjecture for n sufficiently large. Our proof makes use of the classical representation theory of S_n, or more precisely, (non-Abelian) Fourier Analysis on S_n. One of our key tools will be an extremal result for cross-intersecting families of permutations. (A pair of families of permutations $\mathcal{A}, \mathcal{B} \subset S_n$ is said to be *cross-intersecting* if for any $\sigma \in \mathcal{A}$ and $\pi \in \mathcal{B}$, σ and π agree at some point, i.e. there is some $i \in [n]$ such that $\sigma(i) = \pi(i)$.) We prove that for $n \geq 4$, a cross-intersecting pair of families of permutations $\mathcal{A}, \mathcal{B} \subset S_n$ satisfies $|\mathcal{A}||\mathcal{B}| \leq ((n-1)!)^2$, with equality iff $\mathcal{A} = \mathcal{B} = \{\sigma \in S_n : \sigma(i) = j\}$ for some $i, j \in [n]$. This was a conjecture of Leader [24]. The proof uses an eigenvalue bound applied to the *derangement graph* Γ on S_n, in which two permutations are joined if they disagree at every point; the eigenvalues of this graph can be analyzed using the representation theory of S_n.

In Chapter 2, we consider a natural generalization of these questions. A family of permutations $\mathcal{A} \subset S_n$ is said to be *t-intersecting* if any two permutations in $\mathcal{A}$ agree in at least t places, i.e. for any $\sigma, \pi \in \mathcal{A}$, $|\{i \in [n] : \sigma(i) = \pi(i)\}| \geq t$. Deza and Frankl [8] conjectured in 1977 that if n is sufficiently large depending on t, then a t-intersecting family $\mathcal{A} \subset S_n$ has size at most $(n-t)!$.

We prove this conjecture using a combination of an eigenvalue method, the classical representation theory of S_n, and a combinatorial construction. A very similar argument proves a cross-intersecting version of the conjecture. (We say that a pair of families $\mathcal{A}, \mathcal{B} \subset S_n$ is *t-cross-intersecting* if $|\{i \in [n] : \sigma(i) = \pi(i)\}| \geq t$ for any $\sigma \in \mathcal{A},\ \pi \in \mathcal{B}$.) We show that if n is sufficiently large depending on t, then any t-cross-intersecting pair of families $\mathcal{A}, \mathcal{B} \subset S_n$ satisfy $|\mathcal{A}||\mathcal{B}| \leq ((n-t)!)^2$. Ehud Friedgut and Haran Pilpel independently discovered very similar proofs of these results, and we have now written a joint paper [9].

We also obtain a Hilton-Milner type result for t-intersecting families, generalizing the Cameron-Ku conjecture. We prove that for n sufficiently large depending on t, if $\mathcal{A} \subset S_n$ is a t-intersecting family which is not contained within a coset of the stabilizer of t points, then $\mathcal{A}$ cannot be larger than the family

$$\{\sigma : \sigma(i) = i\ \forall i \leq t,\ \sigma(j) = j \text{ for some } j > t+1\} \cup \{(1\ t+1), \ldots, (t\ t+1)\}$$

which has size $(1 - 1/e + o(1))(n-t)!$. The extremal families are precisely the double translates of this family. Our proof uses similar ideas to that of the Cameron-Ku conjecture in Chapter 1, but the representation theory of S_n and the combinatorial arguments become more involved.

It follows immediately that provided n is sufficiently large depending on t, if $\mathcal{A} \subset S_n$ is a t-intersecting family of maximum size $(n-t)!$ then $\mathcal{A}$ is a coset of the stabilizer of t points (or 't-coset' for short). This was proved in a different way by Friedgut and Pilpel (although their methods did not yield stability results or Hilton-Milner type results.)

Using very similar methods, we are able to prove analogues of all these results for the alternating group A_n. Firstly, we show that provided n is sufficiently large depending on t, a t-intersecting family $\mathcal{A} \subset A_n$ has size at most $(n-t)!/2$. We have been unable to find a direct proof that equality holds only if $\mathcal{A}$ is a coset of the stabilizer of t points; instead, we deduce this from a Hilton-Milner type result for A_n, namely that for n sufficiently large depending on t, if $\mathcal{A} \subset A_n$ is a t-intersecting family which is not contained within a coset of the stabilizer of t points, then $\mathcal{A}$ cannot be larger than the family

$$\begin{aligned} \mathcal{B} &= \{\sigma \in A_n : \sigma(i) = i \ \forall i \leq t,\ \sigma(j) = (n-1\ n)(j) \text{ for some } j > t+1\} \\ &\quad \cup \{(1\ t+1)(n-1\ n), (1\ t+1)(n-1\ n), \ldots, (t\ t+1)(n-1\ n)\} \end{aligned}$$

which has size $(1 - 1/e + o(1))(n-t)!/2$. Again, the extremal families are precisely the double translates of this family.

In Chapter 3, we change tack and consider an extremal problem in the discrete cube $\{0,1\}^n$. We say a family $\mathcal{A}$ of k-dimensional subcubes of $\{0,1\}^n$ is *irredundant* if none is contained in the union of the others. How large can an irredundant family be? Aharoni and Holzman [1] conjectured that for $k > n/2$, the answer is $\binom{n}{k}$ (which is attained by the family of all k-subcubes through a fixed point). Meshulam [27] proved that for any $k \leq n$, an irredundant family of k-subcubes of Q_n has size at most

$$\frac{2^n}{\sum_{i=0}^{k} \binom{n}{i}} \binom{n}{k}$$

for all $k \leq n$. This implies Aharoni and Holzman's conjecture for $k > 0.9n$. We first give an alternative proof of Meshulam's upper bound for general k, which we believe to be more transparent. Meshulam's bound implies that if $n = 2k+1$, then an irredundant family has size at most $2\binom{n}{k}$, which is attained by the family of subcubes which go through either the top or bottom point. In the light of this, it is natural to ask whether we can find a large irredundant family of subcubes that go through either the top or bottom point for $k > n/2$.

Our main result is that for $k \geq n/2$, any irredundant family in which all subcubes go through either the top or bottom point has size at most $\binom{n}{k}$. The proof involves a slightly unexpected linear algebra argument. The conjecture of Aharoni and Holzman remains open.

In Chapter 4, we consider the problem of generating all the subsets of a finite set from unions of small numbers of other sets. We say that a family $\mathcal{F}$ of subsets of $\{1,2,\ldots,n\}$ is a 2-*base* if any subset of $\{1,2,\ldots,n\}$ is a union of at most two sets in $\mathcal{F}$. Erdős conjectured that the smallest 2-bases are of the form

$$\mathcal{F} = \mathbb{P}V_1 \cup \mathbb{P}V_2 \setminus \{\emptyset\}$$

where (V_1, V_2) is a partition of $\{1, 2, \ldots, n\}$ into two classes of sizes $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$. In other words, it is best to split $\{1, 2, \ldots, n\}$ into two classes of sizes as equal as possible, and take the union of the power-sets of the two classes.

Similarly, we say that a family $\mathcal{F}$ of subsets of $\{1, 2, \ldots, n\}$ is a *k-base* if any subset of $\{1, 2, \ldots, n\}$ is a union of at most k sets in $\mathcal{G}$. Generalizing Erdős' conjecture above, Frein, Lévêque and Sebő [25] conjectured that for any $k \leq n$, the smallest k-bases are of the form

$$\mathcal{F}_{n,k} = \bigcup_{i=1}^{k} \mathbb{P}V_i \setminus \{\emptyset\} \tag{1}$$

where (V_i) is a partition of $[n]$ into k classes of sizes as equal as possible. In other words, it is best to split $\{1, 2, \ldots, n\}$ into k classes of sizes as equal as possible, and take the union of the power-sets of the k classes.

This conjecture seems hard, and in an effort to make progress, Frein, Lévêque and Sebő [25] asked what happens if we demand the unions to be disjoint. We say that a family $\mathcal{G}$ of subsets of $\{1, 2, \ldots, n\}$ is a *k-generator* if any subset of $\{1, 2, \ldots, n\}$ can be expressed as a union of at most k *disjoint* sets in $\mathcal{G}$. Frein, Lévêque and Sebő made the weaker conjecture that for any $n \geq k$, such a family must be at least as large as the k-generator (1). We generalize a theorem of Alon and Frankl [2] in order to show that for fixed k, any k-generator of the subsets of $[n]$ must have size at least $k2^{n/k}(1 - o(1))$, thereby verifying the conjecture asymptotically for multiples of k.

Chapter 1

Cross-intersecting families of permutations and the Cameron-Ku conjecture

1.1 Introduction

A family of permutations $\mathcal{A} \subset S_n$ is said to be *intersecting* if any two permutations in $\mathcal{A}$ agree at some point, i.e. for any $\sigma, \pi \in \mathcal{A}$, there is some $i \in [n]$ such that $\sigma(i) = \pi(i)$. For example, if every permutation in $\mathcal{A}$ maps i to j for some $i, j \in [n]$ (we call such a family '*centred*'), then $\mathcal{A}$ is intersecting. Deza and Frankl [8] showed that if $\mathcal{A} \subset S_n$ is intersecting, then $|\mathcal{A}| \leq (n-1)!$; this is known as the Deza-Frankl Theorem. They gave a short, direct proof: take any n-cycle ρ, and let H be the cyclic group of order n generated by ρ. For any left coset σH of H, any two distinct permutations in σH disagree at every point, and therefore σH contains at most 1 member of $\mathcal{A}$. Since the left cosets of H partition S_n, it follows that $|\mathcal{A}| \leq (n-1)!$.

It turned out to be surprisingly hard to prove that if equality holds then $\mathcal{A} = \{\sigma \in S_n : \sigma(i) = j\}$ for some $i, j \in [n]$, but this was eventually done by Cameron and Ku [6]; Larose and Malvenuto [23] independently found a different proof. Cameron and Ku [6] then asked how large a non-centred intersecting family can be. They noted that if an intersecting family $\mathcal{A}$ contains a permutation τ not fixing 1, say, then the number of permutations in $\mathcal{A}$ fixing 1 is at most $(n-1)! - d_{n-1} - d_{n-2} = (1 - 1/e + o(1))(n-1)!$, where d_n is the number of permutations of $[n]$ with no fixed point. They conjectured that in fact, any non-centred intersecting family has size at most $(n-1)! - d_{n-1} - d_{n-2} + 1 = (1 - 1/e + o(1))(n-1)!$, with equality iff $\mathcal{A}$ is a double translate of the family $\mathcal{B} = \{\sigma \in S_n : \sigma(1) = 1, \sigma(i) = i \text{ for some } i > 2\} \cup \{(12)\}$, meaning that $\mathcal{A} = \pi\mathcal{B}\tau$ for some $\pi, \tau \in S_n$. In other words, if we demand that our intersecting family be non-centred, its maximum size decreases by a factor of about $1 - 1/e$, and it is best to take $\mathcal{A}$ centred apart from just one permutation. We prove this conjecture for n sufficiently large; our proof works for $n \geq 50,000$. This implies the weaker 'stability' conjecture of Cameron and Ku [6] that there exists a constant $c > 0$ such that any intersecting family $\mathcal{A} \subset S_n$ of size at least $(1-c)(n-1)!$ is centred.

Our proof makes use of the classical representation theory of S_n, or more precisely, (non-Abelian) Fourier Analysis on S_n. One of our key tools will be an extremal result for cross-intersecting families of permutations. A pair of families of permutations $\mathcal{A}, \mathcal{B} \subset S_n$ is said to be *cross-intersecting* if for any $\sigma \in \mathcal{A}, \tau \in \mathcal{B}$, σ and τ agree at some point, i.e. there is some $i \in [n]$ such that $\sigma(i) = \tau(i)$. Leader [24] conjectured that for $n \geq 4$, for such a pair, $|\mathcal{A}||\mathcal{B}| \leq ((n-1)!)^2$, with equality iff $\mathcal{A} = \mathcal{B} = \{\sigma \in S_n : \sigma(i) = j\}$ for some $i, j \in [n]$. We prove this conjecture using an eigenvalue argument, combined with representation theory of S_n. (Note that the statement does not hold for $n = 3$, as the pair $\mathcal{A} = \{(1), (123), (321)\}$, $\mathcal{B} = \{(12), (23), (31)\}$ is cross-intersecting.)

Calculating the least eigenvalue of the derangment graph and applying Hoffman's bound (see the following section) gives another proof of the Deza-Frankl Theorem. This calculation is non-trivial; it was first done by P. Renteln [29], using symmetric functions, and independently and slightly later by Friedgut and Pilpel [13], and by Godsil and Meager [15]. The method used by the author below uses a 'trick' to simultaneously bound all eigenvalues except 4, which makes the calculations cleaner, although in a certain sense it is less direct than the approach of Friedgut and Pilpel, for example.

1.2 Cross-intersecting families and the derangement graph

Let Γ be the *derangement graph* on S_n, where we join two permutations iff they disagree at every point, i.e. $\sigma\tau \in E(\Gamma) \Leftrightarrow \sigma(i) \neq \tau(i)\ \forall i \in [n]$. Observe that Γ is the Cayley graph[1] generated by the set $\mathcal{D}_n$ of derangements, or permutations without fixed points. Let $d_n = |\mathcal{D}_n|$ be the number of derangements of $[n]$; then Γ is d_n-regular. As is well known, by the inclusion-exclusion formula,

$$d_n = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!} = (1/e + o(1))n!.$$

For $n \geq 4$, Γ is connected, since the derangements generate S_n. Note that for $n = 3$, Γ is the disjoint union of two triangles with vertex sets $A_3 = \{\mathrm{id}, (123), (321)\}$ and $S_3 \setminus A_3 = \{(12), (23), (31)\}$.

Clearly, a family of permutations is intersecting if it is an independent set in the graph Γ; a pair of families of permutations is cross-intersecting if there are no edges of Γ between them.

The following well-known theorem of Hoffman [18] gives an upper bound on the size of the largest independent set in a regular graph in terms of its eigenvalues:

Theorem 1.1 (Hoffman)**.** *Let Γ be a d-regular graph on N vertices, whose adjacency matrix A has eigenvalues $d = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N$. Let $X \subset V(\Gamma)$*

[1]Recall that if G is a finite group, and $X \subset G$ is an inverse-closed subset of G, then the *Cayley graph* on G generated by X is the graph on vertex-set G where g is joined to gx for all $g \in G$ and $x \in X$.

be an independent set; then

$$|X| \leq \frac{|\lambda_N|}{d + |\lambda_N|} N. \tag{1.1}$$

Let $\mathbf{f}$ *be the all-1's vector in* $\mathbb{C}^N$*, let* $U = \mathit{Span}\{\mathbf{f}\} \oplus E(\lambda_N)$ *be the direct sum of the subspace of constant vectors and the* λ_N*-eigenspace of* A*, and let* v_X *be the characteristic vector of* X*. If equality holds in (1.1), then the shifted characteristic vector* $v_X - (|X|/N)\mathbf{f}$ *is an eigenvector of* A *with eigenvalue* λ_N*, i.e.* $v_X \in U$*.*

The following 'cross-independent' version of Hoffman's Theorem is less well-known; a variant can be found in [3]:

Theorem 1.2. *(i) Let* Γ *be a* d*-regular graph on* N *vertices, whose adjacency matrix* A *has eigenvalues* $\lambda_1 = d \geq \lambda_2 \geq \ldots \geq \lambda_N$*. Let* $\nu = \max(|\lambda_2|, |\lambda_N|)$*. Suppose* X *and* Y *are sets of vertices of* Γ *with no edges between them, i.e.* $xy \notin E(\Gamma)$ *for every* $x \in X$ *and* $y \in Y$*. Then*

$$\sqrt{|X||Y|} \leq \frac{\nu}{d+\nu} N \tag{1.2}$$

(ii) Suppose further that $|\lambda_2| \neq |\lambda_N|$*, and let* λ' *be the larger in modulus of the two. Let* v_X, v_Y *be the characteristic vectors of* X, Y *and let* $\mathbf{f}$ *denote the all-1's vector in* $\mathbb{C}^N$*. If we have equality in (1.2), then* $|X| = |Y|$*, and the shifted characteristic vectors* $v_X - (|X|/N)\mathbf{f}, v_Y - (|Y|/N)\mathbf{f}$ *are eigenvectors of* A *with eigenvalue* λ'*, so* $v_X, v_Y \in \mathit{Span}\{\mathbf{f}\} \oplus E(\lambda')$*, the direct sum of the* d*- and* λ'*-eigenspaces of* A*.*

The proof is an easy extension of the proof of Hoffman's Theorem, with an additional application of the Cauchy-Schwarz Inequality; we give it for completeness.

Proof. Equip $\mathbb{C}^N$ with the inner product:

$$\langle x, y \rangle = \frac{1}{N} \sum_{i=1}^{N} \bar{x}_i y_i,$$

and let

$$||x|| = \sqrt{\frac{1}{N} \sum_{i=1}^{N} |x_i|^2}$$

be the induced norm. Let $u_1 = \mathbf{f}, u_2, \ldots, u_N$ be an orthonormal basis of real eigenvectors of A corresponding to the eigenvalues $\lambda_1 = d, \lambda_2, \ldots, \lambda_N$. Let X and Y be as above; write

$$v_X = \sum_{i=1}^{N} \xi_i u_i, \quad v_Y = \sum_{i=1}^{N} \eta_i u_i$$

as linear combinations of the eigenvectors of A. We have $\xi_1 = \alpha$, $\eta_1 = \beta$,

$$\sum_{i=1}^{N} \xi_i^2 = ||v_X||^2 = |X|/N = \alpha, \quad \sum_{i=1}^{N} \eta_i^2 = ||v_Y||^2 = |Y|/N = \beta.$$

Since there is no edge of Γ between X and Y, we have the crucial property:

$$\begin{aligned} 0 &= \sum_{x \in X, y \in Y} A_{x,y} = v_Y^\top A v_X \\ &= \sum_{i=1}^{N} \lambda_i \xi_i \eta_i = d\alpha\beta + \sum_{i=2}^{N} \lambda_i \xi_i \eta_i \\ &\geq d\alpha\beta - \nu \left| \sum_{i=2}^{N} \xi_i \eta_i \right|. \end{aligned} \tag{1.3}$$

Provided $|\lambda_2| \neq |\lambda_N|$, if we have equality above, then $\xi_i = \eta_i = 0$ unless $\lambda_i = d$ or λ', so $v_X - (|X|/N)\mathbf{f}, v_Y - (|Y|/N)\mathbf{f}$ are λ'-eigenvectors, so $v_X, v_Y \in \mathrm{Span}\{\mathbf{f}\} \oplus E(\lambda')$.

The Cauchy-Schwarz inequality gives:

$$\left| \sum_{i=2}^{N} \xi_i \eta_i \right| \leq \sqrt{\sum_{i=2}^{N} \xi_i^2 \sum_{i=2}^{N} \eta_i^2} = \sqrt{(\alpha - \alpha^2)(\beta - \beta^2)}.$$

Substituting this into (1.3) gives:

$$d\alpha\beta \leq \nu \sqrt{(\alpha - \alpha^2)(\beta - \beta^2)},$$

so

$$\frac{\alpha\beta}{(1-\alpha)(1-\beta)} \leq (\nu/d)^2.$$

By the AM/GM inequality, $(\alpha + \beta)/2 \geq \sqrt{\alpha\beta}$ with equality iff $\alpha = \beta$, so

$$\frac{\alpha\beta}{(1 - \sqrt{\alpha\beta})^2} = \frac{\alpha\beta}{1 - 2\sqrt{\alpha\beta} + \alpha\beta} \leq \frac{\alpha\beta}{1 - \alpha - \beta + \alpha\beta} \leq (\nu/d)^2,$$

implying that

$$\sqrt{\alpha\beta} \leq \frac{\nu}{d+\nu}.$$

Hence, we have

$$\sqrt{|X||Y|} \leq \frac{\nu}{d+\nu}N.$$

Provided $|\lambda_2| \neq |\lambda_N|$, we have equality only if $|X| = |Y| = \frac{\nu}{d+\nu}N$ and $v_X - (|X|/N)\mathbf{f}, v_Y - (|Y|/N)\mathbf{f}$ are eigenvectors of A with eigenvalue λ', as required. □

We will show that for $n \geq 5$, the derangement graph satisfies the hypotheses of this result with $\nu = d_n/(n-1)$; in fact, $\lambda_N = -\frac{d_n}{n-1}$ and all other eigenvalues are $O((n-2)!)$. Note that the eigenvalues of the derangement graph (focussing on the least eigenvalue) have been investigated by Ku and Wales [22], Renteln [29], and Godsil and Meagher [15]. The difference between our approach and theirs is that we employ a short-cut (Lemma 1.9) to bound all eigenvalues of high multiplicity. We also believe that our presentation is natural from an algebraic viewpoint.

Background from non-Abelian Fourier analysis

In this section, we gather the facts we need from non-Abelian Fourier analysis.

If G is a finite group, a *representation* of G is a vector space W together with a group homomorphism $\rho : G \to \mathrm{GL}(W)$ from G to the group of all automorphisms of W, or equivalently a linear action of G on W. If $W = \mathbb{C}^m$, then $\mathrm{GL}(W)$ can be identified with the group of all complex invertible $m \times m$ matrices; we call ρ a *complex matrix representation* of degree (or dimension) m. Note that ρ makes $\mathbb{C}^m$ into a $\mathbb{C}G$-module of dimension m.

We say a representation (ρ, W) is *irreducible* if it has no proper subrepresentation, i.e. no proper subspace of W is fixed by $\rho(g)$ for every $g \in G$. We say that two (complex) representations (ρ, W) and (ρ', W') are *equivalent*, or *isomorphic*, if there exists a linear isomorphism $\phi : W \to W'$ such that $\rho'(g) \circ \phi = \phi \circ \rho(g)$ for every $g \in G$.

For any finite group G, there are only finitely many equivalence classes of irreducible complex representations of G. Let $(\rho_1, \rho_2, \ldots, \rho_k)$ be a complete set of pairwise non-equivalent complex irreducible matrix representations of G (i.e. containing one from each equivalence class of complex irreducible representations).

Definition 1.3. *The* (non-Abelian) Fourier transform *of a function* $f : G \to \mathbb{C}$ *at the irreducible representation* ρ_i *is the matrix*

$$\hat{f}(\rho_i) = \frac{1}{|G|} \sum_{g \in G} f(g) \rho_i(g).$$

If G is a finite group and Γ is a graph on G, the adjacency matrix A of Γ is a linear operator on $\mathbb{C}[G]$, the vector space of all complex-valued functions on G. Recall the following

Definition 1.4. *For a finite group* G*, the* group module $\mathbb{C}G$ *is the complex vector space with basis* G *and multiplication defined by extending the group multiplication linearly; explicitly,*

$$\left(\sum_{g \in G} x_g g \right) \left(\sum_{h \in G} y_h h \right) = \sum_{g,h \in G} x_g y_h (gh).$$

Identifying a function $f : G \to \mathbb{C}$ with $\sum_{g \in G} f(g) g$, we may consider $\mathbb{C}[G]$ as the group module $\mathbb{C}G$.

Let U_i be the subspace of functions in $\mathbb{C}[G]$ whose Fourier transform is concentrated on ρ_i, i.e. with $\hat{f}(\rho_j) = 0$ for each $j \neq i$. Using the above identification, U_i is the sum of all submodules of the group module isomorphic to the module defined by ρ_i; it is a 2-sided ideal of $\mathbb{C}S_n$, and has dimension $\dim(U_i) = (\dim(\rho_i))^2$. The group module decomposes as

$$\mathbb{C}G = \bigoplus_{i=1}^{k} U_i.$$

A function $f : G \to \mathbb{C}$ may be recovered from its Fourier transform using the Fourier Inversion Formula:

$$f(g) = \sum_{i=1}^{k} \dim(\rho_i) \mathrm{Tr} \left(\hat{f}(\rho_i) \rho_i(g^{-1}) \right)$$

where $\mathrm{Tr}(M)$ denotes the trace of the matrix M. It follows from this that the projection of f onto U_i has g-coordinate

$$P_{U_i}(f)_g = \frac{\dim(\rho_i)}{|G|} \sum_{h \in G} f(h) \mathrm{Tr}(\rho_i(hg^{-1})) = \frac{\dim(\rho_i)}{|G|} \sum_{h \in G} f(h) \chi_{\rho_i}(hg^{-1}), \tag{1.4}$$

where $\chi_{\rho_i}(g) = \mathrm{Tr}(\rho_i(g))$ denotes the character of the representation ρ_i.

We say that Γ is a *normal* Cayley graph if its generating set is a union of conjugacy-classes of G. The set of derangements is a union of conjugacy classes of S_n, so the derangement graph is a normal Cayley graph. The following result gives an explicit 1-1 correspondence between the equivalence classes of irreducible representations of G and the eigenvalues of Γ:

Theorem 1.5. *(Frobenius-Schur-others) Let G be a finite group; let $X \subset G$ be an inverse-closed, conjugation-invariant subset of G and let Γ be the Cayley graph on G with generating set X. Let $(\rho_1, V_1), \ldots, (\rho_k, V_k)$ be a complete set of pairwise non-equivalent complex irreducible representations of G. Let U_i be the sum of all submodules of the group module $\mathbb{C}G$ which are equivalent to (ρ_i, V_i). Then we have the decomposition*

$$\mathbb{C}G = \bigoplus_{i=1}^{k} U_i;$$

each U_i is an eigenspace of A with dimension $\dim(V_i)^2$ and eigenvalue

$$\lambda_{\rho_i} = \frac{1}{\dim(V_i)} \sum_{g \in X} \chi_{\rho_i}(g).$$

Background on the representation theory of S_n

In this section, we gather the facts we need about the representation theory of S_n; our treatment follows [19] and [30].

A *partition* of n is a non-increasing sequence of positive integers summing to n, i.e. a sequence $\alpha = (\alpha_1, \ldots, \alpha_k)$ with $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_l \geq 1$ and $\sum_{i=1}^{l} \alpha_i = n$; we write $\alpha \vdash n$. For example, $(3, 2, 2) \vdash 7$; we sometimes use the shorthand $(3, 2, 2) = (3, 2^2)$.

The *cycle-type* of a permutation $\sigma \in S_n$ is the partition of n obtained by expressing σ as a product of disjoint cycles and listing its cycle-lengths in non-increasing order. The conjugacy-classes of S_n are precisely

$$\{\sigma \in S_n : \text{ cycle-type}(\sigma) = \alpha\}_{\alpha \vdash n}.$$

Moreover, there is an explicit 1-1 correspondence between irreducible representations of S_n (up to isomorphism) and partitions of n, which we now describe.

Let $\alpha = (\alpha_1, \ldots, \alpha_l)$ be a partiton of n. The *Young diagram* of α is an array of n dots, or cells, having l left-justified rows where row i contains α_i

dots. For example, the Young diagram of the partition $(3, 2^2)$ is

$$\begin{array}{ccc} \bullet & \bullet & \bullet \\ \bullet & \bullet & \\ \bullet & \bullet & \end{array}$$

If the array contains the numbers $\{1, 2, \ldots, n\}$ in some order in place of the dots, we call it an α-*tableau*; for example,

$$\begin{array}{ccc} 6 & 1 & 7 \\ 5 & 4 & \\ 3 & 2 & \end{array}$$

is a $(3, 2^2)$-tableau. Two α-tableaux are said to be *row-equivalent* if for each row, they have the same numbers in that row. If an α-tableau s has rows $R_1, \ldots, R_l \subset [n]$ and columns $C_1, \ldots, C_k \subset [n]$, we let $R_s = S_{R_1} \times S_{R_2} \times \ldots \times S_{R_l}$ be the row-stablizer of s and $C_s = S_{C_1} \times S_{C_2} \times \ldots \times S_{C_k}$ be the column-stabilizer.

An α-*tabloid* is an α-tableau with unordered row entries (or formally, a row-equivalence class of α-tableaux); given a tableau s, we write $[s]$ for the tabloid it produces. For example, the $(3, 2^2)$-tableau above produces the following $(3, 2^2)$-tabloid

$$\begin{array}{lll} \{1 & 6 & 7\} \\ \{4 & 5\} & \\ \{2 & 3\}. & \end{array}$$

Consider the natural left action of S_n on the set X^α of all α-tabloids; let $M^\alpha = \mathbb{C}[X^\alpha]$ be the corresponding permutation module, i.e. the complex vector space with basis X^α and S_n action given by extending this action linearly. Given an α-tableau s, we define the corresponding α-*polytabloid*

$$e_s := \sum_{\pi \in C_s} \epsilon(\pi) \pi [s].$$

We define the *Specht module* S^α to be the submodule of M^α spanned by the α-polytabloids:

$$S^\alpha = \text{Span}\{e_s : \ s \text{ is an } \alpha\text{-tableau}\}.$$

A central observation in the representation theory of S_n is that *the Specht modules are a complete set of pairwise non-equivalent, irreducible representations of* S_n. Hence, any irreducible representation ρ of S_n is isomorphic to some S^α. For example, $S^{(n)} = M^{(n)}$ is the trivial representation; $M^{(1^n)}$ is the left-regular representation, and $S^{(1^n)}$ is the sign representation S.

We say that a tableau is *standard* if the numbers strictly increase along each row and down each column. It turns out that for any partition α of n,

$$\{e_t : t \text{ is a standard } \alpha\text{-tableau}\}$$

is a basis for the Specht module S^α.

Given a partition α of n, for each cell (i, j) in its Young diagram, we define the 'hook-length' ($h^\alpha_{i,j}$) to be the number of cells in its 'hook' (the set of cells in the same row to the right of it or in the same column below it, including itself) — for example, the hook-lengths of $(3, 2^2)$ are as follows:

$$\begin{matrix} 5 & 4 & 1 \\ 3 & 2 & \\ 2 & 1. & \end{matrix}$$

The dimension f^α of the Specht module S^α is given by the 'Hook Formula':

$$f^\alpha = n!/\prod(\text{hook lengths of } [\alpha]). \tag{1.5}$$

From now on, we will write $[\alpha]$ for the equivalence class of the irreducible representation S^α, χ_α for the irreducible character χ_{S^α}, and ξ_α for the character of the permutation representation M^α. Notice that the set of α-tabloids form a basis for M^α, and therefore $\xi_\alpha(\sigma)$, the trace of the corresponding permutation representation at σ, is precisely the number of α-tabloids fixed by σ.

If $U \in [\alpha]$, $V \in [\beta]$, we define $[\alpha]+[\beta]$ to be the equivalence class of $U \oplus V$, and $[\alpha] \otimes [\beta]$ to be the equivalence class of $U \otimes V$; since $\chi_{U\oplus V} = \chi_U + \chi_V$ and $\chi_{U\otimes V} = \chi_U \cdot \chi_V$, this corresponds to pointwise addition/multiplication of the corresponding characters.

The Branching Rule (see [19] §2.4) states that for any partition α of n, the restriction $[\alpha] \downarrow S_{n-1}$ is isomorphic to a direct sum of those irreducible representations $[\beta]$ of S_{n-1} such that the Young diagram of β can be obtained from that of α by deleting a single dot, i.e., if α^{i-} is the partition whose Young diagram is obtained by deleting the dot at the end of the ith row of that of α, then

$$[\alpha] \downarrow S_{n-1} = \sum_{i:\alpha_i > \alpha_{i-1}} [\alpha^{i-}]. \tag{1.6}$$

For example, if $\alpha = (3, 2^2)$, we obtain

$$[3,2^2] \downarrow S_6 = \begin{bmatrix} \bullet & \bullet & \\ \bullet & \bullet & \\ \bullet & \bullet & \end{bmatrix} + \begin{bmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \\ \bullet & & \end{bmatrix} = [2^3] + [3,2,1].$$

For any partition α of n, we have $S^{(1^n)} \otimes S^{\alpha} \cong S^{\alpha'}$, where α' is the transpose of α, the partition of n with Young diagram obtained by interchanging rows with columns in the Young diagram of α. Hence, $[1^n] \otimes [\alpha] = [\alpha']$, and $\chi_{\alpha'} = \epsilon \cdot \chi_{\alpha}$. For example, we obtain:

$$[3,2,2] \otimes [1^7] = [3,2,2]' = \begin{bmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \\ \bullet & \bullet & \end{bmatrix}' = \begin{bmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & & \end{bmatrix} = [3,3,1]$$

We now explain how the permutation modules M^{β} decompose into irreducibles.

Definition 1.6. *Let α, β be partitions of n. A* generalized α-tableau *is produced by replacing each dot in the Young diagram of α with a number between 1 and n; if a generalized α-tableau has β_i i's $(1 \leq i \leq n)$ it is said to have* content β. *A generalized α-tableau is said to be* semistandard *if the numbers are non-decreasing along each row and strictly increasing down each column.*

Definition 1.7. *Let α, β be partitions of n. The* Kostka number $K_{\alpha,\beta}$ *is the number of semistandard generalized α-tableaux with content β.*

Young's Rule states that for any partition β of n, the permutation module M^{β} decomposes into irreducibles as follows:

$$M^{\beta} \cong \oplus_{\alpha \vdash n} K_{\alpha,\beta} S^{\alpha}.$$

For example, $M^{(n-1,1)}$, which corresponds to the natural permutation action of S_n on $[n]$, decomposes as

$$M^{(n-1,1)} \cong S^{(n-1,1)} \oplus S^{(n)},$$

and therefore

$$\xi_{(n-1,1)} = \chi_{(n-1,1)} + 1. \tag{1.7}$$

The eigenvalues of the derangement graph

We now return to considering the derangement graph. Write U_α for the subspace of $\mathbb{C}[S_n]$ consisting of functions whose Fourier transform is concentrated on $[\alpha]$; equivalently, U_α is the sum of all submodules of $\mathbb{C}S_n$ isomorphic to the Specht module S^α. Note that $U_{(n)} = \mathrm{Span}\{\boldsymbol{f}\}$ is the subspace of constant vectors in $\mathbb{C}S_n$. Applying Theorem 1.5 to the derangement graph Γ, we have

$$\mathbb{C}S_n = \bigoplus_{\alpha \vdash n} U_\alpha,$$

and each U_α is an eigenspace of the derangement graph, with dimension $\dim(U_\alpha) = (f^\alpha)^2$ and corresponding eigenvalue

$$\lambda_\alpha = \frac{1}{f^\alpha} \sum_{\sigma \in \mathcal{D}_n} \chi_\alpha(\sigma). \tag{1.8}$$

We will use the following result, a variant of which is proved in [19]; for the reader's convenience, we include a proof using the Branching Rule and the Hook Formula.

Lemma 1.8. *For $n \geq 9$, the only Specht modules S^α of dimension $f^\alpha < \binom{n-1}{2} - 1$ are as follows:*

- *$S^{(n)}$ (the trivial representation), dimension 1;*
- *$S^{(1^n)}$ (the sign representation S), dimension 1;*
- *$S^{(n-1,1)}$, dimension $n-1$;*
- *$S^{(2,1^{n-2})}$ ($\cong S \otimes S^{(n-1,1)}$), dimension $n-1$.*

$(*)$

Proof. By direct calculation using (1.5) the lemma can be verified for $n = 9, 10$. We proceed by induction. Assume the lemma holds for $n-2, n-1$; we will prove it for n. Let α be a partition of n such that $f^\alpha < \binom{n-1}{2} - 1$. Consider the restriction $[\alpha] \downarrow S_{n-1}$, which has the same dimension. First suppose $[\alpha] \downarrow S_{n-1}$ is reducible. If it has one of our 4 irreducible representations $(*)$ as a constituent, then by (1.6), the possibilies for α are as follows:

constituent	possibilies for α
$[n-1]$	$(n), (n-1, 1)$
$[1^{n-1}]$	$(1^n), (2, 1^{n-1})$
$[n-2, 1]$	$(n-1, 1), (n-2, 2), (n-2, 1, 1)$
$[2, 1^{n-3}]$	$(2, 1^{n-2}), (2, 2, 1^{n-4}), (3, 1^{n-3})$

But using (1.5), the new irreducible representations above all have dimension $\geq \binom{n-1}{2} - 1$:

α	f^α
$(n-2, 2), (2, 2, 1^{n-4})$	$\binom{n-1}{2} - 1$
$(n-2, 1, 1), (3, 1^{n-3})$	$\binom{n-1}{2}$

so none of these are constituents of $[\alpha] \downarrow S_{n-1}$. So we may assume that the irreducible constituents of $[\alpha] \downarrow S_{n-1}$ do not include any of our 4 irreducible representations $(*)$, hence by the induction hypothesis for $n-1$, each has dimension $\geq \binom{n-2}{2} - 1$. But $2(\binom{n-2}{2} - 1) \geq \binom{n-1}{2} - 1$ provided $n \geq 11$, hence there is just one, i.e. $[\alpha] \downarrow S_{n-1}$ is irreducible. Therefore $[\alpha] = [s^t]$ for some $s, t \in \mathbb{N}$ with $st = n$, i.e. it has square Young diagram. Now consider

$$[\alpha] \downarrow S_{n-2} = [s^{t-1}, s-2] + [s^{t-2}, s-1, s-1].$$

Note that neither of these 2 irreducible constituents are any of our 4 irreducible representations $(*)$, hence by the induction hypothesis for $n-2$, each has dimension $\geq \binom{n-3}{2} - 1$, but $2(\binom{n-3}{2} - 1) \geq \binom{n-1}{2} - 1$ for $n \geq 11$, contradicting $\dim([\alpha] \downarrow S_{n-2}) < \binom{n-1}{2} - 1$. □

If α is any partition of n whose Specht module has high dimension $f^\alpha \geq \binom{n-1}{2} - 1$, we may bound $|\lambda_\alpha|$ using the following trick:

Lemma 1.9. *Let Γ be a graph on N vertices whose adjancency matrix A has eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N$; then*

$$\sum_{i=1}^{N} \lambda_i^2 = 2e(\Gamma).$$

This is well-known; we include a proof for completeness.

Proof. Diagonalize A: there exists a real invertible matrix P such that $A =$

$P^{-1}DP$, where D is the diagonal matrix

$$D = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \dots & \lambda_N \end{pmatrix}.$$

We have $A^2 = P^{-1}D^2P$, and therefore

$$2e(\Gamma) = \sum_{i,j=1}^{N} A_{i,j} = \sum_{i,j=1}^{N} A_{i,j}^2 = \mathrm{Tr}(A^2) = \mathrm{Tr}(P^{-1}D^2P) = \mathrm{Tr}(D^2) = \sum_{i=1}^{N} \lambda_i^2$$

as required. □

Hence, the eigenvalues of the derangement graph satisfy:

$$\sum_{\alpha \vdash n} (f^{\alpha}\lambda_{\alpha})^2 = 2e(\Gamma) = n!d_n = (n!)^2(1/e + o(1)),$$

so for each partition α of n,

$$|\lambda_{\alpha}| \leq \frac{\sqrt{n!d_n}}{f^{\alpha}} = \frac{n!}{f^{\alpha}}\sqrt{1/e + o(1)}.$$

Therefore, if S^{α} has dimension $f^{\alpha} \geq \binom{n-1}{2} - 1$, then $|\lambda_{\alpha}| \leq O((n-2)!)$. For each of the Specht modules $(*)$, we now explicitly calculate the corresponding eigenvalue using (1.8).

For the trivial module, $\chi_{(n)} \equiv 1$, so

$$\lambda_{(n)} = d_n.$$

For the sign module $S^{(1^n)}$, $\chi_{(1^n)} = \epsilon$, so

$$\lambda_{(1^n)} = \sum_{\sigma \in \mathcal{D}_n} \epsilon(\sigma) = e_n - o_n$$

where e_n, o_n are the number of even and odd derangements of $[n]$, respectively. It is well known that for any $n \in \mathbb{N}$,

$$e_n - o_n = (-1)^{n-1}(n-1). \tag{1.9}$$

To see this, note that an odd permutation $\sigma \in S_n$ without fixed points can be written as $(i\ n)\rho$, where $\sigma(n) = i$, and ρ is either an even permutation of $[n-1]\setminus\{i\}$ with no fixed points (if $\sigma(i)=n$), or an even permutation of $[n-1]$ with no fixed points (if $\sigma(i)\neq n$). Conversely, for any $i\neq n$, if ρ is any even permutation of $[n-1]$ with no fixed points or any even permutation of $[n-1]\setminus\{i\}$ with no fixed points, then $(i\ n)\rho$ is a permutation of $[n]$ with no fixed points taking $n\mapsto i$. Hence, for all $n\geq 3$,

$$o_n=(n-1)(e_{n-1}+e_{n-2}).$$

Similarly,

$$e_n=(n-1)(o_{n-1}+o_{n-2}).$$

(1.9) follows by induction on n.

Hence, we have:

$$\lambda_{(1^n)}=(-1)^{n-1}(n-1).$$

For the partition $(n-1,1)$, from (1.7), we have:

$$\chi_{(n-1,1)}(\sigma)=\xi_{(n-1,1)}(\sigma)-1=\#\{\text{fixed points of }\sigma\}-1$$

giving:

$$\lambda_{(n-1,1)}=\frac{1}{n-1}\sum_{\sigma\in\mathcal{D}_n}(-1)=-\frac{d_n}{n-1}.$$

For $S^{(2,1^{n-2})}\cong S^{(1^n)}\otimes S^{(n-1,1)}$, $\chi_{(2,1^{n-2})}=\epsilon\cdot\chi_{(n-1,1)}$, so

$$\chi_{(2,1^{n-2})}(\sigma)=\epsilon(\sigma)(\#\{\text{fixed points of }\sigma\}-1)$$

and therefore

$$\lambda_{(2,1^{n-2})}=\frac{1}{n-1}\sum_{\sigma\in\mathcal{D}_n}-\epsilon(\sigma)=-\frac{e_n-o_n}{n-1}=(-1)^n.$$

To summarize, we obtain:

α	λ_α
(n)	d_n
(1^n)	$(-1)^{n-1}(n-1)$
$(n-1,1)$	$-d_n/(n-1)$
$(2,1^{n-2})$	$(-1)^n$

Hence, $U_{(n)}$ is the d_n-eigenspace, $U_{(n-1,1)}$ is the $-d_n/(n-1)$-eigenspace, and all other eigenvalues are $O((n-2)!)$. Hence, Leader's conjecture follows (for n sufficiently large) by applying Theorem 1.2 to the derangement graph. It is easy to check that $\nu=d_n/(n-1)$ for all $n\geq 4$, giving

Theorem 1.10. *If $n \geq 4$, then any cross-intersecting pair of families $\mathcal{A}, \mathcal{B} \subset S_n$ satisfy*
$$|\mathcal{A}||\mathcal{B}| \leq ((n-1)!)^2$$

If equality holds, then by Theorem 1.2 part *(ii)*, the characteristic vectors $v_{\mathcal{A}}, v_{\mathcal{B}}$ must lie in the direct sum of the d_n and $-d_n/(n-1)$-eigenspaces. It can be checked that for $n \geq 5$, $|\lambda_\alpha| < d_n/(n-1)\ \forall \alpha \neq (n), (n-1,1)$, so the d_n eigenspace is precisely $U_{(n)}$ and the $-d/(n-1)$-eigenspace is precisely $U_{(n-1,1)}$. We will see in Chapter 2 (Lemma 2.5) that $U_{(n)} \oplus U_{(n-1,1)}$ is precisely the span of the characteristic vectors of the 1-cosets of S_n. Hence, for $n \geq 5$, if equality holds in Theorem 1.10, then the characteristic vectors of $\mathcal{A}$ and $\mathcal{B}$ are linear combinations of the characteristic vectors of the 1-cosets. We will see in Chapter 2 (Corollary 2.14) that if the characteristic vector of $\mathcal{A} \subset S_n$ is a linear combination of the characteristic vectors of the 1-cosets, then $\mathcal{A}$ is a disjoint union of 1-cosets. It follows that for $n \geq 5$, if equality holds in Theorem 1.10, then $\mathcal{A}$ and $\mathcal{B}$ are both disjoint unions of 1-cosets. Since they are cross-intersecting, they must both be equal to the same 1-coset, i.e.
$$\mathcal{A} = \mathcal{B} = \{\sigma \in S_n :\ \sigma(i) = j\}$$
for some $i, j \in [n]$. It is easily checked that the same conclusion holds when $n = 4$, so we have the following characterization of the case of equality in Leader's conjecture:

Theorem 1.11. *For $n \geq 4$, if $\mathcal{A}, \mathcal{B} \subset S_n$ is a cross-intersecting pair of families satisfying*
$$|\mathcal{A}||\mathcal{B}| = ((n-1)!)^2,$$
then
$$\mathcal{A} = \mathcal{B} = \{\sigma \in S_n :\ \sigma(i) = j\}$$
for some $i, j \in [n]$.

1.3 Stability

We now turn to the question of stability. First, we prove a 'rough' stability result: for any positive constant $c > 0$, if $\mathcal{A}$ is an intersecting family of permutations of size $|\mathcal{A}| \geq c(n-1)!$, then there exist i and j such that all but $O((n-2)!)$ permutations in $\mathcal{A}$ map i to j, i.e. $\mathcal{A}$ is 'almost' centred. In other words, writing $\mathcal{A}_{i \mapsto j}$ for the collection of all permutations in $\mathcal{A}$ mapping i to j, $|\mathcal{A} \setminus \mathcal{A}_{i \mapsto j}| \leq O((n-2)!)$. To prove this, we will first show that if $\mathcal{A}$ is an intersecting family of size at least $c(n-1)!$, then the characteristic vector $v_{\mathcal{A}}$ of $\mathcal{A}$ cannot be too far from the subspace U spanned by the characteristic vectors of the 1-cosets, the intersecting families of maximum size $(n-1)!$. This relies on a straightforward consequence of the proof of Hoffman's Theorem (Theorem 1.1). Hoffman's Theorem states that if equality holds in (1.1), then $v_X \in U$. We need a 'softened' version of this statement:

Lemma 1.12. *Let Γ, A, X and U be as in Theorem 1.1. Let $\alpha = |X|/N$. Let λ_L be the negative eigenvalue of second largest modulus. Equip $\mathbb{C}^N$ with the inner product:*

$$\langle x, y\rangle = \frac{1}{N}\sum_{i=1}^{N} \bar{x}_i y_i,$$

and let

$$||x|| = \sqrt{\frac{1}{N}\sum_{i=1}^{N} |x_i|^2}$$

be the induced norm. Let D be the Euclidean distance from the characteristic vector v_X of X to the subspace U, i.e. the norm $||P_{U^\perp}(v_X)||$ of the projection of v_X onto $U^\perp$. Then

$$D^2 \leq \frac{(1-\alpha)|\lambda_N| - \lambda_1\alpha}{|\lambda_N| - |\lambda_L|}\alpha.$$

For completeness, we include a proof:

Proof. Let $u_1 = \mathbf{f}, u_2, \ldots, u_N$ be an orthonormal basis of real eigenvectors of A corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_N$. Write

$$v_X = \sum_{i=1}^{N} \xi_i u_i$$

as a linear combination of the eigenvectors of A; we have $\xi_1 = \alpha$ and

$$\sum_{i=1}^{N} \xi_i^2 = ||v_X||^2 = |X|/N = \alpha.$$

Then we have the crucial property:

$$0 = \sum_{x,y\in X} A_{x,y} = v_X^\top A v_X = \sum_{i=1}^{N} \lambda_i \xi_i^2 \geq \lambda_1 \xi_1^2 + \lambda_N \sum_{i:\lambda_i=\lambda_N} \xi_i^2 + \lambda_L \sum_{i>1:\lambda_i\neq\lambda_N} \xi_i^2.$$

Note that

$$\sum_{i>1:\lambda_i \neq \lambda_N} \xi_i^2 = D^2,$$

and

$$\sum_{i:\lambda_i = \lambda_N} \xi_i^2 = \alpha - \alpha^2 - D^2,$$

so we have

$$0 \geq \lambda_1 \alpha^2 + \lambda_N(\alpha - \alpha^2 - D^2) + \lambda_L D^2.$$

Rearranging, we obtain:

$$D^2 \leq \frac{(1-\alpha)|\lambda_N| - \lambda_1 \alpha}{|\lambda_N| - |\lambda_L|} \alpha,$$

as required. □

The derangement graph Γ has $\lambda_N = -d_n/(n-1)$ and $|\lambda_L| = O((n-2)!)$, so applying the above result to an intersecting family $\mathcal{A} \subset S_n$ gives:

$$||P_{U_t^\perp}(v_{\mathcal{A}})||^2 \leq (1 - |\mathcal{A}|/(n-1)!)(1 + O(1/n))|\mathcal{A}|/n!. \quad (1.10)$$

We will use this to show that there exist $i, j \in [n]$ such that $|\mathcal{A}_{i \mapsto j}| \geq \omega((n-2)!)$. Clearly, for any fixed $i \in [n]$,

$$\sum_{j=1}^{n} |\mathcal{A}_{i \mapsto j}| = |\mathcal{A}|,$$

and therefore the average size of an $|\mathcal{A}_{i \mapsto k}|$ is $|\mathcal{A}|/n$; $|\mathcal{A}_{i \mapsto j}|$ is ω of the average size. This statement would at first seem too weak to help us, but combining it with the fact that $\mathcal{A}$ is intersecting, we may 'boost' it to the much stronger statement $|\mathcal{A}_{i \mapsto j}| \geq (1 - o(1))|\mathcal{A}|$. In detail, we will deduce from Theorem 1.10 that for any $j \neq k$,

$$|\mathcal{A}_{i \mapsto j}||\mathcal{A}_{i \mapsto k}| \leq ((n-2)!)^2,$$

giving $|\mathcal{A}_{i \mapsto k}| \leq o((n-2)!)$ for any $k \neq j$. Summing over all $k \neq j$ will give $|\mathcal{A} \setminus \mathcal{A}_{i \mapsto j}| \leq o((n-1)!)$, enabling us to complete the proof.

Here, then, is our rough stability result:

Theorem 1.13. *Let $c > 0$ be a positive constant. If $\mathcal{A} \subset S_n$ is an intersecting family of permutations of size $|\mathcal{A}| \geq c(n-1)!$, then there exist $i, j \in [n]$ such that all but at most $O((n-2)!)$ permutations in $\mathcal{A}$ map i to j.*

Proof. Let $\mathcal{A} \subset S_n$ be an intersecting family of permutations of size $|\mathcal{A}| \geq c(n-1)!$. Write $|\mathcal{A}| = (1-\delta)(n-1)!$, where $\delta \leq 1-c$. Then by (1.10),

$$D^2 = ||P_{U^\perp}(v_{\mathcal{A}})||^2 \leq \delta(1+O(1/n))|\mathcal{A}|/n!. \tag{1.11}$$

The projection of $v_{\mathcal{A}}$ onto $U_{(n)} = \text{Span}\{\mathbf{f}\}$ is clearly $(|\mathcal{A}|/n!)\mathbf{f}$. By equation (1.4), the projection of $v_{\mathcal{A}}$ onto $U_{(n-1,1)}$ has σ-coordinate

$$\begin{aligned}
P_{U_{(n-1,1)}}(v_{\mathcal{A}})_\sigma &= \frac{n-1}{n!}\sum_{\rho\in\mathcal{A}}\chi_{(n-1,1)}(\rho\sigma^{-1}) \\
&= \frac{n-1}{n!}\sum_{\rho\in\mathcal{A}}(\xi_{(n-1,1)}(\rho\sigma^{-1})-1) \\
&= \frac{n-1}{n!}\sum_{\rho\in\mathcal{A}}(\#\{\text{fixed points of } \rho\sigma^{-1}\}-1) \\
&= \frac{n-1}{n!}(\#\{(\rho,i) : \rho\in\mathcal{A},\ i\in[n],\ \rho(i)=\sigma(i)\} - |\mathcal{A}|) \\
&= \frac{n-1}{n!}\sum_{i=1}^{n}|\mathcal{A}_{i\mapsto\sigma(i)}| - \frac{n-1}{n!}|\mathcal{A}|.
\end{aligned}$$

Hence, the σ-coordinate P_σ of the projection of $v_{\mathcal{A}}$ onto $U = U_{(n)} \oplus U_{(n-1,1)}$ is given by

$$P_\sigma = \frac{n-1}{n!}\sum_{i=1}^{n}|\mathcal{A}_{i\mapsto\sigma(i)}| - \frac{(n-2)}{n!}|\mathcal{A}|,$$

which is a linear function of the number of times σ agrees with a permutation in $\mathcal{A}$.

From (1.11),

$$\sum_{\sigma\in\mathcal{A}}(1-P_\sigma)^2 + \sum_{\sigma\notin\mathcal{A}}P_\sigma^2 \leq |\mathcal{A}|\delta(1+O(1/n)). \tag{1.12}$$

Choose $C > 0$ such that $|\mathcal{A}|(1-1/n)\delta(1+C/n)$ is at least the right-hand-side of (1.12); then $(1-P_\sigma)^2 < \delta(1+C/n)$ for at least $|\mathcal{A}|/n$ permutations in $\mathcal{A}$, so the subset

$$\mathcal{A}' := \{\sigma\in\mathcal{A} : (1-P_\sigma)^2 < \delta(1+C/n)\}$$

has size at least $|\mathcal{A}|/n$. Similarly, $P_\sigma^2 < 2\delta/n$ for all but at most

$$n|\mathcal{A}|(1+O(1/n))/2 = (1-\delta)n!(1+O(1/n))/2$$

permutations $\sigma \notin \mathcal{A}$, so the subset $\mathcal{T} = \{\sigma \notin \mathcal{A} : P_\sigma^2 < 2\delta/n\}$ has size

$$|\mathcal{T}| \geq n! - (1-\delta)(n-1)! - (1-\delta)n!(1+O(1/n))/2.$$

The permutations $\sigma \in \mathcal{A}'$ have P_σ close to 1, and the permutations $\pi \in \mathcal{T}$ have P_π close to 0. Using only the lower bounds on the sizes of $\mathcal{A}'$ and $\mathcal{T}$, we may prove the following:

Claim: There exist permutations $\sigma \in \mathcal{A}'$, $\pi \in \mathcal{T}$ such that $\sigma^{-1}\pi$ is a product of at most $h = h(n)$ transpositions, where $h = 2\sqrt{2(n-1)\log n}$.

Proof of Claim: Let H be the *transposition graph* on S_n, the Cayley graph on S_n generated by the transpositions, i.e. $V(H) = S_n$ and $\sigma\pi \in E(H)$ iff $\sigma^{-1}\pi$ is a transposition. We use an isoperimetric inequality for H, essentially the martingale inequality of Maurey:

Theorem 1.14. *Let $X \subset V(H)$ with $|X| \geq an!$ where $0 < a < 1$. Then for any $h \geq h_0 := \sqrt{\frac{1}{2}(n-1)\log\frac{1}{a}}$,*

$$|N_h(X)| \geq \left(1 - e^{-\frac{2(h-h_0)^2}{n-1}}\right) n!.$$

For a proof, see for example [26]. Applying Theorem 1.14 to the set $\mathcal{A}'$, which has $|\mathcal{A}'| \geq \frac{c(n-1)!}{n} \geq \frac{n!}{n^4}$, with $a = 1/n^4$, $h = 2h_0$, gives $|N_h(\mathcal{A}')| \geq (1-n^{-4})n!$, so certainly $N_h(\mathcal{A}') \cap \mathcal{T} \neq \emptyset$, proving the claim.

We now have two permutations $\sigma \in \mathcal{A}$, $\pi \notin \mathcal{A}$ which are 'close' to one another in H (differing in only $O(\sqrt{n\log n})$ transpositions) such that $P_\sigma > 1 - \sqrt{\delta(1+C/n)}$ and $P_\pi < \sqrt{2\delta/n}$, and therefore $P_\sigma - P_\pi > 1 - \sqrt{\delta} - O(1/\sqrt{n})$, i.e. σ agrees many more times than π with permutations in $\mathcal{A}$:

$$\sum_{i=1}^{n} |\mathcal{A}_{i\mapsto\sigma(i)}| - \sum_{i=1}^{n} |\mathcal{A}_{i\mapsto\pi(i)}| \geq (n-1)!(1-\sqrt{\delta} - O(1/\sqrt{n})).$$

Suppose for this pair we have $\pi = \sigma\tau_1\tau_2\ldots\tau_l$ for transpositions $\tau_1,\ldots,\tau_l$, where $l \leq t$. Let I be the set of numbers appearing in these transpositions; then $|I| \leq 2l \leq 2t$, and $\sigma(i) = \pi(i)$ for each $i \notin I$. Hence,

$$\sum_{i\in I} |\mathcal{A}_{i\mapsto\sigma(i)}| - \sum_{i\in I} |\mathcal{A}_{i\mapsto\pi(i)}| \geq (n-1)!(1-\sqrt{\delta} - O(1/\sqrt{n})),$$

so certainly,

$$\sum_{i \in I} |\mathcal{A}_{i \mapsto \sigma(i)}| \geq (n-1)!(1 - \sqrt{\delta} - O(1/\sqrt{n})).$$

By averaging,

$$\begin{aligned} |\mathcal{A}_{i \mapsto \sigma(i)}| &\geq \frac{1}{|I|}(n-1)!(1-\sqrt{\delta}-O(1/\sqrt{n})) \\ &\geq \frac{(n-1)!}{4\sqrt{2(n-1)\log n}}(1-\sqrt{\delta}-O(1/\sqrt{n})) \end{aligned}$$

for some $i \in I$. Let $\sigma(i) = j$; then

$$|\mathcal{A}_{i \mapsto j}| \geq \frac{(n-1)!}{4\sqrt{2(n-1)\log n}}(1-\sqrt{1-c}-O(1/\sqrt{n})) = \omega((n-2)!).$$

We will now use Theorem 1.10 to show that $|\mathcal{A}_{i \mapsto k}|$ is small for each $k \neq j$. Notice that for each $k \neq j$, the pair $\mathcal{A}_{i \mapsto j}, \mathcal{A}_{i \mapsto k}$ is cross-intersecting.

Lemma 1.15. *Let $\mathcal{A} \subset S_n$ be an intersecting family; then for all i, j and k with $k \neq j$,*

$$|\mathcal{A}_{i \mapsto j}||\mathcal{A}_{i \mapsto k}| \leq ((n-2)!)^2.$$

Proof. By double translation, we may assume that $i = j = 1$ and $k = 2$. Let $\sigma \in \mathcal{A}_{1 \mapsto 1}$ and $\pi \in \mathcal{A}_{1 \mapsto 2}$; then there exists $p \neq 1$ such that $\sigma(p) = \pi(p) > 2$. Hence, the translates $\mathcal{E} = \mathcal{A}_{1 \mapsto 1}$ and $\mathcal{F} = (1\ 2)\mathcal{A}_{1 \mapsto 2}$ are families of permutations fixing 1 and cross-intersecting on the domain $\{2, 3, \ldots, n\}$. Deleting 1 from each permutation in the two families gives a cross-intersecting pair $\mathcal{E}', \mathcal{F}'$ of families of permutations of $\{2, 3, \ldots, n\}$; applying Theorem 1.10 gives:

$$|\mathcal{A}_{1 \mapsto 1}||\mathcal{A}_{1 \mapsto 2}| = |\mathcal{E}'||\mathcal{F}'| \leq ((n-2)!)^2.$$

□

Since $|\mathcal{A}_{i \mapsto j}| \geq \omega((n-2)!)$, $|\mathcal{A}_{i \mapsto k}| \leq o((n-2)!)$ for all $k \neq j$, so summing over all $k \neq j$ gives

$$|\mathcal{A} \setminus \mathcal{A}_{i \mapsto j}| = \sum_{k \neq j} |\mathcal{A}_{i \mapsto k}| \leq o((n-1)!),$$

and therefore

$$|\mathcal{A}_{i \mapsto j}| = |\mathcal{A}| - |\mathcal{A} \setminus \mathcal{A}_{i \mapsto j}| \geq (c - o(1))(n-1)!. \tag{1.13}$$

Applying Lemma 1.15 again gives

$$|\mathcal{A}_{i\mapsto k}| \leq O((n-3)!)$$

for all $k \neq j$; summing over all $k \neq j$ gives

$$|\mathcal{A} \setminus \mathcal{A}_{i\mapsto j}| \leq O((n-2)!),$$

proving Theorem 1.13. □

The stability conjecture of Cameron and Ku follows easily:

Corollary 1.16. *Let $c > 1 - 1/e$; then for n sufficiently large depending on c, any intersecting family $\mathcal{A} \subset S_n$ of size $|\mathcal{A}| \geq c(n-1)!$ is centred.*

Proof. By Theorem 1.13, there exist $i, j \in [n]$ such that $|\mathcal{A} \setminus \mathcal{A}_{i\mapsto j}| \leq O((n-2)!)$, and therefore

$$|\mathcal{A}_{i\mapsto j}| \geq (c - O(1/n))(n-1)!. \tag{1.14}$$

Suppose for a contradiction that $\mathcal{A}$ is non-centred. Then there exists a permutation $\tau \in \mathcal{A}$ such that $\tau(i) \neq j$. Any permutation in $\mathcal{A}_{i\mapsto j}$ must agree with τ at some point. But for any $i, j \in [n]$ and any $\tau \in S_n$ such that $\tau(i) \neq j$, the number of permutations in S_n which map i to j and agree with τ at some point is

$$(n-1)! - d_{n-1} - d_{n-2} = (1 - 1/e - o(1))(n-1)!$$

(By double translation, we may assume that $i = j = 1$ and $\tau = (1\ 2)$. Observe that a permutation fixing 1 intersects $(1\ 2)$ if and only if it has a fixed point greater than 2, so the number of permutations fixing 1 and intersecting $(1\ 2)$ is $(n-1)! - d_{n-1} - d_{n-2}$.) This contradicts (1.14) provided n is sufficiently large depending on c. □

We now use our rough stability result to prove the Hilton-Milner type conjecture of Cameron and Ku, for n sufficiently large. First, we introduce an extra notion which will be useful in the proof. Following Cameron and Ku [6], given a permutation $\pi \in S_n$ and $i \in [n]$, we define the *i-fix* of π to be the permutation π_i which fixes i, maps the preimage of i to the image of i, and agrees with π at all other points of $[n]$, i.e.

$$\pi_i(i) = i;\ \pi_i(\pi^{-1}(i)) = \pi(i);\ \pi_i(k) = \pi(k)\ \forall k \neq i, \pi^{-1}(i)$$

In other words, $\pi_i = \pi(\pi^{-1}(i)\ i)$. We inductively define

$$\pi_{i_1,\ldots,i_l} = (\pi_{i_1,\ldots,i_{l-1}})_{i_l}$$

Notice that if σ fixes j, then σ agrees with π_j wherever it agrees with π.

Theorem 1.17. *For n sufficiently large, if $\mathcal{A} \subset S_n$ is a non-centred intersecting family, then $\mathcal{A}$ is at most as large as the family*

$$\mathcal{C} = \{\sigma \in S_n : \sigma(1) = 1, \sigma(i) = i \text{ for some } i > 2\} \cup \{(12)\}$$

which has size $(n-1)! - d_{n-1} - d_{n-2} + 1 = (1 - 1/e + o(1))(n-1)!$. Equality holds iff $\mathcal{A}$ is a double translate of $\mathcal{C}$, i.e. $\mathcal{A} = \pi\mathcal{C}\tau$ for some $\pi, \tau \in S_n$.

Proof. Let $\mathcal{A}$ be a non-centred intersecting family the same size as $\mathcal{C}$; we must show that $\mathcal{A}$ is a double translate of $\mathcal{C}$. By Theorem 1.13, there exist $i, j \in [n]$ such that $|\mathcal{A} \setminus \mathcal{A}_{i \mapsto j}| \leq O((n-2)!)$, and therefore

$$|\mathcal{A}_{i \mapsto j}| \geq (n-1)! - d_{n-1} - d_{n-2} + 1 - O(n-2)! = (1 - 1/e - o(1))(n-1)!.$$

Since $\mathcal{A}$ is non-centred, it must contain some permutation ρ such that $\rho(i) \neq j$. By double translation, we may assume that $i = j = 1$ and $\rho = (1\ 2)$; we will show that under these hypotheses, $\mathcal{A} = \mathcal{C}$. We have

$$|\mathcal{A}_{1 \mapsto 1}| \geq (1 - 1/e - o(1))(n-1)! \tag{1.15}$$

and $(1\ 2) \in \mathcal{A}$. Note that every permutation in $\mathcal{A}$ must intersect $(1\ 2)$, and therefore

$$\mathcal{A}_{1 \mapsto 1} \cup \{(1\ 2)\} \subset \mathcal{C}.$$

We need to show that $(1\ 2)$ is the only permutation in $\mathcal{A}$ that does not fix 1. Suppose for a contradiction that $\mathcal{A}$ contains some other permutation π not fixing 1. Then π must shift some point $p > 2$. If σ fixes both 1 and p, then σ agrees with $\pi_{1,p} = (\pi_1)_p$ wherever it agrees with π. There are exactly d_{n-2} permutations which fix 1 and p and disagree with $\pi_{1,p}$ at every point of $\{2, \ldots, n\} \setminus \{p\}$; each disagrees everywhere with π, so none are in $\mathcal{A}$, and therefore

$$|\mathcal{A}_{1 \mapsto 1}| \leq (n-1)! - d_{n-1} - 2d_{n-2}.$$

Hence, by assumption,

$$|\mathcal{A} \setminus \mathcal{A}_{1 \mapsto 1}| \geq d_{n-2} + 1 = \Omega((n-2)!).$$

Notice that we have the following trivial bound on the size of a t-intersecting family $\mathcal{F} \subset S_n$:

$$|\mathcal{F}| \leq \binom{n}{t}(n-t)! = n!/t!$$

since every permutation in $\mathcal{F}$ must agree with a fixed $\rho \in \mathcal{F}$ in at least t places.

Hence, $\mathcal{A} \setminus \mathcal{A}_{1 \mapsto 1}$ cannot be $(\log n)$-intersecting and therefore contains two permutations ρ, τ agreeing on at most $\log n$ points. The number of permutations fixing 1 and agreeing with both τ_1 and τ_2 at one of these points is at most $(\log n)(n-2)!$. All other permutations in $\mathcal{A} \cap \mathcal{C}$ agree with ρ and τ at two separate points of $\{2, \ldots, n\}$, and by the above argument, the same holds for the 1-fixes ρ_1 and τ_1. The number of permutations fixing 1 that agree with ρ_1 and τ_1 at two separate points of $\{2, \ldots, n\}$ is at most $((1-1/e)^2 + o(1))(n-1)!$ (it is easily checked that given two fixed permutations, the probability that a uniform random permutation agrees with them at separate points is at most $(1-1/e)^2 + o(1)$). Hence,

$$\begin{aligned} |\mathcal{A}_{1 \mapsto 1} &\leq ((1-1/e)^2 + o(1))(n-1)! + (\log n)(n-2)! \\ &= ((1-1/e)^2 + o(1))(n-1)! \end{aligned}$$

contradicting (1.15) provided n is sufficiently large.

Hence, $(1\ 2)$ is the only permutation in $\mathcal{A}$ that does not fix 1, so $\mathcal{A} = \mathcal{A}_{1 \mapsto 1} \cup \{(1\ 2)\} \subset \mathcal{C}$; since $|\mathcal{A}| = |\mathcal{C}|$, we have $\mathcal{A} = \mathcal{C}$ as required. □

We now perform a very similar stability analysis for cross-intersecting families. First, we prove a 'rough' stability result analogous to Theorem 1.13, namely that for any positive constant $c > 0$, if $\mathcal{A}, \mathcal{B} \subset S_n$ is a pair of cross-intersecting families of permutations with $\sqrt{|\mathcal{A}||\mathcal{B}|} \geq c(n-1)!$, then there exist $i, j \in [n]$ such that all but at most $O((n-2)!)$ permutations in $\mathcal{A}$ and all but at most $O((n-2)!)$ permutations in $\mathcal{B}$ map i to j.

Theorem 1.18. *Let $c > 0$ be a positive constant. If $\mathcal{A}, \mathcal{B} \subset S_n$ is a cross-intersecting pair of families with $\sqrt{|\mathcal{A}||\mathcal{B}|} \geq c(n-1)!$, then there exist $i, j \in [n]$ such that all but at most $O((n-2)!)$ permutations in $\mathcal{A}$ and all but at most $O((n-2)!)$ permutations in $\mathcal{B}$ map i to j.*

Proof. Let $|\mathcal{A}| \leq |\mathcal{B}|$. First we examine the proof of Theorem 1.2 to bound $D = ||P_{U^\perp}(v_X)||$, $E = ||P_{U^\perp}(v_Y)||$. This time, we have

$$\sum_{i>1:\lambda_i \neq \lambda_N} \xi_i^2 = D^2,$$

$$\sum_{i>1:\lambda_i \neq \lambda_N} \eta_i^2 = E^2,$$

$$\sum_{i>1:\lambda_i = \lambda_N} \xi_i^2 = \alpha - \alpha^2 - D^2,$$

$$\sum_{i>1:\lambda_i=\lambda_N} \eta_i^2 = \beta - \beta^2 - E^2.$$

Substituting into (1.3) gives:

$$\begin{aligned}
d\alpha\beta &= -\sum_{i>1:\lambda_i\neq\lambda_N} \lambda_i\xi_i\eta_i - \lambda_N \sum_{i>1:\lambda_i=\lambda_N} \xi_i\eta_i \\
&\leq \mu \sum_{i>1:\lambda_i\neq\lambda_N} |\xi_i||\eta_i| + |\lambda_N| \sum_{i>1:\lambda_i=\lambda_N} |\xi_i||\eta_i| \\
&\leq \mu \sqrt{\sum_{i>1:\lambda_i\neq\lambda_N} \xi_i^2}\sqrt{\sum_{i>1:\lambda_i\neq\lambda_N} \eta_i^2} + |\lambda_N|\sqrt{\sum_{i>1:\lambda_i=\lambda_N} \xi_i^2}\sqrt{\sum_{i>1:\lambda_i=\lambda_N} \eta_i^2} \\
&= \mu DE + |\lambda_N|\sqrt{\alpha-\alpha^2-D^2}\sqrt{\beta-\beta^2-E^2}
\end{aligned}$$

where $\mu = \max_{i>1:\lambda_i\neq\lambda_N} |\lambda_i|$. Note that the derangement graph Γ has $\mu \leq O((n-2)!)$. Hence, applying the above result to a cross-intersecting pair $\mathcal{A}, \mathcal{B} \subset S_n$ with $\sqrt{|\mathcal{A}||\mathcal{B}|} = (1-\delta)(n-1)!$, we obtain

$$\sqrt{1-\alpha-D^2/\alpha}\sqrt{1-\beta-E^2/\beta} \geq \frac{d_n\sqrt{\alpha\beta} - \mu(D/\sqrt{\alpha})(E/\sqrt{\beta})}{|\lambda_N|} \geq 1-\delta-O(1/n)$$

and therefore $1-\alpha-D^2/\alpha \geq (1-\delta)^2 - O(1/n)$, so $D^2 \leq \alpha(2\delta-\delta^2+O(1/n))$. Replacing δ with $2\delta-\delta^2+O(1/n)$ in the proof of Theorem 1.13, we see that there exist $i, j \in [n]$ such that

$$|\mathcal{A}_{i\mapsto j}| \geq \frac{(n-1)!}{4\sqrt{2(n-1)\log n}}(1-\sqrt{2\delta-\delta^2} - O(1/\sqrt{n})) = \omega((n-2)!)$$

since $\delta \leq 1-c$. For each $k \neq j$, the pair $\mathcal{A}_{i\mapsto j}, \mathcal{B}_{i\mapsto k}$ is cross-intersecting, so as in Lemma 1.15, we have:

$$|\mathcal{A}_{i\mapsto j}||\mathcal{B}_{i\mapsto k}| \leq ((n-2)!)^2.$$

Hence, for all $k \neq j$,

$$|\mathcal{B}_{i\mapsto k}| \leq o((n-2)!),$$

so summing over all $j \neq k$ gives

$$|\mathcal{B} \setminus \mathcal{B}_{i\mapsto j}| \leq o((n-1)!).$$

Since $|\mathcal{B}| \geq |\mathcal{A}|$, $|\mathcal{B}| \geq c(n-1)!$, and therefore

$$|\mathcal{B}_{i\mapsto j}| \geq (c-o(1))(n-1)!.$$

For each $k \neq j$, the pair $\mathcal{A}_{i\mapsto k}, \mathcal{B}_{i\mapsto j}$ is cross-intersecting, so as before, we have:

$$|\mathcal{A}_{i\mapsto k}||\mathcal{B}_{i\mapsto j}| \leq ((n-2)!)^2.$$

Hence, for all $k \neq j$,

$$|\mathcal{A}_{i\mapsto k}| \leq O((n-3)!),$$

so summing over all $j \neq k$ gives

$$|\mathcal{A} \setminus \mathcal{A}_{i\mapsto j}| \leq O((n-2)!).$$

Also, $|\mathcal{B}| = |\mathcal{B}_{i\mapsto j}|+|\mathcal{B}\setminus\mathcal{B}_{i\mapsto j}| \leq (1+o(1))(n-1)!$, so $|\mathcal{A}| \geq c^2(1-o(1))(n-1)!$. Hence,

$$|\mathcal{A}_{i\mapsto j}| \geq c^2(1-o(1))(n-1)!,$$

so by the same argument as above,

$$|\mathcal{B}_{i\mapsto k}| \leq O((n-3)!)$$

for all $k \neq j$, and therefore

$$|\mathcal{B} \setminus \mathcal{B}_{i\mapsto j}| \leq O((n-2)!)$$

as well, proving Theorem 1.18. □

We may use Theorem 1.18 to deduce two Hilton-Milner type results for cross-intersecting families:

Theorem 1.19. *For n sufficiently large, if $\mathcal{A}, \mathcal{B} \subset S_n$ is a cross-intersecting pair of families which are not both contained within the same 1-coset, then* $\min(|\mathcal{A}|, |\mathcal{B}|) \leq |\mathcal{C}| = (n-1)! - d_{n-1} - d_{n-2} + 1$. *Equality holds if and only if*

$$\begin{aligned} \mathcal{A} &= \{\sigma \in S_n : \sigma(i) = j,\ \sigma \textit{ intersects } \tau\} \cup \{\rho\}, \\ \mathcal{B} &= \{\sigma \in S_n : \sigma(i) = j, \sigma \textit{ intersects } \rho\} \cup \{\tau\} \end{aligned}$$

for some $i, j \in [n]$ and some $\tau, \rho \in S_n$ which intersect and do not map i to j.

Proof. Suppose $\min(|\mathcal{A}|, |\mathcal{B}|) \geq |\mathcal{C}|$. Applying Theorem 1.18 with any $c < 1 - 1/e$, we see that there exist $i, j \in [n]$ such that

$$|\mathcal{A} \setminus \mathcal{A}_{i\mapsto j}|, |\mathcal{B} \setminus \mathcal{B}_{i\mapsto j}| \leq O((n-2)!).$$

By double translation, we may assume that $i = j = 1$, so

$$|\mathcal{A} \setminus \mathcal{A}_{1\mapsto 1}|, |\mathcal{B} \setminus \mathcal{B}_{1\mapsto 1}| \leq O((n-2)!).$$

Assume $\mathcal{A}$ is not contained within the 1-coset $\{\sigma \in S_n : \sigma(1) = 1\}$; let ρ be a permutation in $\mathcal{A}$ not fixing 1. Suppose for a contradiction that $\mathcal{A}$ contains another permutation π not fixing 1. As in the proof of Theorem 1.17, this implies that

$$|\mathcal{B}_{1\mapsto 1}| \leq (n-1)! - d_{n-1} - 2d_{n-2},$$

and so by assumption,

$$|\mathcal{B} \setminus \mathcal{B}_{1\mapsto 1}| \geq d_{n-2} + 1,$$

so $\mathcal{B} \setminus \mathcal{B}_{1\mapsto 1}$ cannot be $(\log n)$-intersecting. As in the proof of Theorem 1.17, this implies that

$$|\mathcal{A}_{1\mapsto 1}| \leq ((1-1/e)^2 + o(1))(n-1)!,$$

giving

$$|\mathcal{A}| \leq ((1-1/e)^2 + o(1))(n-1)! < |\mathcal{C}|$$

—a contradiction. Hence,

$$\mathcal{A} = \mathcal{A}_{1\mapsto 1} \cup \{\rho\}.$$

If $\mathcal{B}$ were centred, then every permutation in $\mathcal{B}$ would have to fix 1 and intersect ρ, and we would have $|\mathcal{B}| = |\mathcal{B}_{1\mapsto 1}| \leq (n-1)! - d_{n-1} - d_{n-2} < |\mathcal{C}|$, a contradiction. Hence, $\mathcal{B}$ is also non-centred. Repeating the above argument with $\mathcal{B}$ in place of $\mathcal{A}$, we see that $\mathcal{B}$ contains just one permutation not fixing 1, τ say. Hence,

$$\mathcal{B} = \mathcal{B}_{1\mapsto 1} \cup \{\tau\}.$$

Since $\min(|\mathcal{A}|, |\mathcal{B}|) \geq |\mathcal{C}|$, we have

$$\begin{aligned} \mathcal{A}_{1\mapsto 1} &= \{\sigma \in S_n : \sigma(1) = 1,\ \sigma \text{ intersects } \tau\}, \\ \mathcal{B}_{1\mapsto 1} &= \{\sigma \in S_n : \sigma(1) = 1,\ \sigma \text{ intersects } \rho\}, \end{aligned}$$

proving the theorem. □

Similarly, we may prove

Theorem 1.20. *For n sufficiently large, if $\mathcal{A},\mathcal{B} \subset S_n$ is a cross-intersecting pair of families which are not both contained within the same 1-coset, then*

$$|\mathcal{A}||\mathcal{B}| \leq ((n-1)! - d_{n-1} - d_{n-2})((n-1)! + 1).$$

Equality holds if and only if

$$\mathcal{A} = \{\sigma \in S_n : \sigma(i) = j,\ \sigma \text{ intersects } \rho\}, \quad \mathcal{B} = \{\sigma \in S_n : \sigma(i) = j\} \cup \{\rho\}$$

for some $i, j \in [n]$ and some $\rho \in S_n$ with $\rho(i) \neq j$.

Proof. Let $\mathcal{A},\mathcal{B}$ be a cross-intersecting pair of families, not both centred, with $|\mathcal{A}||\mathcal{B}| \geq ((n-1)! - d_{n-1} - d_{n-2})((n-1)! + 1)$. We have

$$\sqrt{|\mathcal{A}||\mathcal{B}|} \geq (\sqrt{1 - 1/e} - O(1/n))(n-1)!$$

so applying Theorem 1.18 with any $c < \sqrt{1 - 1/e}$, we see that there exist $i, j \in [n]$ such that

$$|\mathcal{A} \setminus \mathcal{A}_{i \mapsto j}|, |\mathcal{B} \setminus \mathcal{B}_{i \mapsto j}| \leq O((n-2)!).$$

By double translation, we may assume that $i = j = 1$, so

$$|\mathcal{A} \setminus \mathcal{A}_{1 \mapsto 1}|, |\mathcal{B} \setminus \mathcal{B}_{1 \mapsto 1}| \leq O((n-2)!).$$

Therefore,

$$\sqrt{|\mathcal{A}_{1 \mapsto 1}||\mathcal{B}_{1 \mapsto 1}|} \geq (\sqrt{1 - 1/e} - O(1/n))(n-1)!. \tag{1.16}$$

If $\mathcal{B}$ contains some permutation ρ not fixing 1, then

$$\mathcal{A}_{1 \mapsto 1} \subset \{\sigma \in S_n : \sigma(1) = 1, \sigma \text{ intersects } \rho\}$$

and therefore

$$|\mathcal{A}_{1 \mapsto 1}| \leq (n-1)! - d_{n-1} - d_{n-2} = (1 - 1/e + o(1))(n-1)!.$$

Similarly, if $\mathcal{A}$ contains a permutation not fixing 1, then

$$|\mathcal{B}_{1 \mapsto 1}| \leq (1 - 1/e + o(1))(n-1)!.$$

By (1.16), both statements cannot hold (provided n is large), so we may assume that every permutation in $\mathcal{A}$ fixes 1, and that $\mathcal{B}$ contains some permutation ρ not fixing 1. Hence,

$$\mathcal{A} \subset \{\sigma \in S_n : \sigma(1) = 1, \sigma \text{ intersects } \rho\}$$

and

$$|\mathcal{A}| \leq (n-1)! - d_{n-1} - d_{n-2} = (1 - 1/e + o(1))(n-1)!. \tag{1.17}$$

So by assumption,

$$|\mathcal{B}| \geq (n-1)! + 1. \tag{1.18}$$

Suppose for a contradiction that $\mathcal{B}$ contains another permutation $\pi \neq \rho$ such that $\pi(1) \neq 1$. Then, by the same argument as in the proof of Theorem 1.17, we would have

$$|\mathcal{A}| = |\mathcal{A}_{1 \mapsto 1}| \leq (n-1)! - d_{n-1} - 2d_{n-2}$$

so by assumption,

$$|\mathcal{B}| \geq \frac{((n-1)! - d_{n-1} - d_{n-2})((n-1)! + 1)}{(n-1)! - d_{n-1} - 2d_{n-2}} = (n-1)! + \Omega((n-2)!).$$

This implies that $|\mathcal{B} \setminus \mathcal{B}_{1 \mapsto 1}| = \Omega((n-2)!)$, so $\mathcal{B} \setminus \mathcal{B}_{1 \mapsto 1}$ cannot be $(\log n)$-intersecting. Hence, by the same argument as in the proof of Theorem 1.17,

$$|\mathcal{A}_{1 \mapsto 1}| \leq ((1-1/e)^2 + o(1))(n-1)!.$$

Therefore,

$$\sqrt{|\mathcal{A}_{1 \mapsto 1}||\mathcal{B}_{1 \mapsto 1}|} \leq (1 - 1/e + o(1))(n-1)!$$

— contradicting (1.16). Hence, ρ is the only permutation in $\mathcal{B}$ not fixing 1, i.e.

$$\mathcal{B} = \mathcal{B}_{1 \mapsto 1} \cup \{\rho\}.$$

So we must have equality in (1.18), i.e.

$$\mathcal{B}_{1 \mapsto 1} = \{\sigma \in S_n : \ \sigma(1) = 1\}.$$

But then we must also have equality in (1.17), i.e.

$$\mathcal{A} = \{\sigma \in S_n : \ \sigma(1) = 1, \ \sigma \text{ intersects } \rho\},$$

proving the theorem. □

Chapter 2

A Proof of the Deza-Frankl Conjecture

2.1 Introduction

In Chapter 1, we dealt with intersecting families of permutations; in this chapter we consider a natural generalization. A family of permutations $\mathcal{A} \subset S_n$ is said to be *t-intersecting* if any two permutations in $\mathcal{A}$ agree in at least t places, i.e. for any $\sigma, \pi \in \mathcal{A}$, $|\{i \in [n] : \sigma(i) = \pi(i)\}| \geq t$. In 1977, Deza and Frankl [8] conjectured that provided n is sufficiently large depending on t, a t-intersecting family $\mathcal{A} \subset S_n$ has size at most $(n-t)!$. Note that equality holds if $\mathcal{A}$ is a coset of the stabilizer of t points, i.e. of the form $\{\sigma \in S_n : \sigma(i_k) = j_k \ \forall k \in [t]\}$ for some distinct $i_1, \ldots, i_t$ and distinct $j_1, \ldots, j_t$. We call such a family a 't-coset' for short.

We remark in passing that for some values of t and n, there is a Katona-type proof of the Deza-Frankl conjecture. A subset $H \subset S_n$ is said to be *t-transitive* if for any distinct $i_1, \ldots, i_t \in [n]$ and any distinct $j_1, \ldots, j_t \in [n]$, there exists $\sigma \in H : \sigma(i_k) = j_k$ $(1 \leq k \leq t)$; it is said to be *sharply t-transitive* if there exists a unique such σ. Note that a t-transitive subset $H \subset S_n$ is sharply t-transitive iff it has size $n(n-1)\ldots(n-t+1)$. Deza and Frankl pointed out that their conjecture holds if S_n has a sharply t-transitive subset H, as then any left translate σH of H is also sharply t-transitive, so any two distinct permutations in σH agree in at most $t-1$ places, and therefore $|\mathcal{A} \cap \sigma H| \leq 1$; averaging over all left translates gives $|\mathcal{A}| \leq (n-t)!$.

For $t = 2$ and $n = q$ a prime power, S_n has a sharply 2-transitive *subgroup* H: identify the ground set with the finite field $\mathbb{F}_q$ of order q, and take H to be the group of all affine maps $x \mapsto ax + b$ $(a \in \mathbb{F}_q \setminus \{0\},\ b \in \mathbb{F}_q)$. Any two distinct permutations in H agree in at most 1 point, so an intersecting family $\mathcal{A}$ contains at most 1 element of each left coset of H. Since the left cosets of H partition S_n, it follows that $|\mathcal{A}| \leq (n-2)!$.

For $t = 3$ and $n = q+1$ (where q is a prime power), S_n has a sharply 3-transitive subgroup: identify the ground set with $\mathbb{F}_q \cup \{\infty\}$ and take H to be the group of all Mobius transformations

$$x \mapsto \frac{ax+b}{cx+d} \quad (a, b, c, d \in \mathbb{F}_q,\ ad - bc \neq 0)$$

However, it is a classical result of C. Jordan that the only sharply t-transitive permutation groups for $t \geq 4$ are S_t (for $t \geq 4$), A_{t-2} (for $t \geq 8$), M_{11} (for $t = 4$) and M_{12} (for $t = 5$), where M_{11}, M_{12} are the Matthieu groups. Moreover, sharply t-transitive subsets of S_n have not been found for any other values of n and t. Thus, it seems unlikely that this approach can work in general.

In this chapter, we will use an eigenvalue argument together with representation theory of S_n to prove the conjecture. A very similar argument proves a t-cross-intersecting version of the conjecture. (We say that a pair of families $\mathcal{A}, \mathcal{B} \subset S_n$ are *t-cross-intersecting* if $|\{i \in [n] : \sigma(i) = \pi(i)\}| \geq t$ for any $\sigma \in \mathcal{A},\ \pi \in \mathcal{B}$.) We show that for $t \in \mathbb{N}$, and n sufficiently large depending on t, if $\mathcal{A}, \mathcal{B} \subset S_n$ are t-cross-intersecting, then $|\mathcal{A}||\mathcal{B}| \leq ((n-t)!)^2$.

We also obtain a Hilton-Milner type result for t-intersecting families, generalizing the Cameron-Ku conjecture (see Chapter 1). We prove that for n sufficiently large depending on t, if $\mathcal{A} \subset S_n$ is a t-intersecting family which is not contained within a t-coset, then $\mathcal{A}$ cannot be larger than the family

$$\begin{aligned} \mathcal{B} &= \{\sigma : \sigma(i) = i\ \forall i \leq t,\ \sigma(j) = j \text{ for some } j > t+1\} \\ &\cup \{(1\ t+1), (2\ t+1), \ldots, (t\ t+1)\} \end{aligned}$$

which has size $(1 - 1/e + o(1))(n-t)!$. The extremal families are precisely the double translates $\{\pi \mathcal{B} \tau : \ \pi, \tau \in S_n\}$ of this family. Note that it follows immediately that provided n is sufficiently large depending on t, the t-intersecting families in S_n of maximum size $(n-t)!$ are precisely the t-cosets.

To prove our Hilton-Milner type result, we perform a stability analysis on our proof of the Deza-Frankl conjecture, making use of our t-cross-intersecting result to show that a large t-intersecting family is close to being contained within a t-coset. Our method is based on the ideas in Chapter 1, but both the representation theory and the combinatorial arguments are significantly more involved.

The Deza-Frankl conjecture was proved independently by Ehud Friedgut and Haran Pilpel at about the same time using an equivalent argument. We have now submitted a joint paper [9] on the Deza-Frankl conjecture, and where indicated, the proof of the Deza-Frankl conjecture in this chapter is presented as in [9] rather than as in the author's original proof. The only fundamental difference between the author's approach and that of Friedgut and Pilpel was that they proved the equality case more directly. Interestingly, the direct proof of equality does not seem to work for A_n.

The plan of this chapter is as follows. In section 2.2, we give an overview of the proof of the Deza-Frankl conjecture. Section 2.3 contains the relevant background from the representation theory of S_n and some simple consequences. Section 2.4 contains the main combinatorial work of the proof. In section 2.5, we prove the analogous result for the alternating group A_n, namely that a t-intersecting family $\mathcal{A} \subset A_n$ has size at most $(n-t)!/2$.

In section 2.6, we make a digression and describe Friedgut and Pilpel's proof that the t-intersecting families in S_n of maximum size are precisely the

t-cosets of S_n. (Their argument uses the Strong Duality Theorem of Linear Programming; a statement and proof of this is included in the Appendix for the reader's convenience.) We outline brifely why the same method of proof does not work for A_n.

In section 2.7, we prove the t-cross-intersecting result mentioned above. In section 2.8, we perform a stability analysis and hence prove the Hilton-Milner type result. We note that this immediately gives the equality case in the Deza-Frankl conjecture, for n sufficiently large. In section 2.9, we apply the same techniques to obtain an analogous Hilton-Milner type result for t-intersecting families in A_n, and hence deduce that for n sufficiently large depending on t), a t-intersecting family in A_n of maximum size must be a coset of the stabilizer of t points.

2.2 An overview of the proof

Let Γ_t be the 't-derangement graph' on S_n, where we join two permutations iff they agree at less than t points. A t-intersecting family of permutations is simply an independent set of vertices in this graph. Observe that Γ_t is the Cayley graph on S_n generated by the set $\mathcal{D}_{n,t}$ of all permutations of $[n]$ with less than t fixed points, which is a union of conjugacy-classes of S_n, so Γ_t is a normal Cayley graph. Applying Theorem 1.5 to Γ_t, we see that its eigenvalues are given by:

$$\lambda_\alpha = \frac{1}{f^\alpha} \sum_{\sigma \in \mathcal{D}_{n,t}} \chi_\alpha(\sigma) \quad (\alpha \vdash n).$$

In order to deduce the Deza-Frankl conjecture by applying Hoffman's Theorem (Theorem 1.1) to the t-derangement graph, one would need its eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N$ to satisfy

$$|\lambda_N|/\lambda_1 = \mu_{n,t} := \frac{1}{(n)_t - 1} = \frac{1}{n(n-1)\ldots(n-t+1)-1}. \tag{2.1}$$

However, this fails for the t-derangement graph when $t > 1$. Instead, we will appeal to the following 'weighted' version of Hoffman's Theorem:

Theorem 2.1. *Let M be a real, symmetric, $N \times N$ matrix with real eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N$ (where $\lambda_1 > 0$), such that the all-1's vector $\mathbf{f}$ is an eigenvector of M with eigenvalue λ_1, i.e. all row and column sums of M equal λ_1. Let $X \subset [N]$ such that $M_{x,y} = 0$ for any $x, y \in X$. Let U be the* *direct sum of the subspace of constant vectors and the λ_N-eigenspace. Then*

$$|X| \leq \frac{|\lambda_N|}{\lambda_1 + |\lambda_N|} N.$$

Equality holds only if the characteristic vector v_X lies in the subspace U.

(The proof is an easy extension of that of Hoffman's Theorem, and we omit it.)

To make use of this, we will construct a real, symmetric matrix $(M_{\sigma,\pi})_{\sigma,\pi \in S_n}$ such that $M_{\sigma,\pi} = 0$ whenever $|\{i \in [n] : \sigma(i) = \pi(i)\}| \geq t$, with eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N$ which do satisfy (2.1) for n sufficiently large depending on t. The subspace U, the direct sum of the subspace of constant vectors and the λ_N-eigenspace, will necessarily be the subspace spanned by the characteristic vectors of the t-cosets of S_n. The Deza-Frankl conjecture will follow immediately.

To make the eigenvalues easy to analyse, we will choose M to be of the form $M_{\sigma,\pi} = w(\sigma\pi^{-1})$, for w a suitable real-valued class function (i.e. constant on the conjugacy classes of S_n). Equivalently, identifying w with the corresponding element of the group module $\mathbb{C}S_n$,

$$w = \sum_{\sigma \in S_n} w(\sigma)\sigma,$$

our M will be the matrix of the linear map

$$x \mapsto wx \tag{2.2}$$

on $\mathbb{C}S_n$, with respect to the standard basis S_n. Observe that the class functions correspond to the centre $Z(\mathbb{C}S_n)$ of the group module, so $w \in Z(\mathbb{C}S_n)$. Since w is inverse- invariant, such an M is symmetric:

$$M_{\sigma,\pi} = w(\sigma\pi^{-1}) = w(\pi\sigma^{-1}) = M_{\pi,\sigma}.$$

We will need the following trivial extension of Theorem 1.5:

Theorem 2.2. *Let G be a finite group; let w be an inverse-invariant, real-valued class-function on G, and let M be the matrix defined by $M_{g,h} = w(gh^{-1})$. Let $(\rho_1, V_1), \ldots, (\rho_k, V_k)$ be a complete set of pairwise non-isomorphic complex irreducible representations of G. Let U_i be the sum of all submodules of the group module $\mathbb{C}G$ which are isomorphic to (ρ_i, V_i). Then we have the decomposition*

$$\mathbb{C}G = \bigoplus_{i=1}^{k} U_i;$$

each U_i is an eigenspace of M with dimension $\dim(V_i)^2$ *and eigenvalue*

$$\lambda_{\rho_i} = \frac{1}{\dim(V_i)} \sum_{g \in G} w(g)\chi_{\rho_i}(g).$$

Proof. Observe that M is a real linear combination of the adjacency matrices of normal Cayley graphs, and apply Theorem 1.5. □

To guarantee that $M_{\sigma,\pi} = 0$ whenever $|\{i \in [n] : \sigma(i) = \pi(i)\}| \geq t$, we must choose w to be zero on all permutations with at least t fixed points, i.e. supported on the conjugacy classes of permutations with less than t fixed points. We will use the representation theory of S_n to show that there exists a class function w such that (2.2) has eigenvalues satisfying (2.1).

2.3 Preliminary results on S_n

In addition to the background in Chapter 1, we some more preliminary results on S_n.

The following two orders on the partitions of n will be useful.

Definition 2.3. *(Dominance order) Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ and $\mu = (\mu_1, \ldots, \mu_s)$ be partitions of n. We say that $\lambda \trianglerighteq \mu$ (λ dominates μ) if $\sum_{j=1}^{i} \lambda_i \geq \sum_{j=1}^{i} \mu_i$ $\forall i$ (where we define $\lambda_i = 0$ $\forall i > r$, $\mu_j = 0$ $\forall j > s$).*

It is easy to see that this is a partial order.

Definition 2.4. *(Lexicographic order) Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ and $\mu = (\mu_1, \ldots, \mu_s)$ be partitions of n. We say that $\lambda > \mu$ if $\lambda_j > \mu_j$, where $j = \min\{i \in [n] : \lambda_i \neq \mu_i\}$.*

It is easy to see that this is a total order which extends the dominance order.

Recall Young's Rule from Chapter 1: for any partition μ of n, the permutation module M^μ decomposes into irreducibles as follows:

$$M^\mu \cong \oplus_{\lambda \vdash n} K_{\lambda,\mu} S^\lambda,$$

where the Kostka number $K_{\lambda,\mu}$ is the number of semistandard generalized λ-tableaux with content μ. We now examine the Kostka matrix $(K_{\lambda,\mu})_{\lambda,\mu \vdash n}$ more closely.

Clearly, if $K_{\lambda,\mu} \neq 0$ then $\lambda \trianglerighteq \mu$ (if s is a semistandard generalized λ-tableau of content μ, then all μ_j j's must appear in the first j rows of t, so $\sum_{j=0}^{i} \mu_j \leq \sum_{j=0}^{i} \lambda_j \ \forall i$). Also, $K_{\lambda,\lambda} = 1 \ \forall \lambda \vdash n$, since the generalized λ-tableau with λ_i i's in the ith row is the unique semistandard generalized λ-tableau of content λ. Order the partitions of n under the lexicographic order on partitions, greatest first. Since the lexicographic order extends the dominance order, K is upper-triangular, and it has 1's all along the diagonal. Hence, K is invertible, so any irreducible character χ_α can be expressed as a linear combination of permutation characters ξ_β. For any finite group G, the irreducible characters of G form a basis for the space of complex-valued class functions on G. The characters of S_n are real-valued, so the irreducible characters form a basis for the space of real-valued class functions. Hence, by the above argument, so do the permutation characters. This will be useful to bear in mind in section 2.4.

Let

$$\mathcal{F}_{n,t} = \{\alpha \vdash n : \alpha \geq (n-t, 1^t)\} = \{\alpha \vdash n : \alpha_1 \geq n-t\}$$

be the collection of all partitions of n whose Young diagram has first row of length at least $n-t$; we call these partitions the 'fat' partitions of n. Observe that for $n \geq 2t$,

$$|\mathcal{F}_{n,t}| = \sum_{s=0}^{t} p(s),$$

where $p(s)$ denotes the number of partitions of s. This grows very rapidly with t, but (as will be crucial for our proof) it is independent of n for $n \geq 2t$.

Let U_t denote the subspace of $\mathbb{C}S_n$ spanned by the characteristic vectors of the t-cosets of S_n. The following observation demonstrates the importance of the fat partitions:

Lemma 2.5.

$$U_t = \bigoplus_{\alpha \in \mathcal{F}_{n,t}} U_\alpha.$$

Proof. Recall that for each partition $\alpha \vdash n$, U_α is a 2-sided ideal of $\mathbb{C}S_n$. Hence, so is

$$V := \bigoplus_{\alpha \in \mathcal{F}_{n,t}} U_\alpha.$$

Now notice that the set of vectors

$$\left\{ \sum \{\sigma : \sigma(1) = j_1, \ldots, \sigma(t) = j_t\} : \ j_1, \ldots, j_t \text{ are distinct} \right\}$$

is a basis for a copy W of the permutation module $M^{(n-t,1^t)}$ in the group module $\mathbb{C}S_n$; U_t is precisely the subspace spanned by all right translates of

W. Observe that $\{S^\alpha : \alpha \in \mathcal{F}_{n,t}\}$ are precisely the irreducible constituents of $M^{(n-t,1^t)}$, since $K_{\alpha,(n-t,1^t)} \geq 1$ iff there exists a semistandard generalized α-tableau of content $(n-t, 1^t)$, i.e. iff $\alpha_1 \geq n-t$. Hence, $W \leq V$. Since V is a 2-sided ideal of $\mathbb{C}S_n$, $U_t \leq V$. Moreover, W contains a copy of S^α for each $\alpha \in \mathcal{F}_{n,t}$. Recall that if G is any finite group, and T, T' are two isomorphic submodules of $\mathbb{C}G$, then there exists $s \in \mathbb{C}G$ such that the right multiplication map $x \mapsto xs$ is an isomorphism from T to T'. Hence, the sum of all right translates of W contains all submodules of $\mathbb{C}S_n$ isomorphic to S^α, for each $\alpha \in \mathcal{F}_{n,t}$, i.e. $U_\alpha \leq U_t$ $\forall \alpha \in \mathcal{F}_{n,t}$. Hence, $V \leq U_t$, so $V = U_t$, as required. □

We will need the following well-known result:

Lemma 2.6. *Let μ be a partition of n, and let ξ_μ be the character of the permutation module M^μ. Let $\sigma \in S_n$. If $\xi_\mu(\sigma) \neq 0$, then cycle-type$(\sigma) \trianglelefteq \mu$.*

Proof. The set of μ-tabloids is a basis for the permutation module M^μ. Thus, $\xi_\mu(\sigma)$, which is the trace of the corresponding representation on the permutation σ, is simply the number of μ-tabloids fixed by σ. If $\xi_\mu(\sigma) \neq 0$, then σ fixes some μ-tabloid T. Hence, every row of length l in T is a union of the sets of numbers in a collection of disjoint cycles of total length l in σ. Thus, the cycle-type of σ is a refinement of μ, so clearly, cycle-type$(\sigma) \trianglelefteq \rho$, as required. □

This is needed for the following:

Lemma 2.7. *Let A be the permutation-character table of S_n, with rows and columns indexed by partitions / conjugacy classes in the lexicographic order, greatest first (so $A_{\lambda,\mu} = \xi_\lambda(\sigma_\mu)$ where σ_μ is a permutation with cycle-type μ, and the top-left corner of A is $\xi_{(n)}(\sigma_{(n)})$.) Then the top-left minor A_t with rows and columns indexed by the partitions in $\mathcal{F}_{n,t}$ is upper-triangular with positive entries on the diagonal, and does not depend on n provided $n > 2t$.*

The proof of this is straightforward; we follow [9]:

Proof. By Lemma 2.6, and the fact that the lexicographic order extends the dominance order, A_t is upper-triangular. It is easy to see that a permutation σ with cycle-type λ fixes precisely those λ-tabloids with each row being a cycle of σ, so A_t has positive entries along the diagonal.

To see that A_t is independent of n for $n > 2t$, let $\mu = (n-s, \mu_2, \ldots, \mu_m)$, $\lambda = (n-r, \lambda_2, \ldots, \lambda_l) \geq (n-t, 1^t)$ be partitions of n, so $r, s \leq t$, and $(\mu_2, \ldots, \mu_m) \vdash s$, $(\lambda_2, \ldots, \lambda_l) \vdash r$. Let σ be a permutation with cycle-type

μ; note that as $n > 2t$, we have $n - s > r$, so if σ fixes the λ-tabloid T, then $r < s$, and the $n - s$ numbers in the longest cycle of σ must all appear in the longest row of T (which has length $n - r$). Delete the longest cycle of σ to form a permutation $\tilde{\sigma}$ with cycle-type $\tilde{\mu} = (\mu_2, \ldots, \mu_m)$. We see that σ fixes a λ-tabloid T iff $r < s$, the $n - s$ numbers in the longest cycle of σ all appear in the longest row of T, and the $(s - r, \lambda_2, \ldots, \lambda_l)$-'tabloid' $\tilde{T}$ produced by deleting these numbers from the first row of T is fixed by $\tilde{\sigma}$. (Note that $\tilde{T}$ may not be a genuine tabloid, as $s - r$ may not be the largest part.) Hence, $\xi_\lambda(\sigma_\mu)$ depends only on $r, s, \lambda_2, \ldots, \lambda_l, \mu_2, \ldots, \mu_m$, and not on n, provided $n > 2t$. Moreover, the lexicographic ordering on partitions of the form $(n - r, \lambda_2, \ldots, \lambda_l)$ where $r \leq t$ depends only on $r, \lambda_2, \ldots, \lambda_l$ and not on n, provided $n > 2t$. This proves the lemma. □

We need one more preliminary result. Write $D_{n,t}$ for the number of permutations in S_n with no cycle of length $\leq t$, and $E_{n,t}, O_{n,t}$ for the number of these permutations which are even/odd respectively.

Lemma 2.8. *Let $t \in \mathbb{N}$ be fixed; then there exists a positive constant $C = C_t$ such that $E_{n,t}, O_{n,t} \geq C_t n!$ $\forall n \geq 2t + 2$.*

Proof. First suppose $2t + 2 \leq n \leq 3t + 2$; if n is odd, then the even permutations with no cycles of length $\leq t$ are precisely the n-cycles, and the odd ones are precisely the permutations with exactly two cycles, both of length $\geq t + 1$; if n is odd, the situation is reversed. Choose $C_t > 0$ such that $E_{n,t}, O_{n,t} \geq C_t n!$ whenever $2t + 2 \leq n \leq 3t + 2$; it is easy to check that we can take $C_t = \frac{2}{(3t+2)^2}$.

We now derive recurrence relations for $E_{n,t}, O_{n,t}$. Let σ be an even permutation with no cycle of length $\leq t$. Let $i = \sigma(n)$; then we may write $\sigma = (ni)\rho$ where ρ is an odd permutation of $[n-1]$ and has no cycle of length $\leq t$ except possibly a t-cycle containing i. Conversely, given any such pair ρ, i, $(ni)\rho$ has no cycle of length $\leq t$. Hence, we have

$$E_{n,t} = (n-1)(O_{n-1,t} + (n-2)(n-3)\ldots(n-t)E_{n-t-1,t}) \quad \text{if } t \text{ is even;}$$
$$E_{n,t} = (n-1)(O_{n-1,t} + (n-2)(n-3)\ldots(n-t)O_{n-t-1,t}) \quad \text{if } t \text{ is odd.}$$

Similarly,

$$O_{n,t} = (n-1)(E_{n-1,t} + (n-2)(n-3)\ldots(n-t)O_{n-t-1,t}) \quad \text{if } t \text{ is even;}$$
$$O_{n,t} = (n-1)(E_{n-1,t} + (n-2)(n-3)\ldots(n-t)E_{n-t-1,t}) \quad \text{if } t \text{ is odd.}$$

We can now prove the lemma by induction on n. Let $n \geq 3t+3$ and assume the statement is true for smaller values of n; the recurrence relations above give

$$E_{n,t}, O_{n,t} \geq (n-1)(C_t(n-1)! + (n-2)(n-3)\dots(n-t)C_t(n-t-1)!) = C_t n!$$

as required. □

2.4 Construction of the class function

We now proceed to construct a real-valued class function w, supported on conjugacy-classes of permutations with less than t fixed points, such that the linear operator

$$\begin{aligned} M : \mathbb{C}S_n &\rightarrow \mathbb{C}S_n \\ x &\mapsto wx \end{aligned}$$

has eigenvalues satisfying (2.1). Since rescaling makes no difference, we will demand that $\lambda_1 = 1$ and $\lambda_N = -\mu_{n,t}$. By Theorem 2.2, the eigenvalues of such an M are given by

$$\lambda_\alpha^{(w)} = \frac{1}{f^\alpha} \sum_{\sigma \in S_n} w(\sigma)\chi_\alpha(\sigma) = \frac{n!}{f^\alpha} \langle w, \chi_\alpha \rangle \quad (\alpha \vdash n),$$

where $\langle , \rangle$ denotes the standard inner product on $\mathbb{C}S_n$:

$$\langle \phi, \psi \rangle = \frac{1}{n!} \sum_{\sigma \in S_n} \overline{\phi(\sigma)}\psi(\sigma).$$

The λ-eigenspace is

$$\bigoplus_{\alpha \vdash n : \lambda_\alpha^{(w)} = \lambda} U_\alpha,$$

where U_α denotes the sum of all copies of S^α in $\mathbb{C}S_n$, and has dimension

$$n_\lambda = \sum_{\alpha \vdash n : \lambda_\alpha^{(w)} = \lambda} (f^\alpha)^2.$$

We want the 1-eigenspace to be $U_{(n)}$, the subspace of constant vectors in $\mathbb{C}S_n$, and we want the direct sum of 1- and $-\mu_{n,t}$ eigenspaces to be the subspace U_t spanned by the characteristic vectors of the t-cosets of S_n. Hence, by Lemma 2.5, we need the $-\mu_{n,t}$-eigenspace of M to be

$$\bigoplus_{(n) > \alpha \geq (n-t, 1^t)} U_\alpha,$$

i.e. we must have $\lambda_{\alpha}^{(w)} = -\mu_{n,t}$ for all partitions $\alpha \vdash n$ such that $(n) > \alpha \geq (n-t, 1^t)$.

Let

$$\mathcal{F}'_{n,t} = \{\beta' : \beta \in \mathcal{F}_{n,t}\}$$

be the set of transposes of fat partitions of n, i.e. the set of all partitions whose Young diagram has first column of depth at least $n-t$; we call these the 'tall' partitions of n. Partitions of n which are neither fat nor tall will be called 'medium'. We will ensure that $\lambda_{\alpha}^{(w)} = 0\ \forall \alpha \in \mathcal{F}'_{n,t}$. We will see that if n is sufficiently large depending on t, then all medium partitions $\alpha \notin \mathcal{F}_{n,t} \cup \mathcal{F}'_{n,t}$ have

$$f^{\alpha} \geq c_t n^{-(t+1)}$$

for some constant c_t depending only on t, i.e. their Specht modules have high dimension. Provided w is suitably bounded, this will imply that $|\lambda_{\alpha}^{(w)}| \leq O(n^{-(t+1)})$ — i.e. the corresponding eigenvalues are small. (For $t=1$, Lemma 1.8 states that all medium partitions α of n satisfy $f^{\alpha} \geq \binom{n-1}{2} - 1$; we were able to use this to bound the corresponding eigenvalues of the derangement graph. We use an analogous argument here.)

More precisely, we will construct a real-valued class function w supported on conjugacy-classes of permutations with less than t fixed points, satisfying:

1. $|w_{\sigma}| \leq K_t/n!\ \forall \sigma \in S_n$, where K_t is some constant depending only on t;
2. $\lambda_{(n)}^{(w)} = \sum_{\sigma \in S_n} w_{\sigma} = 1$;
3. $\lambda_{\alpha}^{(w)} = \frac{n!}{f^{\alpha}} \langle w, \chi_{\alpha} \rangle = -\mu_{n,t} \quad \forall \alpha \in \mathcal{F}_{n,t} \setminus \{(n)\}$;
4. $\lambda_{\alpha}^{(w)} = \frac{n!}{f^{\alpha}} \langle w, \chi_{\alpha} \rangle = 0 \quad \forall \alpha \in \mathcal{F}'_{n,t}$.

(†)

Since $[\alpha'] \cong S \otimes [\alpha]\ \forall \alpha \vdash n$, we have $\chi_{[\alpha']} = \epsilon \cdot \chi_{[\alpha]}$, so condition 4 becomes:

$$\langle w, \epsilon \cdot \chi_{\alpha} \rangle = 0\ \forall \alpha \in \mathcal{F}_{n,t}.$$

In fact, similarly to in [9], we will construct two real-valued class functions w^+ and w^-, supported on conjugacy-classes of respectively even/odd permutations with less than t fixed points, such that for $u = w^+, w^-$,

1. $|u_{\sigma}| \leq K_t/n!\ \forall \sigma \in S_n$;

2. $\lambda_{(n)}^{(u)} = \sum_{\sigma \in S_n} u_\sigma = 1$;

3. $\lambda_\alpha^{(u)} = \frac{n!}{f^\alpha}\langle u, \chi_\alpha \rangle = -\mu_{n,t} \quad \forall \alpha \in \mathcal{F}_{n,t} \setminus \{(n)\}$.

$$(*)$$

We will then have

$$\langle w^+, \epsilon \cdot \chi_\alpha \rangle = \langle w^+, \chi_\alpha \rangle = -\mu_{n,t} f^\alpha / n! \quad \forall \alpha \in \mathcal{F}_{n,t};$$
$$\langle w^-, \epsilon \cdot \chi_\alpha \rangle = -\langle w^-, \chi_\alpha \rangle = \mu_{n,t} f^\alpha / n! \quad \forall \alpha \in \mathcal{F}_{n,t}.$$

Taking $w = \frac{1}{2}(w^+ + w^-)$ will satisfy the conditions (†).

We now show that condition 1 implies that all the eigenvalues corresponding to medium partitions of n are small. Using the Hook Formula (1.5) and the Branching Rule (1.6), we obtain the following analogue of Lemma 1.8:

Lemma 2.9. *Let $t \in \mathbb{N}$ be fixed; then there exist constants $n_0(t) \in \mathbb{N}$ and $c_t > 0$ such that if $n \geq n_0(t)$, all partitions $\alpha \vdash n$ whose Specht module has dimension $f^\alpha < c_t n^{t+1}$ either have $\alpha_1 \geq n - t$ (i.e. $\alpha \in \mathcal{F}_{n,t}$) or $\alpha'_1 \geq n - t$ (i.e. $\alpha \in \mathcal{F}'_{n,t}$).*

Proof. First, choose $n_0(t) > 2n - 4$ such that for $n \geq n_0(t)$, we have $2(n-2)^{t+1} \geq n^{t+1}$ and $(n-t-1)(n-t-2)\ldots(n-2t-1) \geq \frac{1}{2}n^{t+1}$. Now choose $c_t \leq \frac{1}{2(t+1)!}$ sufficiently small that the statement of the lemma holds for $n = n_0(t), n_0(t)+1$.

We proceed by induction on n. Assume the statement holds for $n-2, n-1$; we will prove it for n. Let $\alpha \vdash n$ with $f^\alpha < c_t n^{t+1}$. Consider the restriction $[\alpha] \downarrow S_{n-1}$, which has dimension f^α.

First suppose that $[\alpha] \downarrow S_{n-1}$ is reducible. Suppose it has $[\beta]$ as a constituent for some $\beta \in \mathcal{F}_{n-1,t} \cup \mathcal{F}'_{n-1,t}$. If $\beta_1 \geq n - t$, then $\alpha_1 \geq n - t$, so $\alpha \in \mathcal{F}_t$; if $\beta'_1 \geq n - t$, then $\alpha'_1 \geq n - t$, so $\alpha \in \mathcal{F}'_{n,t}$.

Suppose then that $\beta_1 = n - t - 1$ and $\alpha_1 = n - t - 1$; we will bound f^α from below using the Hook Formula. Notice that for $j \geq t+2$, $\alpha'_j \leq 1$, so the hook lengths of $[\alpha]$ satisfy $h^\alpha_{1,j} = n - t - j$; for $1 \leq j \leq t+1$ we trivially have $h^\alpha_{1,j} \leq n + 1 - j$. Also, since there are just $t+1$ spaces below the first row of $[\alpha]$, $\prod_{i \geq 2, j \geq 1} h^\alpha_{i,j} \leq (t+1)!$. Hence, the product of the hook lengths satisfies:

$$\prod_{i,j} h^\alpha_{i,j} \leq n(n-1)\ldots(n-t)(n-2t-2)!(t+1)!$$

and therefore

$$f^\alpha \geq \frac{(n-t-1)(n-t-2)\dots(n-2t-1)}{(t+1)!} \geq \frac{1}{2(t+1)!}n^{t+1}$$

for $n \geq n_0(t)$. By symmetry, the same conclusion holds if $\beta'_1 = n-t-1$ and $\alpha'_1 = n-t-1$.

Hence, we may assume that the irreducible constituents of $[\alpha] \downarrow S_{n-1}$ do not include any $[\beta]$ such that $\beta \in \mathcal{F}_{n-1,t} \cup \mathcal{F}'_{n-1,t}$, so by the induction hypothesis for $n-1$, each has dimension $\geq c_t(n-1)^{t+1}$. But $2c_t(n-1)^{t+1} \geq c_t n^{t+1}$ for $n \geq n_0(t)$, hence there is just one, i.e. $[\alpha] \downarrow S_{n-1}$ is irreducible. Therefore $[\alpha] = [a^b]$ for some $a, b \in \mathbb{N} : ab = n$, i.e. it has rectangular Young diagram. Since $b \geq 2$, $a \leq n/2 < n-2-t$ provided $n > 2t+4$; similarly, $b < n-2-t$.

Now consider the restriction

$$[\alpha] \downarrow S_{n-2} = [a^{b-1}, a-2] + [a^{b-2}, a-1, a-1].$$

Note that both of these irreducible constituents have Young diagram with first row of length $\leq a < n-2-t$ and first column of length $\leq b < n-2-t$, and therefore by the induction hypothesis for $n-2$, they have dimension $\geq c_t(n-2)^{t+1}$. But $2c_t(n-2)^{t+1} \geq c_t n^{t+1}$ for $n \geq n_0(t)$, contradicting $\dim([\alpha] \downarrow S_{n-2}) < c_t n^{t+1}$. □

We now appeal to the following easy generalization of Lemma 1.9:

Lemma 2.10. *Let M be a real, symmetric $N \times N$ matrix with eigenvalues $\lambda_1, \dots, \lambda_N$. Then*

$$\sum_{i=1}^{N} \lambda_i^2 = \sum_{x,y \in [N]} M_{x,y}^2.$$

Proof. Diagonalize M: there exists a real invertible matrix P such that $A = P^{-1}DP$, where D is the diagonal matrix

$$D = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \dots & \lambda_N \end{pmatrix}.$$

We have $M^2 = P^{-1}D^2P$, and therefore

$$\sum_{x,y \in [N]} M_{x,y}^2 = \mathrm{Tr}(M^2) = \mathrm{Tr}(P^{-1}D^2P) = \mathrm{Tr}(D^2) = \sum_{i=1}^{N} \lambda_i^2$$

as required. $\square$

Hence, in our case, we get

$$\sum_{\lambda} n_\lambda \lambda^2 = n! \sum_{\sigma \in S_n} w_\sigma^2 \leq (n!)^2 (K_t/n!)^2 = K_t^2$$

so for every medium partition α, the corresponding eigenvalue is small:

$$|\lambda| \leq \frac{K_t}{c_t n^{t+1}} = O(\mu_{n,t}/n) \tag{2.3}$$

Hence, the conditions $(*)$ will guarantee that our operator M satisies (2.1), provided n is sufficiently large depending on t.

It remains to construct u for $u = w^+, w^-$. First, we rephrase the conditions $(*)$ on u in terms of its inner-products with the permutation characters $\{\xi_\beta : \ \beta \in \mathcal{F}_{n,t}\}$. Notice that since the Kostka matrix K is upper triangular with 1's on the diagonal, so is the top-left minor $K^{(t)}$ with rows and columns indexed by the partitions in $\mathcal{F}_{n,t}$. So $K^{(t)}$ is invertible, and the inner products of a class function with $\{\xi_\beta : \ \beta \in \mathcal{F}_{n,t}\}$ determine those with $\{\chi_\alpha : \ \alpha \in \mathcal{F}_{n,t}\}$. So conditions 2 and 3 of $(*)$ are equivalent to:

$$\begin{aligned}
\langle u, \xi_\beta \rangle &= \frac{1}{n!}\left(1 - \mu_{n,t} \sum_{(n) > \alpha \geq (n-t,1^t)} K_{\alpha,\beta} f^\alpha\right) \\
&= \frac{1}{n!}\left(1 - \mu_{n,t} \sum_{(n) > \alpha \geq (n-t,1^t)} K_{\alpha,\beta} \chi_\alpha(1)\right) \\
&= \frac{1}{n!}(1 - \mu_{n,t}(\xi_\beta(1) - K_{(n),\beta}\chi_{(n)}(1))) \\
&= \frac{1}{n!}(1 - \mu_{n,t}(\xi_\beta(1) - 1)) \\
&= \frac{1}{n!}\frac{(n)_t - \xi_\beta(1)}{(n)_t - 1}
\end{aligned}$$

for each $\beta \in \mathcal{F}_{n,t}$. (Here, we have used the fact that for any partition $\beta \vdash n$, the multiplicity $K_{(n),\beta}$ of the trivial representation in M^β is 1, since there is just one semistandard generalized (n)-tableau of content β.) Hence, we require that

$$\sum_{\sigma \in S_n} u(\sigma)\xi_\beta(\sigma) = \frac{(n)_t - \xi_\beta(1)}{(n)_t - 1} =: \eta_{n,t,\beta} \quad \forall \beta \in \mathcal{F}_{n,t}. \tag{2.4}$$

But $\xi_\beta(\sigma)$ is simply the number of β-tabloids fixed by σ; $\xi_\beta(1)$ is the total number of β-tabloids. Noting that $(n-t, 1^t)$-tabloids correspond to ordered t-tuples of distinct numbers between 1 and n, the total number $\chi_{(n-t,1^t)}(1)$ of $(n-t, 1^t)$-tabloids is simply $(n)_t$; since we are choosing our class functions to be supported on permutations with less than t fixed points, and no such permutation can fix any $(n-t, 1^t)$-tabloid, both sides of (2.4) are automatically zero for $\beta = (n-t, 1^t)$. Any $\beta \in \mathcal{F}_{n,t}$ has first row of length at least $n-t$, so the total number of β-tabloids is at most $(n)_t$, and $\eta_{n,t,\beta}$ is between 0 and 1 for all $\beta \in \mathcal{F}_{n,t}$.

Let B_t be the minor of the permutation-character table with rows and columns indexed by partitions $> (n-t, 1^t)$. From Theorem 2.7, we know that B_t is upper-triangular, has positive entries along the diagonal, and is independent of n for $n > 2t$, so is invertible. For each $\alpha > (n-t, 1^t)$, we will now show that the collection of even / odd conjugacy classes produced by splitting the largest part of α into parts of size $> t$ has union of size $\Theta_t(n!)$, and that ξ_β takes the same value $\xi_\beta(\sigma_\alpha)$ on all of them, for each $\beta > (n-t, 1^t)$. This will allow us to satisfy the required conditions.

Given a partition $\alpha > (n-t, 1^t)$, say $\alpha = (n-s, \alpha_2, \ldots, \alpha_l)$, where $0 \leq s \leq t$ and $\alpha_l \geq 1$, let $\mathcal{P}_{n,t}(\alpha)$ be the collection of partitions obtained by subdividing the part of size $n-s$ into parts of size $> t$. Let $\mathcal{S}_{n,t}(\alpha)$ be the family of permutations with cycle-type in $\mathcal{P}_{n,t}(\alpha)$; let $\mathcal{S}^+_{n,t}(\alpha), \mathcal{S}^-_{n,t}(\alpha)$ be the families of even / odd permutations in $\mathcal{S}_{n,t}(\alpha)$. Notice that the collections $\{\mathcal{P}_{n,t}(\alpha) : \alpha > (n-t, 1^t)\}$ are disjoint, and that $\mathcal{S}^+_{n,t}(\alpha), \mathcal{S}^-_{n,t}(\alpha)$ are unions of conjugacy classes of permutations with less than t fixed points.

Using Lemma 2.8, we have:

$$\begin{aligned}
\min(|\mathcal{S}^+_{n,t}(\alpha)|, |\mathcal{S}^-_{n,t}(\alpha)|) &= \frac{1}{\alpha_2 \ldots \alpha_l \prod_j a_j!} \frac{n!}{(n-s)!} \min(E_{n-s,t}, O_{n-s,t}) \\
&\geq \frac{C_t n!}{\alpha_2 \ldots \alpha_l \prod_j a_j!},
\end{aligned}$$

since to choose a permutation in $\mathcal{S}^+_{n,t}(\alpha)/\mathcal{S}^-_{n,t}(\alpha)$, we can first choose the $\alpha_2, \ldots, \alpha_l$-cycles, and then choose an even/odd permutation of the other $n-s$ numbers which has no cycle of length $\leq t$.

Take any permutation $\sigma \in \mathcal{S}_{n,t}(\alpha)$, where $\alpha = (n-s, \alpha_2, \ldots, \alpha_l)$ with $0 \leq s \leq t$, and let σ' be the permutation with cycle-type α produced from σ by merging all cycles of length $> t$. Take any partition $\beta = (n-r, \beta_2, \ldots, \beta_m)$ with $0 \leq r \leq t$. We wish to evaluate $\xi_\beta(\sigma)$, the number of β-tabloids fixed by σ. Note that σ and σ' fix the same β-tabloids, since if T is a β-tabloid fixed by σ, then all numbers in $(> t)$-cycles of σ must occur in the first row of T, so T must be fixed by σ' as well. Hence,

$$\xi_\beta(\sigma) = \xi_\beta(\sigma') = \xi_\beta(\alpha)$$

—the (β, α)-entry of the permutation-character table of S_n.

We now define the class functions w^+, w^-. We will let w^+ have value x_γ^+ on $\mathcal{S}_{n,t}^+(\gamma)$ for each partition $\gamma > (n-t, 1^t)$, and zero elsewhere; to satisfy condition (2.4), we need:

$$\sum_{\gamma > (n-t,1^t)} x_\gamma^+ |\mathcal{S}_{n,t}^+(\gamma)| (B_t)_{\beta,\gamma} = \eta_\beta \quad \forall \beta > (n-t, 1^t).$$

This system of linear equations has unique solution given by

$$x_\gamma^+ = \sum_{\beta > (n-t,1^t)} \eta_\beta (B_t^{-1})_{\gamma,\beta} / |\mathcal{S}_{n,t}^+(\gamma)| \quad \forall \gamma > (n-t, 1^t).$$

Similarly, we will let w^- have value x_γ^- on $\mathcal{S}_{n,t}^-(\gamma)$ for each partition $\gamma > (n-t, 1^t)$, and zero elsewhere; to satisfy (2.4), we need

$$\sum_{\gamma > (n-t,1^t)} x_\gamma^- |\mathcal{S}_{n,t}^-(\gamma)| (B_t)_{\beta,\gamma} = \eta_\beta \quad \forall \beta > (n-t, 1^t).$$

This system of linear equations has unique solution given by

$$x_\gamma^- = \sum_{\beta > (n-t,1^t)} \eta_\beta (B_t^{-1})_{\gamma,\beta} / |\mathcal{S}_{n,t}^-(\gamma)| \quad \forall \gamma > (n-t, 1^t).$$

Observe that w^+, w^- are class functions supported on conjugacy classes of respectively even/odd permutations with less than t fixed points, uniformly bounded by $K_t/n!$ for some constant K_t depending only on t, and therefore satisfy all the required conditions. We have proved the Deza-Frankl conjecture:

Theorem 2.11. *If n is sufficiently large depending on t, then a t-intersecting family $\mathcal{A} \subset S_n$ has size at most $(n-t)!$.*

We have made no attempt to minimize the value of n for which our proof works. Very crudely, we may take

$$K_t = (9t^2 e^{t/e} t!/2)^{\sum_{k=0}^{t} p(k)}.$$

By (2.3), provided $n \geq K_t/c_t$, all medium partitions α of n satisfy $|\lambda_\alpha| < \mu_{n,t}$. Again, very crudely, taking $c_t = 1/(4t^2)^{t+1}$, we see that our proof works for $n \geq (9t^2 e^{t/e} t!/2)^{\sum_{k=0}^{t} p(k)} (4t^2)^{t+1}$, so certainly for $n \geq (t^2 e^{t/e} t!)^{(t+1)!}$ (using the trivial upper bound $p(k) \leq k!$).

The problem of determining the maximum size of a t-intersecting family in S_n for each value of t and n remains open. We conjecture that this maximum size is always attained by one of the families

$$\{\sigma \in S_n : |f(\sigma) \cap [2i+t]| \geq i+t\} \quad (0 \leq i \leq (n-t)/2),$$

where $f(\sigma)$ is the set of fixed points of σ. This would imply that the maximum size is $(n-t)!$ for $n \geq 2t+2$. It is natural to ask for which values of n and t our method of proof could work, i.e. when is there a real-valued class-function w on S_n supported on conjugacy classes of permutations with less than t fixed points, such that

$$\langle w, \chi_{(n)} \rangle = 1/n! \quad \text{and} \quad \frac{n!}{f^\alpha} \langle w, \chi_\alpha \rangle \geq -\mu_{n,t} \ \forall \alpha \vdash n : \ \alpha \neq (n).$$

Notice that the subspace of class functions supported on conjugacy classes of permutations with less than t fixed points is precisely the subspace

$$\{w : \langle w, \xi_\beta \rangle = 0 \quad \forall \beta \vdash n \text{ with at least } t \text{ parts} = 1\}$$

as clearly, the first subspace is contained in the second, and both have the same dimension, namely the number of partitions of n with less than t parts $= 1$. Hence, we seek a class function w such that

$$\langle w, \xi_\beta \rangle = 0 \quad \forall \beta \vdash n \text{ with at least } t \text{ parts} = 1.$$

Using the facts that

$$\xi_\beta = \sum_{\alpha \vdash n} K_{\alpha,\beta} \chi_\alpha$$

and $K_{(n),\beta} = 1 \ \forall \beta \vdash n$, this condition becomes:

$$1 + n! \sum_{\alpha \neq (n)} K_{\alpha,\beta} \langle w, \chi_\alpha \rangle = 0 \quad (\forall \beta \vdash n \text{ with at least } t \text{ parts} = 1).$$

Writing

$$z_\alpha = \frac{((n)_t - 1)\langle w, \chi_\alpha \rangle + f^\alpha}{(n)_t} n!,$$

we seek $(z_\alpha \geq 0 : \alpha \vdash n, \ \alpha \neq (n))$ such that

$$(n)_t - 1 + (n)_t \sum_{\alpha \neq (n)} K_{\alpha,\beta} z_\alpha = \sum_{\alpha \neq (n)} K_{\alpha,\beta} f^\alpha \quad (\forall \beta \vdash n \text{ with at least } t \text{ parts} = 1).$$

Notice that the right-hand-side is simply $\xi_\beta(1) - 1$, so this simplifies to:

$$1 + \sum_{\alpha \neq (n)} K_{\alpha,\beta} z_\alpha = \xi_\beta(1)/(n)_t \quad (\forall \beta \vdash n \text{ with at least } t \text{ parts} = 1).$$

It is easy to see that this cannot be achieved for $t = 2, n = 6$, so our method of proof cannot be used to show that a 2-intersecting family in S_6 has size at most $4! = 24$.

2.5 A Deza-Frankl type result for A_n

An analogous result holds for the alternating group A_n, the index-2 subgroup of S_n consisting of the even permutations of $\{1, 2, \ldots, n\}$. As for S_n, by a '*t-coset* of A_n' we will mean a coset of the stabilizer of t points.

Theorem 2.12. *If n is sufficiently large depending on t, then any t-intersecting family $\mathcal{A} \subset A_n$ has size $|\mathcal{A}| \leq (n-t)!/2$.*

Remark: This implies the Deza-Frankl conjecture. To see this, let $\mathcal{A} \subset S_n$ be t-intersecting; then $\mathcal{A} \cap A_n$ and $(\mathcal{A} \setminus A_n)(1\ 2)$ are both t-intersecting families of permutations in A_n, so by Theorem 2.12, both have size at most $(n-t)!/2$. Hence,

$$|\mathcal{A}| = |\mathcal{A} \cap A_n| + |\mathcal{A} \setminus A_n| \leq (n-t)!.$$

Proof. Let

$$M^+_{\sigma,\pi} = w^+_{\sigma\pi^{-1}} \quad (\sigma,\ \pi \in S_n);$$

then M^+ is the (real, symmetric) matrix of the linear map

$$x \mapsto w^+ x$$

on $\mathbb{C}S_n$, with respect to the basis S_n. It follows from $(*)$ that the 1-eigenspace of M^+ is $U_{(n)} \oplus U_{(1^n)}$, i.e. the subspace of $\mathbb{C}S_n$ spanned by the all-1's vector and the sign vector, the least eigenvalue is $-\mu_{n,t}$, and all other eigenvalues are $\leq O(\mu_{n,t}/n)$.

Observe that $M^+_{\sigma,\pi} = 0$ whenever $\sigma \in A_n$ and $\pi \notin A_n$, since w^+ is supported on A_n. Moreover,

$$M^+_{\sigma(1\ 2),\pi(1\ 2)} = M^+_{\sigma,\pi}.$$

Order S_n by putting A_n first in any order and then $S_n \setminus A_n$ in the order induced from the ordering on A_n by the map $\sigma \mapsto \sigma(1\ 2)$; then the matrix M^+ has the block-diagonal form:

$$M^+ = \left(\begin{array}{c|c} L & 0 \\ \hline 0 & L \end{array}\right),$$

where L is the restriction of M^+ to A_n. Therefore, the eigenvalues of M^+ are the same as the eigenvalues of L, with double the multiplicities. Hence, 1-eigenspace of L consists of the constant vectors in $\mathbb{C}A_n$, the least eigenvalue is $-\mu_{n,t}$, and all other eigenvalues are $O(\mu_{n,t}/n)$. Applying Theorem 2.1 to the matrix L with X a t-intersecting family in A_n proves Theorem 2.12. □

2.6 The extremal families

We now describe the approach of Friedgut and Pilpel (see [9]) to proving that the t-intersecting families in S_n of maximum size are precisely the t-cosets of S_n (provided n is sufficiently large depending on t). Notice that we have equality in Theorem 2.11 only if the characteristic vector $v_{\mathcal{A}} \in E_M(\lambda_1) \oplus E_M(\lambda_N) = U_t$, the subspace of $\mathbb{C}S_n$ spanned by the characteristic vectors of the t-cosets of S_n. We will show that if $v_{\mathcal{A}} \in U_t$, then $\mathcal{A}$ is a disjoint union of t-cosets of S_n. To do this, we will show that if $f : S_n \to \mathbb{R}_{\geq 0}$ is a linear combination of the characteristic vectors of the t-cosets of S_n, then it can be expressed as a linear combination of them with non-negative coefficients. To illustrate the ideas more clearly, we will begin with the $t = 1$ case.

Theorem 2.13 (Friedgut, Pilpel)**.** *If $u : S_n \to \mathbb{R}_{\geq 0}$ is in U_1, the subspace of $\mathbb{C}S_n$ spanned by the 1-cosets, then it is a non-negative linear combination of 1-cosets, i.e. there exist $(\beta_{i,j} \geq 0 :\ i,j \in [n])$ such that*

$$u = \sum_{i,j} \beta_{i,j} v_{i \mapsto j}.$$

Proof. Let $u : S_n \to \mathbb{R}_{\geq 0}$ be in U_1. We say a real $n \times n$ matrix A *represents* u if

$$u = \sum_{i,j} a_{i,j} v_{i \mapsto j},$$

i.e.

$$u(\sigma) = \sum_{i=1}^{n} a_{i,\sigma(i)} \quad \forall \sigma \in S_n.$$

Let A be a matrix representing u. Our task is to find a non-negative matrix B which also represents u. Notice that B and A represent the same function iff their difference $D := B - A$ represents the zero function, i.e.

$$\sum_{i=1}^{n} d_{i,\sigma(i)} = 0 \quad \forall \sigma \in S_n.$$

Notice that if D is any matrix such that

$$d_{i,j} = x_i + y_j \ \forall i, j \in [n] \quad \text{where} \quad \sum_i x_i + \sum_j y_j = 0,$$

then D represents the zero function. The subspace $\mathcal{D}$ of all such matrices D has dimension $2n - 2$, and U_1 has dimension $(n-1)^2 + 1 = n^2 - (2n-2)$, and therefore $\mathcal{D}$ is precisely the subspace of matrices representing the zero function. Hence, our task is to find such a matrix D such that $A + D$ has non-negative entries, i.e. to solve the system of inequalities

$$x_i + y_j \geq -a_{i,j} \ (1 \leq i, j \leq n) \quad \text{subject to} \quad \sum_i x_i + \sum_j y_j = 0. \tag{2.5}$$

By the Strong Duality Theorem of Linear Programming (see Appendix A), this is unsolvable if and only if there exist $c_{i,j} \geq 0$ such that

$$\sum_j c_{i,j} = 1 \ (1 \leq i \leq n), \quad \sum_i c_{i,j} = 1 \ (1 \leq j \leq n), \quad \sum_{i,j} c_{i,j} a_{i,j} < 0.$$

Suppose for a contradiction that this holds. The matrix $C = (c_{i,j})_{i,j \in [n]}$ is bistochastic, and therefore by Birkhoff's theorem it can be written as a convex combination of permutation matrices,

$$C = \sum_{k=1}^{N} r_k P_{\sigma_k}$$

where $\sigma_1, \ldots, \sigma_k \in S_n$ and $r_k \geq 0$ $(1 \leq k \leq N)$, $\sum_{k=1}^{N} r_k = 1$. But then $\sum_{i,j} c_{i,j} a_{i,j}$ is a convex combination of values of u:

$$\sum_{i,j} c_{i,j} a_{i,j} = \sum_{k=1}^{N} r_k \sum_{i,j} (P_{\sigma_k})_{i,j} a_{i,j} = \sum_{k=1}^{N} r_k \sum_{i=1}^{n} a_{i,\sigma_k(i)} = \sum_{k=1}^{N} r_k u(\sigma_k)$$

and is therefore non-negative, a contradiction. □

Corollary 2.14. *Let $\mathcal{A} \subset S_n$ be a family of permutations whose characteristic vector is a linear combination of the characteristic vectors of the 1-cosets of S_n, i.e. $v_{\mathcal{A}} \in U_1$; then $\mathcal{A}$ is a disjoint union of 1-cosets of S_n.*

Proof. By induction on $|\mathcal{A}|$. Let $\mathcal{A} \subset S_n$ be a non-empty family of permutations such that $v_{\mathcal{A}} \in U_1$. Suppose the statement is true for all smaller families. Express

$$v_{\mathcal{A}} = \sum_{i,j} b_{i,j} v_{i \mapsto j},$$

where $b_{i,j} \geq 0 \ \forall i,j \in [n]$. Since $\mathcal{A} \neq \emptyset$, $v_{\mathcal{A}} \neq 0$, and therefore $b_{i,j} > 0$ for some $i,j \in [n]$. But then the 1-coset $V_{i \mapsto j} = \{\sigma \in S_n : \ \sigma(i) = j\}$ is contained in $\mathcal{A}$. Remove this 1-coset: consider the family $\mathcal{A} \setminus V_{i \mapsto j}$. This also has characteristic vector in U_1 and so by the induction hypothesis is a disjoint union of 1-cosets. Therefore, so is $\mathcal{A}$, proving the corollary. □

This immediately implies the result of Cameron and Ku / Larose and Malvenuto (see respectively [6],[23]) that the 1-intersecting families in S_n of maximum size are precisely the 1-cosets.

We now give the argument of Friedgut and Pilpel for general t:

Theorem 2.15 (Friedgut, Pilpel)**.** *If $u : S_n \to \mathbb{R}_{\geq 0}$ is in U_t, the subspace of $\mathbb{C}S_n$ spanned by the t-cosets, then it is a non-negative linear combination of t-cosets.*

Proof. Let

$$I_t = \{(i_1, \ldots, i_t) : \ i_1, \ldots, i_t \text{ are distinct}\}$$

be the set of all ordered t-tuples of distinct numbers between 1 and n; then

$$|I_t| = (n)_t = n(n-1)\ldots(n-t+1).$$

Given t-tuples $\alpha = (\alpha_1, \ldots, \alpha_t), \ \beta = (\beta_1, \ldots, \beta_t) \in I_t$, we denote the corresponding t-coset as

$$V_{\alpha \mapsto \beta} = \{\sigma \in S_n : \sigma(\alpha_k) = \beta_k \ \forall k \in [t]\},$$

and its characteristic vector as $v_{\alpha \mapsto \beta}$.

Given a permutation $\sigma \in S_n$, let $P_\sigma^{(t)}$ be the $(n)_t \times (n)_t$ matrix of the permutation of I_t induced by σ.

Let $u : S_n \to \mathbb{R}_{\geq 0}$ be in U_t. Then there exists a matrix A with rows and columns indexed by I_t such that

$$u = \sum_{\alpha,\beta \in I_t} a_{\alpha,\beta} v_{\alpha \mapsto \beta}.$$

Let A be an $(n)_t \times (n)_t$ matrix with rows and columns indexed by I_t. Choose an ordering of the t coordinates, or equivalently a permutation $\pi \in S_t$, and consider the natural lexicographic ordering on I_t induced by this ordering, i.e. $\alpha < \beta$ iff $\alpha_{\pi(k)} < \beta_{\pi(k)}$, where $k = \min\{l : \alpha_{\pi(l)} \neq \beta_{\pi(l)}\}$. This lexicographic ordering on I_t recursively partitions A into blocks: first it partitions A into n^2 $(n-1)_{t-1} \times (n-1)_{t-1}$ blocks $B_{i,j}$ according to the $\pi(1)$-coordinate of each t-tuple; then it partitions each block $B_{i,j}$ into $(n-1)^2$ $(n-2)_{t-2} \times (n-2)_{t-2}$ sub-blocks according to the $\pi(2)$-coordinate of each t-tuple, and so on.

We define a *t-line* in A recursively as follows. For $t = 1$, a 1-line in $I_1 \times I_1$ is simply a set of the form $\{(i,j) : j \in [n]\}$ or $\{(i,j) : i \in [n]\}$, and a 1-line in A is just a row or column of A. Given a matrix A as above, a t-line in A is given by choosing an ordering π of the t-coordinates, partitioning A into blocks according to π as above, choosing a row or column of $(n-1)_{t-1} \times (n-1)_{t-1}$ blocks $B_{i,j}$, and then taking a union of $(t-1)$-lines, one from each block. Let $\mathcal{L}_t$ be the set of t-lines. Given a t-line L, the corresponding t-cosets

$$\{V_{\alpha \mapsto \beta} : (\alpha, \beta) \in L\}$$

form a partition of S_n. Hence, if a matrix E is obtained from A by adding x_L to every entry in the t-line L, for each L in turn, where

$$\sum_L x_L = 0,$$

then E represents the same function as A. We wish to find a non-negative such E, i.e. we wish to solve the system of linear inequalities

$$\sum_{L:(\alpha,\beta)\in L} x_L \geq -a_{\alpha,\beta} \ (L \in \mathcal{L}_t) \quad \text{subject to} \quad \sum_L x_L = 0.$$

Again, by the Strong Duality Theorem of Linear Programming, this is possible unless there exists an $(n)_t \times (n)_t$ matrix C such that $\sum_{(\alpha,\beta)\in L} c_{\alpha,\beta} = 1 \ \forall L \in \mathcal{L}_t$. We call such a matrix *t-bistochastic.* Observe that the following definition is equivalent:

Definition 2.16. *For $t = 1$, an $n \times n$ matrix is* 1-bistochastic *if it is bistochastic. For $t > 1$, an $(n)_t \times (n)_t$ matrix is* t-bistochastic *if, for any partition into $n \times n$ blocks $B_{i,j}$ of size $(n-1)_{t-1} \times (n-1)_{t-1}$ according to a lexicographic order on the t-tuples induced by any of the $t!$ orderings of the coordinates, there exists a bistochastic $n \times n$ matrix $R = (r_{i,j})$ and n^2 $(t-1)$-bistochastic matrices $M_{i,j}$ of order $(n-1)$ such that $B_{i,j} = r_{i,j} M_{i,j}$.*

One may generalize Birkhoff's Theorem to t-bistochastic matrices as follows:

Theorem 2.17 (Benabbas, Friedgut, Pilpel)**.** *An $(n)_t \times (n)_t$ matrix M is t-bistochastic if and only if it is a convex combination of $P_\sigma^{(t)}$'s, i.e. $(n)_t \times (n)_t$ matrices of permutations of I_t induced by permutations of $\{1, 2, \dots, n\}$.*

Proof. By induction on t. For $t = 1$, this is Birkhoff's theorem. For the induction step, let $M = r_{i,j} \cdot M_{i,j}$ be a block decomposition of M as in the definition of t-bistochasticity, according to the natural order of the coordinates, i.e. $\pi = \mathrm{id}$. By Birkhoff's theorem, $R = (r_{i,j})$ is a convex combination of permutation matrices, and thus it is either a permutation matrix or else we can write a convex relation $R = sP + (1-s)T$, with P a permutation matrix and T bistochastic with strictly more zero entries than R. Treating P separately and proceeding in this manner by induction, we may thus assume that R is a permutation matrix, and without loss of generality we may assume that $R = I$, the identity. So in M we have that every non-zero entry is indexed by a pair of t-tuples of the form $((i, *, *, \dots, *), (i, *, *, \dots, *))$ for some i. Now reorder the rows and columns of M with a lexicographic order on I_t determined by the permutation $\pi = (1\ 2)$. Consider now an off-diagonal block in this ordering of the form $((*, i, *, *, \dots, *), (*, j, *, *, \dots, *))$ with $i \neq j$. It breaks into $(n-1) \times (n-1)$ sub-blocks of size $(n-2)_{t-2} \times (n-2)_{t-2}$, but only $n-2$ of them have nonzero entries, namely the sub-blocks of the form $((k, i, *, *, \dots, *), (k, j, *, *, \dots, *))$ for some k different from i and j. Hence, the block has a row of zero sub-blocks, namely the sub-blocks $((j, i, *, *, \dots, *), (l, j, *, *, \dots, *))$ for $l \neq j$, and a column of zero sub-blocks, namely the sub-blocks $((l, i, *, *, \dots, *), (i, j, *, *, \dots, *))$ for $l \neq i$. Since the block is $(t-1)$-bistochastic, it must be the zero matrix.

As for the diagonal blocks of M in the new ordering of I_t, blocks of the form $((*, i, *, *, \dots, *), (*, i, *, *, \dots, *))$, they can only have nonzero sub-blocks on the diagonal. Since they are $(t-1)$-bistochastic, by the induction hypothesis they are each equal to a positive convex combination of $P_\sigma^{(t-1)}$'s where each σ is a permutation of $[n] \setminus \{i\}$. But if any of these σ's has $\sigma(k) \neq k$ for some k, then the off-diagonal sub-block

$$((k, i, *, *, \dots, *), (\sigma(k), i, *, *, \dots, *))$$

is non-zero, a contradiction. Hence each permutation σ must be the identity, and therefore all the diagonal blocks are copies of the $(n-1)_{t-1} \times (n-1)_{t-1}$ identity matrix. Hence, M is the identity matrix, and we are done. □

We can now complete the proof of Theorem 2.15. By the Generalized Birkhoff Theorem, C is in the convex hull of permutation matrices induced by permutations of $\{1,\ldots,n\}$. This means, as before, that

$$\sum_{\alpha,\beta\in I_t} c_{\alpha,\beta}a_{\alpha,\beta}$$

is a convex combination of values of u, hence non-negative, which yields the required contradiction. $\square$

One uses the same argument as before to deduce:

Corollary 2.18. *Let $\mathcal{A} \subset S_n$ be a family of permutations whose characteristic vector is a linear combination of the characteristic vectors of the t-cosets of S_n, i.e. $v_{\mathcal{A}} \in U_t$; then $\mathcal{A}$ is a disjoint union of t-cosets of S_n.*

This immediately implies that equality holds in Theorem 2.11 precisely for the t-cosets of S_n.

Interestingly, the same approach does not work for A_n: there exist non-negative functions $u : A_n \to \mathbb{R}_{\geq 0}$ in the subspace $\tilde{U}_1$ of $\mathbb{C}A_n$ spanned by the characteristic vectors of the 1-cosets of A_n, which cannot be written as a non-negative linear combination of the characteristic vectors of the 1-cosets of A_n. As before, we say a real $n \times n$ matrix A *represents* u if

$$u(\sigma) = \sum_{i=1}^{n} a_{i,\sigma(i)} \quad \forall \sigma \in A_n.$$

It is easy to see that $\dim(\tilde{U}_1) = \dim(U_1)$, so as before, B and A represent the same function if and only if $B - A \in \mathcal{D}$. But observe that if A is the matrix

$$A = \begin{pmatrix} 1 & -1/2 & 1 & 1 & \ldots & 1 \\ -1/2 & 1 & 1 & 1 & \ldots & 1 \\ 1 & 1 & 0 & 1 & \ldots & 1 \\ \vdots & \vdots & & \ddots & & \vdots \\ 1 & 1 & \ldots & & & 0 \end{pmatrix}$$

and τ is the transposition $(1\ 2)$, then

$$\sum_{i=1}^{n} a_{i,\tau(i)} = -1 \quad \text{and} \quad \sum_{i=1}^{n} a_{i,\sigma(i)} \geq 0\ \forall \sigma \in A_n.$$

Recall that any matrix $D \in \mathcal{D}$ satisfies

$$\sum_{i=1}^{n} d_{i,\sigma(i)} = 0 \quad \forall \pi \in S_n,$$

and therefore any matrix B representing the same function as A must also satisfy

$$\sum_{i=1}^{n} b_{i,\tau(i)} = -1,$$

so cannot have non-negative entries. Hence, the non-negative function represented by A cannot be written as a non-negative linear combination of the characteristic vectors of the 1-cosets of A_n.

However, it will follow straight away from our Hilton-Milner type theorem for A_n that equality holds in Theorem 2.12 only for the t-cosets of A_n: we just need to use more information than the fact that $v_{\mathcal{A}}$ is a linear combination of the characteristic vectors of the t-cosets of A_n.

2.7 t-cross-intersecting families of permutations

Recall that we say a pair of families $\mathcal{A}, \mathcal{B} \subset S_n$ is *t-cross-intersecting* if $|\{i \in [n] : \sigma(i) = \pi(i)\}| \geq t$ for any $\sigma \in \mathcal{A},\ \pi \in \mathcal{B}$. We have the following:

Theorem 2.19. *If n is sufficiently large depending on t, then any t-cross-intersecting pair of families $\mathcal{A}, \mathcal{B} \subset S_n$ satisfy $|\mathcal{A}||\mathcal{B}| \leq ((n-t)!)^2$.*

This follows from a 'weighted' version of Theorem 1.2:

Theorem 2.20. *Let M be a real, symmetric N by N matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N$, where $\lambda_1 > 0$ has only constant eigenvectors, and $\lambda_N < 0$. Let $\nu = \max(|\lambda_2|, |\lambda_N|)$. If $X, Y \subset [N]$ satisfy $M_{x,y} = 0\ \forall x \in X,\ y \in Y$, then*

$$\sqrt{|X||Y|} \leq \frac{\nu}{\lambda_1 + \nu} N. \tag{2.6}$$

Suppose further that $|\lambda_2| \neq |\lambda_N|$, and let λ' be the larger in modulus of the two. Let v_X, v_Y be the characteristic vectors of X, Y and let $\mathbf{f}$ denote the all-1's vector. If equality holds in (2.6), then $|X| = |Y|$, and the shifted characteristic vectors $v_X - (|X|/N)\mathbf{f}$ and $v_Y - (|Y|/N)\mathbf{f}$ are eigenvectors of M with eigenvalue λ'.

The proof is a straightforward generalization of the proof of Theorem 1.2, and we omit it.

Applying Theorem 2.20 to the matrix M constructed above, which has $\lambda_1 = 1,\ \nu = \mu_{n,t}$, with $X, Y \subset S_n$ cross-intersecting, proves Theorem 2.19. If we have equality in Theorem 2.19, then $v_{\mathcal{A}},\ v_{\mathcal{B}} \in U_t$, and therefore by Corollary 2.18, $\mathcal{A}$ and $\mathcal{B}$ are disjoint unions of t-cosets, and therefore must be the same t-coset. (This can also be deduced from Theorem 2.26 – see later.)

Similarly, applying Theorem 2.20 to the matrix L constructed in Section 2.5 proves:

Theorem 2.21. *If n is sufficiently large depending on t, then any t-cross-intersecting pair of families $\mathcal{A}, \mathcal{B} \subset A_n$ satisfy $|\mathcal{A}||\mathcal{B}| \leq ((n-t)!/2)^2$.*

2.8 Stability for t-intersecting families

Before beginning our stability analysis, we will introduce some more notation.

Given distinct $i_1, \ldots, i_l$ and distinct $j_1, \ldots, j_l$, we write

$$\mathcal{A}_{i_1 \mapsto j_1, i_2 \mapsto j_2, \ldots, i_l \mapsto j_l} := \{\sigma \in \mathcal{A} : \sigma(i_k) = j_k\ \forall k \in [l]\}.$$

Given two permutations $\sigma, \pi \in S_n$, write $\sigma \cap \pi := \{i \in [n] : \sigma(i) = \pi(i)\}$ for the set of points at which they agree.

Our first aim is to prove the following rough stability result:

Theorem 2.22. *Let $t \in \mathbb{N}$ and $c > 0$ be fixed. If $\mathcal{A} \subset S_n$ is a t-intersecting family with $|\mathcal{A}| \geq c(n-t)!$, then there exists a t-coset $\mathcal{C}$ such that $|\mathcal{A} \setminus \mathcal{C}| \leq O((n-t-1)!)$.*

In other words, if $\mathcal{A} \subset S_n$ is a t-intersecting family of size at least a constant proportion of the maximum possible size $(n-t)!$, then there is some t-coset containing all but at most a $O(1/n)$-fraction of $\mathcal{A}$.

To prove this, we will first prove the following weaker statement:

Lemma 2.23. *Let $t \in \mathbb{N}$ and $c > 0$ be fixed. If $\mathcal{A} \subset S_n$ is a t-intersecting family of size at least $c(n-t)!$, then there exist i and j such that all but at most $O((n-t-1)!)$ permutations in $\mathcal{A}$ map i to j.*

In other words, a large t-intersecting family is almost contained within a 1-coset. Theorem 2.22 will follow easily from this by an inductive argument.

We will need the following weighted analogue of Lemma 1.12:

Lemma 2.24. *Let M, X and U be as in Theorem 2.1. Let $\alpha = |X|/N$. Let λ_L be the negative eigenvalue of M with second largest modulus. Equip $\mathbb{C}^N$ with the inner product:*

$$\langle x, y \rangle = \frac{1}{N} \sum_{i=1}^{N} \bar{x}_i y_i$$

and let

$$||x|| = \sqrt{\frac{1}{N} \sum_{i=1}^{N} |x_i|^2}$$

be the induced norm. Let D be the Euclidean distance from the characteristic vector v_X of X to the subspace U, i.e. the norm $||P_{U^\perp}(v_X)||$ of the projection of v_X onto $U^\perp$. Then

$$D^2 \leq \frac{(1-\alpha)|\lambda_N| - \lambda_1 \alpha}{|\lambda_N| - |\lambda_L|} \alpha.$$

The proof is a straightforward generalization of the proof of Lemma 1.12, and we omit it.

Our matrix M has $\lambda_N = -\mu_{n,t}$ and $|\lambda_L| = O(\mu_{n,t}/n)$, so applying Lemma 2.24 to a t-intersecting family $\mathcal{A} \subset S_n$ gives:

$$||P_{U_t^\perp}(v_\mathcal{A})||^2 \leq (1 - |\mathcal{A}|/(n-t)!)(1 + O(1/n))|\mathcal{A}|/n!. \tag{2.7}$$

Lemma 2.23 will be proved using the same general strategy as Lemma 1.13 in Chapter 1. We will first observe from (2.7) that if $\mathcal{A} \subset S_n$ is a t-intersecting family of size at least $c(n-t)!$, then the characteristic vector $v_\mathcal{A}$ of $\mathcal{A}$ is close to the subspace U_t spanned by the characteristic vectors of the t-cosets. We will use this, combined with representation-theoretic arguments, to show that there exists some t-coset $\mathcal{C}_0$ such that

$$|\mathcal{A} \cap \mathcal{C}_0| \geq \omega((n-2t)!)$$

(this step turns out to be significantly harder than for the $t = 1$ case in Chapter 1.) Without loss of generality, we may assume that

$$\mathcal{C}_0 = \{\sigma \in S_n : \sigma(1) = 1, \ldots, \sigma(t) = t\},$$

so

$$|\mathcal{A}_{1 \mapsto 1, 2 \mapsto 2, \ldots, t \mapsto t}| \geq \omega((n-2t)!).$$

Observe that the average size of the intersection of $\mathcal{A}$ with a t-coset is

$$|\mathcal{A}|/n(n-1)\ldots(n-t+1) = \Theta((n-2t)!),$$

so we only know that $\mathcal{A} \cap \mathcal{C}_0$ has size ω of the average size. This statement would at first seem to weak to help us. However, for any distinct $j_1 \neq 1, j_2 \neq 2, \ldots,$ and $j_t \neq t$, the pair of families

$$\mathcal{A}_{1 \mapsto 1, 2 \mapsto 2, \ldots, t \mapsto t}, \quad \mathcal{A}_{1 \mapsto j_1, 2 \mapsto j_2, \ldots, t \mapsto j_t}$$

is t-cross-intersecting, so we may compare their sizes, as for the $t = 1$ case in Chapter 1. In detail, we will deduce from Theorem 2.19 that

$$|\mathcal{A}_{1 \mapsto 1, 2 \mapsto 2, \ldots, t \mapsto t}||\mathcal{A}_{1 \mapsto j_1, 2 \mapsto j_2, \ldots, t \mapsto j_t}| \leq ((n-2t)!)^2,$$

giving $|\mathcal{A}_{1 \mapsto j_1, \ldots, t \mapsto j_t}| \leq o((n-2t)!)$. Summing over all choices of $j_1, \ldots, j_t$ will show that all but at most $o((n-t)!)$ permutations in $\mathcal{A}$ fix some point of $[t]$, enabling us to complete the proof.

Proof of Lemma 2.23:
Let $\mathcal{A} \subset S_n$ be a t-intersecting family of size at least $c(n-t)!$; write $|\mathcal{A}| = (1-\delta)(n-t)!$, where $\delta \leq 1-c$. From (2.7), we know that the Euclidean distance from $v_{\mathcal{A}}$ to U_t is small:

$$||P_{U_t^{\perp}}(v_{\mathcal{A}})||^2 \leq \delta(1+O(1/n))|\mathcal{A}|/n!.$$

From (1.4), the projection of $v_{\mathcal{A}}$ onto U_t has σ-coordinate:

$$P_{U_t}(v_{\mathcal{A}})_\sigma = \frac{1}{n!} \sum_{\text{fat } \alpha} f^\alpha \sum_{\pi \in \mathcal{A}} \chi_\alpha(\pi\sigma^{-1}).$$

Write $P_\sigma = P_{U_t}(v_{\mathcal{A}})_\sigma$; then

$$\frac{1}{n!}\left(\sum_{\sigma \in \mathcal{A}} (1-P_\sigma)^2 + \sum_{\sigma \notin \mathcal{A}} P_\sigma^2\right) \leq \delta(1+O(1/n))|\mathcal{A}|/n!,$$

i.e.

$$\sum_{\sigma \in \mathcal{A}} (1-P_\sigma)^2 + \sum_{\sigma \notin \mathcal{A}} P_\sigma^2 \leq \delta(1+O(1/n))|\mathcal{A}|. \tag{2.8}$$

Choose $C > 0$ such that $|\mathcal{A}|(1-1/n)\delta(1+C/n)$ is at least the right-hand-side of (2.8); then the subset

$$\mathcal{S} := \{\sigma \in \mathcal{A} : (1-P_\sigma)^2 < \delta(1+C/\sqrt{n})\}$$

has size at least $|\mathcal{A}|/n$. Similarly, $P_\sigma^2 < 2\delta/n$ for all but at most

$$n|\mathcal{A}|(1+O(1/n))/2$$

permutations $\sigma \notin \mathcal{A}$. Provided n is sufficiently large, $|\mathcal{A}| \leq (n-t)!$, and therefore the subset $\mathcal{T} = \{\sigma \notin \mathcal{A} : P_\sigma^2 < 2\delta/n\}$ has size

$$|\mathcal{T}| \geq n! - (n-t)! - n(n-t)!(1+O(1/n))/2.$$

The permutations $\sigma \in \mathcal{S}$ have P_σ close to 1, and the permutations $\pi \in \mathcal{T}$ have P_π close to 0. Using only our lower bounds on the sizes of $\mathcal{S}$ and $\mathcal{T}$, we may prove the following:

Claim: There exist permutations $\sigma \in \mathcal{S}$ and $\pi \in \mathcal{T}$ such that $\sigma^{-1}\pi$ is a product of at most $h = h(n)$ transpositions, where $h = \sqrt{2(t+2)(n-1)\log n}$.

Proof of Claim: Apply Theorem 1.14 to the set $\mathcal{S}$, which has $|\mathcal{S}| \geq (1-\delta)(n-t)!/n \geq \frac{n!}{n^{t+2}}$ (provided n is sufficiently large), with $a = 1/n^{t+2}$, $h = 2h_0$. This gives $|N_h(\mathcal{S})| \geq (1-n^{-(t+2)})n!$, so certainly $N_h(\mathcal{S}) \cap \mathcal{T} \neq \emptyset$, proving the claim.

We now have two permutations $\sigma \in \mathcal{A}$, $\pi \notin \mathcal{A}$ which differ in only $O(\sqrt{n\log n})$ transpositions, and satisfy

$$P_\sigma > 1 - \sqrt{\delta(1+C/n)}, \quad P_\pi < \sqrt{2\delta/n},$$

and therefore

$$P_\sigma - P_\pi > 1 - \sqrt{\delta} - O(1/\sqrt{n}).$$

Hence, by averaging, there exist two permutations ρ, τ that differ by just one transposition and satisfy

$$P_\rho - P_\tau > (1-\sqrt{\delta} - O(1/\sqrt{n}))/h \geq \frac{1-\sqrt{\delta}-O(1/\sqrt{n})}{\sqrt{2(t+2)n\log n}},$$

i.e.

$$\sum_{\alpha \in \mathcal{F}_{n,t}} \frac{f^\alpha}{n!} \left(\sum_{\pi \in \mathcal{A}} \chi_\alpha(\pi\rho^{-1}) - \sum_{\pi \in \mathcal{A}} \chi_\alpha(\pi\tau^{-1}) \right) \geq \frac{1-\sqrt{\delta}-O(1/\sqrt{n})}{\sqrt{2(t+2)n\log n}}.$$

By double translation, we may assume without loss of generality that $\rho = \mathrm{Id}$ and $\tau = (1\ 2)$. So we have:

$$\sum_{\alpha \in \mathcal{F}_{n,t}} \frac{f^\alpha}{n!} \left(\sum_{\pi \in \mathcal{A}} \chi_\alpha(\pi) - \sum_{\pi \in \mathcal{A}} \chi_\alpha(\pi(1\ 2)) \right) \geq \frac{1-\sqrt{\delta}-O(1/\sqrt{n})}{\sqrt{2(t+2)n\log n}}.$$

The above sum is over $|\mathcal{F}_{n,t}| = \sum_{s=0}^{t} p(s)$ partitions α of n; this grows very rapidly with t, but is independent of n for $n \geq 2t$. By averaging, there exists some $\alpha \in \mathcal{F}_{n,t}$ such that

$$\begin{aligned}\frac{f^\alpha}{n!}\left(\sum_{\pi\in\mathcal{A}}\chi_\alpha(\pi) - \sum_{\pi\in\mathcal{A}}\chi_\alpha(\pi(1\ 2))\right) &\geq \frac{1-\sqrt{\delta}-O(1/\sqrt{n})}{\sqrt{2(t+2)n\log n}\sum_{s=0}^{t}p(s)} \\ &= \Omega(1/\sqrt{n\log n}).\end{aligned}$$

Recall that the 'fat' irreducible representations $\{[\alpha] : \alpha \in \mathcal{F}_{n,t}\}$ are precisely the irreducible constituents of $M^{(n-t,1^t)}$, so very crudely, for each $\alpha \in \mathcal{F}_{n,t}$,

$$f^\alpha \leq \dim(M^{(n-t,1^t)}) = n(n-1)\dots(n-t+1).$$

Hence,

$$\sum_{\pi\in\mathcal{A}}\chi_\alpha(\pi) - \sum_{\pi\in\mathcal{A}}\chi_\alpha(\pi(1\ 2)) \geq \Omega(1/\sqrt{n\log n})(n-t)!.$$

But for any $\alpha \in \mathcal{F}_{n,t}$, we may express the irreducible character χ_α as a linear combination of permutation characters $\xi_\beta : \beta \in \mathcal{F}_{n,t}$ using the following 'determinantal formula' (see [19]). For any partition α of n,

$$\chi_\alpha = \sum_{\pi\in S_n}\epsilon(\pi)\xi_{\alpha-\mathrm{id}+\pi}.$$

Here, for $\alpha = (\alpha_1,\dots,\alpha_l) \vdash n$, we set $\alpha_i = 0$ $(l < i \leq n)$, we think of α, id and π as sequences of length n, and we define addition and subtraction of these sequences pointwise. In general,

$$\alpha - \mathrm{id} + \pi = (\alpha_1 - 1 + \pi(1), \alpha_2 - 2 + \pi(2), \dots, \alpha_n - n + \pi(n))$$

will be a sequence of n integers with sum n, i.e. a *composition* of n. If λ is a composition of n with all its terms non-negative, then let $\bar{\lambda}$ be the partition of n produced by ordering the terms of λ in non-increasing order, and define $\xi_\lambda = \xi_{\bar{\lambda}}$; if λ has a negative term, we define $\xi_\lambda = 0$. If $\alpha \in \mathcal{F}_{n,t}$, then as $\alpha_1 \geq n-t$, any composition occurring in the above sum has first term at least $n-t$, and therefore ξ_β can only occur in the above sum if $\beta \in \mathcal{F}_{n,t}$. Observe further that since α has at most $t+1$ non-zero parts, $\alpha_i = 0$ for every $i > t+1$, and therefore any permutation $\pi \in S_n$ with $\xi_{\alpha-\mathrm{id}+\pi} \neq 0$ must have $\pi(i) \geq i$ for every $i > t+1$, so must fix $t+2, t+3, \dots,$ and n. Therefore, the above sum is only over $\pi \in S_{\{1,\dots,t+1\}}$, i.e.

$$\chi_\alpha = \sum_{\pi\in S_{t+1}}\epsilon(\pi)\xi_{\alpha-\mathrm{id}+\pi}\ \forall \alpha \in \mathcal{F}_{n,t}.$$

Therefore, χ_α is a (± 1)-linear combination of at most $(t+1)!$ permutation characters ξ_β ($\beta \in \mathcal{F}_{n,t}$), possibly with repeats. Hence, by averaging, there exists some $\beta \in \mathcal{F}_{n,t}$ such that

$$\begin{aligned}\left|\sum_{\pi\in\mathcal{A}} \xi_\beta(\pi) - \sum_{\pi\in\mathcal{A}} \xi_\beta(\pi(1\ 2))\right| &\geq \Omega(1/\sqrt{n\log n})\frac{(n-t)!}{(t+1)!} \\ &= \Omega(1/\sqrt{n\log n})(n-t)!.\end{aligned}$$

Without loss of generality, we may assume that the above quantity is positive, i.e.

$$\sum_{\pi\in\mathcal{A}} \xi_\beta(\pi) - \sum_{\pi\in\mathcal{A}} \xi_\beta(\pi(1\ 2)) \geq \Omega(1/\sqrt{n\log n})(n-t)!.$$

Let $\mathbb{T}_\beta$ be the set of β-tabloids; the left-hand-side is then

$$\begin{aligned}&\#\{(T,\pi) : T \in \mathbb{T}_\beta, \pi \in \mathcal{A}, \pi(T) = T\} \\ - \ &\#\{(T,\pi) : T \in \mathbb{T}_\beta, \pi \in \mathcal{A}, \pi(1\ 2)(T) = T\}.\end{aligned}$$

Interchanging the order of summation, this equals

$$\sum_{T\in\mathbb{T}_\beta} \left(\#\{\pi \in \mathcal{A} : \pi(T) = T\} - \#\{\pi \in \mathcal{A} : \pi(1\ 2)(T) = T\}\right).$$

The above summand is zero for all β-tabloids T with 1 and 2 in the first row of T (as then $(1\ 2)T = T$). Write $\beta = (n-s, \beta_2, \ldots, \beta_l)$, where $0 \leq s \leq t$. The number of β-tabloids with 1 not in the first row is

$$s(n-1)(n-2)\ldots(n-s+1)/\prod_{i=2}^{l}\beta_i!,$$

and therefore the number of β-tabloids with 1 or 2 below the first row is at most

$$\begin{aligned}2s(n-1)(n-2)\ldots(n-s+1)/\prod_{i=2}^{l}\beta_i! &\leq 2t(n-1)(n-2)\ldots(n-s+1) \\ &= \frac{2t(n-1)!}{(n-s)!}.\end{aligned}$$

Hence by averaging, for one such β-tabloid T,

$$\begin{aligned}&\#\{\pi \in \mathcal{A} : \pi(T) = T\} - \#\{\pi \in \mathcal{A} : \pi(1\ 2)(T) = T\} \\ &\qquad \geq \Omega(1/\sqrt{n\log n})\frac{(n-s)!}{2t(n-1)!}(n-t)!,\end{aligned}$$

and therefore the number of permutations in $\mathcal{A}$ fixing T satisfies

$$\#\{\pi \in \mathcal{A} : \pi(T) = T\} \geq \Omega(1/\sqrt{n \log n})\frac{(n-s)!}{2t(n-1)!}(n-t)!.$$

Without loss of generality, we may assume that the first row of T consists of the numbers $\{s+1, \ldots, n\}$. There are $\beta_2!\beta_3! \ldots \beta_l! \leq s! \leq t!$ permutations of $[s]$ fixing the 2^{nd},3^{rd},..., and l^{th} rows of T; any permutation fixing T must agree with one of these permutations on $[s]$. Hence, there exists a permutation ρ of $[s]$ such that at least

$$\Omega(1/\sqrt{n \log n})\frac{(n-s)!(n-t)!}{2t(n-1)!t!}$$

permutations in $\mathcal{A}$ agree with ρ on $[s]$. Without loss of generality, we may assume that $\rho = \text{Id}_{[s]}$, so the number of permutations in $\mathcal{A}$ fixing $[s]$ pointwise satisfies

$$\begin{aligned} |\mathcal{A}_{1\mapsto 1,\ldots,s\mapsto s}| &\geq \Omega(1/\sqrt{n \log n})\frac{(n-s)!(n-t)!}{2t(n-1)!t!} \\ &= \Omega(1/\sqrt{n \log n})\frac{(n-s)!(n-t)!}{(n-1)!}. \end{aligned}$$

We may write $\mathcal{A}_{1\mapsto 1,\ldots,s\mapsto s}$ as a disjoint union

$$\mathcal{A}_{1\mapsto 1,\ldots,s\mapsto s} = \bigcup_{j_{s+1},\ldots,j_t > s \text{ distinct}} \mathcal{A}_{1\mapsto 1\ldots,s\mapsto s,s+1\mapsto j_{s+1},\ldots,t\mapsto j_t},$$

and there are $(n-s)(n-s-1)\ldots(n-t+1)$ choices of $j_{s+1}, \ldots, j_t$, so by averaging, there exists a choice such that

$$|\mathcal{A}_{1\mapsto 1\ldots,s\mapsto s,s+1\mapsto j_{s+1},\ldots,t\mapsto j_t}| \geq \Omega(1/\sqrt{n \log n})\frac{((n-t)!)^2}{(n-1)!}.$$

By translation, we may assume without loss of generality that $j_k = k$ for each k, so

$$\begin{aligned} |\mathcal{A}_{1\mapsto 1,2\mapsto 2,\ldots,t\mapsto t}| &\geq \Omega(1/\sqrt{n \log n})\frac{((n-t)!)^2}{(n-1)!} \\ &= \Omega(\sqrt{n/\log n})(n-2t)! \\ &= \omega((n-2t)!). \end{aligned}$$

We will use this to show that the number of permutations in $\mathcal{A}$ with no fixed point in $[t]$ is small. We may write

$$\mathcal{A} \setminus (\mathcal{A}_{1\mapsto 1} \cup \ldots \cup \mathcal{A}_{t\mapsto t}) = \bigcup_{j_1,\ldots,j_t \text{ distinct } : j_k \neq k \ \forall k \in [t]} \mathcal{A}_{1\mapsto j_1,\ldots,t\mapsto j_t}.$$

We now show that each $\mathcal{A}_{1\mapsto j_1,\ldots,t\mapsto j_t}$ is small using Theorem 2.19. Let $J = \{j_1,\ldots,j_t\}$. Notice that $\mathcal{E} := \mathcal{A}_{1\mapsto 1,\ldots,t\mapsto t}$, $\mathcal{F} := \mathcal{A}_{1\mapsto j_1,\ldots,t\mapsto j_t}$ is a t-cross-intersecting pair of families, so for any $\sigma \in \mathcal{E}$ and $\pi \in \mathcal{F}$, there are t distinct points $i_1, i_2 \ldots, i_t > t$ such that $\sigma(i_k) = \pi(i_k) \notin [t] \cup J$ for each $k \in [t]$. But then

$$(1\ j_1)(2\ j_2)\ldots(t\ j_t)\pi(i_k) = \sigma(i_k) \quad \text{for each } k \in [t],$$

so letting $\mathcal{G} := (1\ j_1)(2\ j_2)\ldots(t\ j_t)\mathcal{F}$, the pair of families $\mathcal{E},\mathcal{G}$ fix $[t]$ pointwise and t-cross-intersect on $\{t+1, t+2, \ldots, n\}$. Deleting $1,\ldots,t$ we obtain a t-cross-intersecting pair $\mathcal{E}', \mathcal{G}'$ of subsets of $S_{\{t+1,\ldots,n\}}$. By Theorem 2.19,

$$|\mathcal{A}_{1\mapsto 1,\ldots,t\mapsto t}||\mathcal{A}_{1\mapsto j_1,\ldots,t\mapsto j_t}| = |\mathcal{E}||\mathcal{G}| = |\mathcal{E}'||\mathcal{G}'| \leq ((n-2t)!)^2.$$

Since

$$|\mathcal{A}_{1\mapsto 1,\ldots,t\mapsto t}| \geq \omega((n-2t)!),$$

we have

$$|\mathcal{A}_{1\mapsto j_1,\ldots,t\mapsto j_t}| \leq o((n-2t)!).$$

There are $\leq n(n-1)(n-2)\ldots(n-t+1)$ possible choices of $j_1,\ldots,j_t$, and therefore the number of permutations in $\mathcal{A}$ with no fixed point in $[t]$ satisfies

$$\begin{aligned} |\mathcal{A} \setminus (\mathcal{A}_{1\mapsto 1} \cup \mathcal{A}_{2\mapsto 2} \cup \ldots \cup \mathcal{A}_{t\mapsto t})| &\leq o((n-2t)!)n(n-1)\ldots(n-t+1) \\ &= o((n-t)!). \end{aligned}$$

Since $|\mathcal{A}| \geq c(n-t)!$, we have

$$|\mathcal{A}_{1\mapsto 1} \cup \mathcal{A}_{2\mapsto 2} \cup \ldots \cup \mathcal{A}_{t\mapsto t}| \geq (c - o(1))(n-t)!.$$

By averaging, there exists some $i \in [t]$ such that

$$|\mathcal{A}_{i\mapsto i}| \geq (c - o(1))(n-t)!/t.$$

We may assume that $i = 1$, so $|\mathcal{A}_{1\mapsto 1}| \geq (c-o(1))(n-t)!/t$. Now, using the same trick as before, we may use Theorem 2.19 to show that $|\mathcal{A} \setminus \mathcal{A}_{1\mapsto 1}| \leq O((n-t-1)!)$. Indeed, write $\mathcal{A} \setminus \mathcal{A}_{1\mapsto 1}$ as a disjoint union

$$\mathcal{A} \setminus \mathcal{A}_{1\mapsto 1} = \bigcup_{j\neq 1} \mathcal{A}_{1\mapsto j}.$$

We will show that each $\mathcal{A}_{1\mapsto j}$ is small. Notice as before that the pair of families $\mathcal{A}_{1\mapsto 1}$, $(1\ j)\mathcal{A}_{1\mapsto j}$ fixes 1 and t-cross-intersects on the domain $\{2,\ldots,n\}$, so Theorem 2.19 gives

$$|\mathcal{A}_{1\mapsto 1}||\mathcal{A}_{1\mapsto j}| \leq ((n-t-1)!)^2.$$

Since $|\mathcal{A}_{1\mapsto 1}| \geq \Omega((n-t)!)$, we obtain $|\mathcal{A}_{1\mapsto j}| \leq O((n-t-2)!)$, and therefore

$$|\mathcal{A} \setminus \mathcal{A}_{1\mapsto 1}| = \sum_{j\neq 1} |\mathcal{A}_{1\mapsto j}| \leq O((n-t-1)!),$$

proving Lemma 2.23.

□

Proof of Theorem 2.22:
By induction on t. The $t=1$ case is the same as that of Lemma 2.23. Assume the theorem is true for $t-1$; we will prove it for t. Let $\mathcal{A} \subset S_n$ be a t-intersecting family of size at least $c(n-t)!$. By Lemma 2.23, there exist i and j such that $|\mathcal{A} \setminus \mathcal{A}_{i\mapsto j}| \leq O((n-t-1)!)$. Without loss of generality we may assume that $i=j=1$, so $|\mathcal{A} \setminus \mathcal{A}_{1\mapsto 1}| \leq O((n-t-1)!)$. Hence, $|\mathcal{A}_{1\mapsto 1}| \geq |\mathcal{A}| - O((n-t-1)!)$. Deleting 1 from each permutation in $\mathcal{A}_{1\mapsto 1}$, we obtain a $(t-1)$-intersecting family $\mathcal{A}' \subset S_{\{2,3,\ldots,n\}}$ of size $\geq (c - O(1/n))(n-t)!$. Choose any positive constant $c' < c$; then provided n is sufficiently large, we have $|\mathcal{A}'| \geq c'(n-t)!$. By the induction hypothesis, there exists a $(t-1)$-coset $\mathcal{C}'$ of $S_{2,\ldots,n}$ such that $|\mathcal{A}' \setminus \mathcal{C}'| \leq O((n-t-1)!)$. Then if $\mathcal{C}$ is the t-coset obtained from $\mathcal{C}'$ by adjoining $1 \mapsto 1$, we have $|\mathcal{A} \setminus \mathcal{C}| \leq O((n-t-1)!)$. This completes the induction and proves Theorem 2.22.

□

As for 1-intersecting families in Chapter 1, we may use our rough stability result to prove a Hilton-Milner type result for t-intersecting families:

Theorem 2.25. *For n sufficiently large depending on t, if $\mathcal{A} \subset S_n$ is a t-intersecting family which is not contained within a t-coset, then $\mathcal{A}$ is no larger than the family*

$$\begin{aligned} \mathcal{D} &= \{\sigma \in S_n : \ \sigma(i) = i \ \forall i \leq t,\ \sigma(j) = j \textit{ for some } j > t+1\} \\ &\quad \cup \{(1\ t+1), (2\ t+1), \ldots, (t\ t+1)\} \end{aligned}$$

which has size $(n-t)! - d_{n-t} - d_{n-t-1} + t = (1 - 1/e + o(1))(n-t)!$. If $\mathcal{A}$ is the same size as $\mathcal{D}$, then $\mathcal{A}$ is a double translate of $\mathcal{D}$, i.e. $\mathcal{A} = \pi\mathcal{D}\tau$ for some $\pi, \tau \in S_n$.

Proof. Suppose $\mathcal{A} \subset S_n$ is a t-intersecting family which is not contained within a t-coset, and has size

$$|\mathcal{A}| \geq (n-t)! - d_{n-t} - d_{n-t-1} + t = (1 - 1/e + o(1))(n-t)!.$$

Applying Theorem 2.22 with any constant c such that $0 < c < 1 - 1/e$, we see that (provided n is sufficiently large) there exists a t-coset $\mathcal{C}$ such that

$$|\mathcal{A} \setminus \mathcal{C}| \leq O(1/n)(n-t)!.$$

By double translation, we may assume without loss of generality that $\mathcal{C} = \{\sigma \in S_n : \sigma(1) = 1, \ldots, \sigma(t) = t\}$. We have:

$$\begin{aligned} |\mathcal{A} \cap \mathcal{C}| &\geq (n-t)! - d_{n-t} - d_{n-t-1} + t - O(1/n)(n-t)! \\ &= (1 - 1/e + o(1))(n-t)!. \end{aligned} \quad (2.9)$$

We now claim that every permutation in $\mathcal{A} \setminus \mathcal{C}$ fixes exactly $t-1$ points of $[t]$. Suppose for a contradiction that $\mathcal{A}$ contains a permutation τ fixing at most $t-2$ points of $[t]$. Then every permutation in $\mathcal{A} \cap \mathcal{C}$ must agree with τ on at least 2 points of $\{t+1, \ldots, n\}$, so

$$|\mathcal{A} \cap \mathcal{C}| \leq \binom{n-t}{2}(n-t-2)! = \tfrac{1}{2}(n-t)!,$$

contradicting (2.9), provided n is sufficiently large.

Since we are assuming that $\mathcal{A}$ is not contained within a t-coset, $\mathcal{A} \setminus \mathcal{C}$ contains some permutation τ; τ must fix all points of $[t]$ except for one. By double translation, we may assume that $\tau = (1\ t+1)$. We will show that under these hypotheses, $\mathcal{A} = \mathcal{D}$.

Every permutation in $\mathcal{A} \cap \mathcal{C}$ must t-intersect $(1\ t+1)$ and must therefore have at least one fixed point $> t+1$, i.e. $\mathcal{A} \cap \mathcal{C}$ is a subset of the family

$$\mathcal{E} := \{\sigma \in S_n : \sigma(i) = i\ \forall i \in [t],\ \sigma(j) = j \text{ for some } j > t+1\},$$

which has size

$$(n-t)! - d_{n-t} - d_{n-t-1}.$$

We now make the following observation:

Claim: $\mathcal{A} \setminus \mathcal{C}$ may only contain the transpositions $\{(i\ t+1)\ : i \in [t]\}$.

Proof of Claim:
Suppose for a contradiction that $\mathcal{A} \setminus \mathcal{C}$ contains a permutation ρ not of this

form. Then $\rho(j) \neq j$ for some $j \geq t+2$. We will show that there are at least d_{n-t-1} permutations in $\mathcal{E}$ which fix j and disagree with ρ at every point of $\{t+1, t+2, \ldots, n\}$, and therefore cannot t-intersect ρ. Let l be the unique point of $[t]$ not fixed by ρ. If σ fixes both l and j, then σ agrees with $\rho_{j,l} = (\rho_j)_l$ wherever it agrees with ρ. Notice that $\rho_{j,l}$ fixes $1, 2, \ldots, t$ and j. There are exactly d_{n-t-1} permutations in $\mathcal{E}$ which fix j and disagree with $\rho_{j,l}$ at every point of $\{t+1, t+2, \ldots, n\} \setminus \{j\}$; each disagrees with ρ at every point of $\{t+1, t+2, \ldots, n\}$. So none t-intersect ρ, so none are in $\mathcal{A}$, and therefore

$$|\mathcal{A} \cap \mathcal{C}| \leq |\mathcal{E}| - d_{n-t-1} = (n-t)! - d_{n-t} - 2d_{n-t-1}$$

Since we are assuming that $|\mathcal{A}| \geq (n-t)! - d_{n-t} - d_{n-t-1} + t$, this means that

$$|\mathcal{A} \setminus \mathcal{C}| \geq d_{n-t-1} + t = (1/e + o(1))(n-t-1)!.$$

Notice that for any $m \leq n$ we have the following trivial upper bound on the size of an m-intersecting family $\mathcal{H} \subset S_n$:

$$|\mathcal{H}| \leq \binom{n}{m}(n-m)! = n!/m!,$$

since every permutation in $\mathcal{H}$ must agree with a fixed permutation in $\mathcal{H}$ in at least m places.

Hence, $\mathcal{A} \setminus \mathcal{C}$ cannot be $(\log n)$-intersecting and therefore contains two permutations π, τ agreeing on at most $\log n$ points. The number of permutations fixing $[t]$ pointwise and agreeing with both π and τ at one of these $\log n$ points is therefore at most $(\log n)(n-t-1)!$. All other permutations in $\mathcal{A} \cap \mathcal{C}$ agree with π and τ at two separate points of $\{t+1, \ldots, n\}$, and by the above argument, the same holds for π_p and τ_q, where p and q are the points of $[t]$ shifted by π and τ respectively. The number of permutations in $\mathcal{C}$ that agree with π_p and τ_q at two separate points of $\{t+1, \ldots, n\}$ is at most $((1-1/e)^2 + o(1))(n-t)!$ (it is easily checked that given two fixed permutations, the probability that a uniform random permutation agrees with them at separate points is at most $(1-1/e)^2 + o(1)$), which implies that

$$\begin{aligned} |\mathcal{A} \cap \mathcal{C}| &\leq ((1-1/e)^2 + o(1))(n-t)! + (\log n)(n-t-1)! \\ &= ((1-1/e)^2 + o(1))(n-t)!, \end{aligned}$$

contradicting (2.9), provided n is sufficiently large. This proves the claim.

Since we are assuming $|\mathcal{A}| \geq |\mathcal{E}| + t$, we must have equality, so $\mathcal{A} = \mathcal{D}$, proving Theorem 2.25. □

Similar arguments give the following stability results for t-cross-intersecting families. We say that two pairs of families $(\mathcal{A},\mathcal{B})$, $(\mathcal{C},\mathcal{D})$ in S_n are *isomorphic* if there exist permutations $\pi,\rho \in S_n$ such that $\mathcal{A} = \pi\mathcal{C}\rho$ and $\mathcal{B} = \pi\mathcal{D}\rho$. We have:

Theorem 2.26. *F n sufficiently large depending on t, if $\mathcal{A},\mathcal{B} \subset S_n$ are t-cross-intersecting but not both contained within the same t-coset, then*

$$\min(|\mathcal{A}|,|\mathcal{B}|) \leq (n-t)! - d_{n-t} - d_{n-t-1} + t.$$

Equality holds if and only if $(\mathcal{A},\mathcal{B})$ is isomorphic to the pair of families

$$\{\sigma : \sigma(i) = i\ \forall i \leq t,\ \sigma(j) = \tau(j) \text{ for some } j > t+1\} \cup \{(i\ t+1) :\ i \in [t]\}$$

$$\{\sigma : \sigma(i) = i\ \forall i \leq t,\ \sigma(j) = j \text{ for some } j > t+1\} \cup \{(1i)\tau(1i) : i \in [t]\}$$

where $\tau(1) \neq 1$ and if $t \geq 2$, τ fixes $2, 3, \ldots, t$ and at least two points $> t+1$, whereas if $t = 1$, τ intersects $(1\ 2)$.

Theorem 2.27. *For n sufficiently large depending on t, if $\mathcal{A},\mathcal{B} \subset S_n$ are t-cross-intersecting but not both contained within the same t-coset, then*

$$|\mathcal{A}||\mathcal{B}| \leq ((n-t)! - d_{n-t} - d_{n-t-1})((n-t)! + t).$$

Equality holds if and only if $(\mathcal{A},\mathcal{B})$ is isomorphic to the pair of families

$$\{\sigma \in S_n : \sigma(i) = i\ \forall i \leq t,\ \sigma(j) = j \text{ for some } j > t+1\}$$

$$\{\sigma \in S_n : \sigma(i) = i\ \forall i \leq t\} \cup \{(1\ t+1), (2\ t+1), \ldots, (t\ t+1)\}$$

The proofs are very similar to the proof of Theorem 2.25, and we omit them.

2.9 Stability for t-intersecting families in A_n

An analogous Hilton-Milner type result holds for A_n. Let e_n, o_n denote the number of respectively even/odd derangements of $[n]$. Recall from Chapter 1 that $e_n - o_n = (-1)^{n-1}(n-1)$ and $d_n = (1/e + o(1))n!$, and therefore $e_n = (1/(2e) + o(1))n!,\ o_n = (1/(2e) + o(1))n!$.

Theorem 2.28. *For n sufficiently large depending on t, if $\mathcal{A} \subset A_n$ is a t-intersecting family which is not contained within a t-coset of A_n, then $\mathcal{A}$ cannot be larger than the family*

$$\begin{aligned} \mathcal{B} \;=\; & \{\sigma \in A_n : \sigma(i) = i\ \forall i \leq t,\ \sigma(j) = (n-1\ n)(j) \text{ for some } j > t+1\} \\ & \cup \{(1\ t+1)(n-1\ n), (2\ t+1)(n-1\ n), \ldots, (t\ t+1)(n-1\ n)\} \end{aligned}$$

which has size $(n-t)!/2 - o_{n-t} - o_{n-t-1} + t = (1-1/e+o(1))(n-t)!/2$*. The extremal families are precisely the double translates* $\{\pi\mathcal{B}\tau : \pi, \tau \in A_n\}$ *of this family.*

Proof. Let $\mathcal{A} \subset A_n$ be a t-intersecting family which is not contained within a t-coset of A_n and has size

$$|\mathcal{A}| \geq (n-t)!/2 - o_{n-t} - o_{n-t-1} + t = (1-1/e+o(1))(n-t)!/2.$$

Applying Theorem 2.22 with any constant c such that $0 < c < (1-1/e)/2$, we see that (provided n is sufficiently large) there exists a t-coset $\mathcal{C}$ such that

$$|\mathcal{A} \setminus \mathcal{C}| \leq \Theta_t(1/n)(n-t)!.$$

By double translation, we may assume without loss of generality that $\mathcal{C} = \{\sigma \in A_n : \sigma(1) = 1, \ldots, \sigma(t) = t\}$. We have:

$$\begin{aligned} |\mathcal{A} \cap \mathcal{C}| &\geq (n-t)!/2 - o_{n-t} - o_{n-t-1} + t - \Theta_t(1/n)(n-t)! \\ &= (1-1/e+o(1))(n-t)!/2. \end{aligned} \tag{2.10}$$

We now claim that every permutation in $\mathcal{A} \setminus \mathcal{C}$ fixes exactly $t-1$ points of $[t]$. Suppose for a contradiction that $\mathcal{A}$ contains a permutation τ fixing at most $t-2$ points of $[t]$. Then every permutation in $\mathcal{A} \cap \mathcal{C}$ must agree with τ in at least 2 points of $\{t+1, \ldots, n\}$, so

$$|\mathcal{A} \cap \mathcal{C}| \leq \binom{n-t}{2}(n-t-2)!/2 = \tfrac{1}{2}(n-t)!/2,$$

contradicting (2.10), provided n is sufficiently large.

Since we are assuming that $\mathcal{A}$ is not contained within a t-coset, $\mathcal{A} \setminus \mathcal{C}$ contains some permutation τ; τ must fix all points of $[t]$ except for one. Without loss of generality, we may assume that $\tau = (1\ t+1)(n-1\ n)$. Every permutation in $\mathcal{A} \cap \mathcal{C}$ must then agree with $(n-1\ n)$ at some point $\geq t+2$, i.e. $\mathcal{A} \cap \mathcal{C}$ is a subset of the family

$$\mathcal{E} := \{\sigma \in A_n : \sigma(i) = i\ \forall i \in [t],\ \sigma(j) = (n-1\ n)(j) \text{ for some } j \geq t+2\},$$

which has size

$$(n-t)!/2 - o_{n-t} - o_{n-t-1}.$$

We now make the following observation:

Claim: $\mathcal{A} \setminus \mathcal{C}$ may only contain permutations of the form $(i\ t+1)(n-1\ n)$

for $i \in [t]$.

Proof of Claim:
Suppose for a contradiction that $\mathcal{A} \setminus \mathcal{C}$ contains a permutation ρ not of this form. Then $\rho(j) \neq (n-1\ n)(j)$ for some $j \geq t+2$, so by a very similar argument as in the proof of Theorem 2.25, there are at least $\min(e_{n-t-1}, o_{n-t-1})$ even permutations which fix $1, 2, \ldots, t$ and agree with $(n-1\ n)$ at j (and are therefore in $\mathcal{E}$) and also disagree with ρ at all points of $\{t+1, t+2, \ldots, n\} \setminus \{j\}$. Since ρ has exactly $t-1$ fixed points in $[t]$, none of these permutations can t-intersect ρ, and therefore

$$\begin{aligned}|\mathcal{A} \cap \mathcal{C}| &\leq |\mathcal{E}| - \min(e_{n-t-1}, o_{n-t-1}) \\ &= (n-t)! - o_{n-t} - o_{n-t-1} - \min(e_{n-t-1}, o_{n-t-1}).\end{aligned}$$

Since we are assuming that $|\mathcal{A}| \geq (n-t)! - o_{n-t} - o_{n-t-1} + t$, this means that

$$|\mathcal{A} \setminus \mathcal{C}| \geq \min(e_{n-t-1}, o_{n-t-1}) + t = (1/e + o(1))(n-t-1)!/2.$$

Notice that for any $m < n$ we have the following trivial upper bound on the size of an m-intersecting family $\mathcal{H} \subset A_n$:

$$|\mathcal{H}| \leq \binom{n}{m}(n-m)!/2 = n!/(2m!),$$

since every permutation in $\mathcal{H}$ must agree with a fixed permutation in $\mathcal{H}$ in at least m places.

Hence, $\mathcal{A} \setminus \mathcal{C}$ cannot be $(\log n)$-intersecting, and therefore contains two permutations π, τ with $|\pi \cap \tau| \leq \log n$. The number of permutations in $\mathcal{C}$ which agree with π and τ simultaneously at some point of $\{t+1, \ldots, n\}$ is therefore at most $(\log n)(n-t-1)!/2$. All other permutations in $\mathcal{A} \cap \mathcal{C}$ agree with π and τ at two separate points of $\{t+1, \ldots, n\}$, and therefore the same holds for π_p and τ_q, where p and q are the unique points of $[t]$ shifted by π and τ respectively. The number of permutations in $\mathcal{C}$ that agree with π_p and τ_q at two separate points of $\{t+1, \ldots, n\}$ is at most $((1-1/e)^2 + o(1))(n-t)!/2$ (it is easily checked that given two fixed permutations, the probability that a uniform random even permutation agrees with them at separate points is $(1-1/e)^2 + o(1)$), which implies that

$$|\mathcal{A} \cap \mathcal{C}| \leq ((1-1/e)^2 + o(1))(n-t)!/2 + (\log n)(n-t-1)!/2$$

contradicting (2.10). This proves the claim.

Since we are assuming $|\mathcal{A}| \geq |\mathcal{E}| + t$, we must have equality, so

$$\begin{aligned} \mathcal{A} &= \{\sigma \in A_n : \sigma(i) = i \ \forall i \leq t,\ \sigma(j) = (n-1\ n)(j) \text{ for some } j > t+1\} \\ &\quad \cup \{(1\ t+1)(n-1\ n), (2\ t+1)(n-1\ n), \ldots, (t\ t+1)(n-1\ n)\}, \end{aligned}$$

proving Theorem 2.28. □

Chapter 3

Irredundant Families of Subcubes

3.1 Introduction

Let $\{0,1\}^n$ denote the n-dimensional discrete cube, the set of all 0-1 vectors of length n. A *k-dimensional subcube* (or *k-subcube*) of $\{0,1\}^n$ is a subset of $\{0,1\}^n$ of the form

$$\{x \in \{0,1\}^n : \ x_i = a_i \ \forall i \in T\}$$

where T is a set of $n-k$ coordinates, called the *fixed coordinates*, and the a_i's are fixed elements of $\{0,1\}$. The other coordinates $S = [n] \setminus T$ are called the *moving coordinates*. We will represent a subcube by an n-tuple of 0's, 1's and $*$'s, where the $*$'s denote moving coordinates and the 0's and 1's denote fixed coordinates. For example, $(*,*,*,0,1)$ denotes a 3-dimensional subcube of $\{0,1\}^5$.

We consider the problem of finding the maximum possible size of a family of k-subcubes of the n-cube $\{0,1\}^n$, none of which is contained in the union of the others. In other words, each has a vertex not contained in any of the others (which we call a 'private' vertex). We will call such a family '*irredundant*', and we write $M(n,k)$ for the maximum size of an irredundant family of k-subcubes of $\{0,1\}^n$.

Let $[n]$ denote the set $\{1,2,\ldots,n\}$. We may identify $\{0,1\}^n$ with $\mathbb{P}[n]$, the set all subsets of $[n]$, by identifying a subset $x \subset [n]$ with its characteristic vector χ_x, defined by

$$\chi_x(i) = 1 \ \forall i \in x, \ \chi_x(i) = 0 \ \forall i \notin x.$$

We write $(0,0,\ldots,0) = \mathbf{0}$ and $(1,1,\ldots,1) = \mathbf{1}$. We will refer to $|x \Delta y|$, the number of coordinates in which x and y differ, as the *Hamming distance* between x and y, and the set

$$\{y \in \{0,1\}^n : |x\Delta y| \leq r\}$$

as the *Hamming ball of centre x and radius r*.

Here are some natural examples of irredundant families:

The family of all translates of a fixed k-subcube,

$$\{A + x : x \in \{0,1\}^n\}$$

where A is a k-subcube of $\{0,1\}^n$ — in other words, the collection of all the subcubes having the same moving coordinates as A. This family partitions $\{0,1\}^n$, so every vertex is a private vertex of its subcube, and it is a maximal irredundant family; it has size 2^{n-k}.

The family $\mathcal{F}_{\mathbf{0}}$ of all k-subcubes containing $\mathbf{0}$, $\{\mathbb{P}x : x \in [n]^{(k)}\}$. Clearly, x is a private vertex of the k-subcube $\mathbb{P}x$; it is the unique such, since any $y \subsetneq x$ can be extended to a different k-set $z \neq x$. This family has size $\binom{n}{k}$. For $k \geq \frac{1}{2}n$ it is maximal, since then any k-subcube contains a k-set. Similarly, for any $v \in Q_n$ we let $\mathcal{F}_v$ be the collection of all k-subcubes through v; we call these the 'principal' irredundant families. Aharoni and Holzman [1] conjectured that for $k > n/2$, there are no larger irredundant families:

Conjecture 3.1 (Aharoni-Holzman, 1991)**.** *If $k > n/2$, any irredundant family of k-subcubes of $\{0,1\}^n$ has size at most $\binom{n}{k}$.*

Aharoni and Holzman (unpublished – see [27]) gave the following general upper bound on the maximum size of an irredundant family of k-subcubes of $\{0,1\}^n$:

$$M(n,k) \leq \sum_{i=k}^{n} \binom{n}{i} \quad \forall k \leq n. \tag{3.1}$$

This may be proved using a short linear independence argument. Mesulam [27] proved the following stronger upper bound using a purely combinatorial argument:

$$M(n,k) \leq \frac{2^n}{\sum_{i=0}^{k} \binom{n}{i}} \binom{n}{k} \quad \forall k \leq n. \tag{3.2}$$

(Intuitively, this is saying that, if there were a partition of $\{0,1\}^n$ into Hamming balls of radius k, it would be best to take the irredundant family of all k-subcubes containing one of the centres of the balls.) We will give a simple proof of Meshulam's bound using Bollobás' Inequality. A variant of this proof shows that if we choose one private vertex for each subcube in an irredundant family, then any Hamming ball of radius k contains at most $\binom{n}{k}$ of these private vertices. (This immediately implies Meshulam's bound by averaging over all Hamming balls of radius k.)

For $k/n > \gamma$, where $\gamma \in (\frac{1}{2}, 1)$ is fixed, Meshulam's bound gives $M(n,k) \leq (1 + o(1))\binom{n}{k}$, i.e. it asymptotically approaches the conjectured bound; if $\gamma \geq \gamma_0 \approx 0.8900$, it gives $M(n,k) < \binom{n}{k} + 1$ for n sufficiently large, proving Conjecture 3.1 in this case.

We observe that equality holds in Meshulam's bound when there is a partition of $\{0,1\}^n$ into Hamming balls of radius k, i.e. in the following cases:

- $k = 1$, $n + 1$ is a power of 2
- $k = 3$, $n = 23$
- $n = 2k + 1$

When $n = 2k + 1$, the irredundant family of all k-subcubes containing either $\mathbf{0}$ or $\mathbf{1}$ has size $2\binom{n}{k}$.

We are then led to investigate the special case when every subcube must go through either $\mathbf{0}$ or $\mathbf{1}$; we prove by an unusual linear algebra argument that for $k \geq n/2$, any irredundant family in which all k-subcubes go through either $\mathbf{0}$ or $\mathbf{1}$ has size at most $\binom{n}{k}$.

Finally, we obtain a general lower bound for all n and k. A probabilistic argument shows that there exists an irredundant family of k-subcubes of $\{0,1\}^n$ of size at least

$$\beta(1-\beta)^{(1-\beta)/\beta} 2^n, \tag{3.3}$$

where

$$\beta := \frac{\binom{n}{k}}{\sum_{i=0}^{k} \binom{n}{i}}.$$

Combining this with Meshulam's bound, we see that

$$\beta(1-\beta)^{(1-\beta)/\beta} 2^n \leq M(n,k) \leq \beta 2^n.$$

The ratio between the upper and lower bound above is at most e for all n and k.

If $k = \lfloor \gamma n \rfloor$ for fixed $\gamma \in (0, \frac{1}{2})$, then

$$\beta = \left(\frac{1-2\gamma}{1-\gamma} \right) (1 + o(1)),$$

so we obtain

$$(1+o(1)) \left(\frac{\gamma}{1-\gamma} \right)^{\frac{\gamma}{1-2\gamma}} \left(\frac{1-2\gamma}{1-\gamma} \right) 2^n \leq M(n, \lfloor \gamma n \rfloor) \leq (1+o(1)) \left(\frac{1-2\gamma}{1-\gamma} \right) 2^n,$$

showing that $M(n, \lfloor \gamma n \rfloor)$ has order of magnitude 2^n.

If $k = o(n)$, we obtain $M(n,k) = (1 - o(1))2^n$.

3.2 Upper bounds

Aharoni and Holzman proved the following:

Proposition 3.2 (Aharoni-Holzman, 1991)**.** *For any $k \leq n$, any irredundant family of k-subcubes of $\{0,1\}^n$ has size at most*

$$\sum_{i=k}^{n} \binom{n}{i}$$

Proof. Let C be a k-subcube of $\{0,1\}^n$; we write $0(C)$ for its set of fixed 0's and $1(C)$ for its set of fixed 1's. The characteristic function χ_C of C can be written as a function of $(x_1, \ldots, x_n) \in \mathbb{R}^n$ as follows:

$$\chi_C(x_1, \ldots, x_n) = \prod_{i \in 0(C)} (1 - x_i) \prod_{i \in 1(C)} x_i \tag{3.4}$$

—for example,

$$\chi_{(1,*,*,*,0)}(x_1, x_2, x_3, x_4, x_5) = x_1(1 - x_5).$$

Now let $\mathcal{A}$ be an irredundant family of k-subcubes of $\{0,1\}^n$. Then

$$\{\chi_C : C \in \mathcal{A}\}$$

is a linearly independent subset of the vector space $\mathbb{R}[x_1, \ldots, x_n]$. To see this, for each $C \in \mathcal{A}$, choose a private vertex $w_C \in C$. Suppose

$$\sum_{C \in \mathcal{A}} a_C \chi_C = 0$$

for some real numbers $\{a_C : C \in \mathcal{A}\}$. Then for any $D \in \mathcal{A}$, evaluating the above on w_D gives:

$$0 = \sum_{C \in \mathcal{A}} a_C \chi_C(w_D) = a_D.$$

It is easy to check that the set of monomials

$$S = \{\prod_{i \in A} x_i : A \in [n]^{(\leq n-k)}\}$$

is a basis for the vector subspace

$$W = \langle \chi_C : C \text{ is a } k\text{-subcube of } \{0,1\}^n \rangle \subset \mathbb{R}[x_1, \ldots, x_n].$$

Hence

$$|\mathcal{A}| \leq \dim(W) = |S| = \sum_{l=0}^{n-k} \binom{n}{l} = \sum_{i=k}^{n} \binom{n}{i},$$

proving the proposition. □

For $k = \lfloor \gamma n \rfloor$, where $\gamma \in (\frac{1}{2}, 1)$, we have:

$$\sum_{i=k}^{n} \binom{n}{i} = \sum_{l=0}^{n-k} \binom{n}{l} \leq \frac{3\gamma - 1}{2\gamma - 1} \binom{n}{\lfloor \gamma n \rfloor},$$

so Proposition 3.2 gives the correct order of magnitude.

For $n = 2k - 1$, however, it only gives $M(2k-1, k) \leq 2^{2k-2}$, compared with $2(1 - o(1))\binom{2k-1}{k}$ from Meshulam's bound.

We now give a proof of Meshulam's bound which we believe to be slightly more intuitive than the proof in [27]. The idea is that for any irredundant family $\mathcal{A}$ and any choice of private vertices, for every $x \in \{0,1\}^n$, the private vertices chosen for the subcubes containing x cannot be too closely packed around x. Our main tool is Bollobás' Inequality:

Theorem 3.3 (Bollobás, 1965)**.** *Let $a_1, \ldots, a_N$ and $b_1, \ldots, b_N$ be subsets of $\{1, 2, \ldots, n\}$ such that $a_i \cap b_j = \emptyset$ if and only if $i = j$. Then*

$$\sum_{i=1}^{N} \binom{|a_i| + |b_i|}{|b_i|}^{-1} \leq 1.$$

Equality holds only if there exists a subset $Y \subset [n]$ and an integer $a \in \mathbb{N}$ such that $\{a_1, \ldots, a_N\} = Y^{(a)}$, and $b_i = Y \setminus a_i$ $\forall i$.

For a proof, we refer the reader to [5].

Given an irredundant family $\mathcal{A}$, we will fix a choice of private vertices, and deduce from Theorem 3.3 an inequality involving the subcubes containing a fixed vertex $x \in Q_n$; we will then sum this inequality over all $x \in Q_n$ to prove bound (3.2).

Theorem 3.4 (Meshulam, 1992)**.** *For any $k \leq n$, if $\mathcal{A}$ is an irredundant family of k-subcubes of $\{0,1\}^n$, then*

$$|\mathcal{A}| \leq \frac{2^n}{\sum_{i=0}^{k} \binom{n}{i}} \binom{n}{k}$$

Proof. Let $\mathcal{A}$ be an irredundant family of k-subcubes of $\{0,1\}^n$, and for each subcube $C \in \mathcal{A}$, choose a private vertex $w_C \in C$.

Claim: For any $x \in \{0,1\}^n$,

$$\sum_{C \in \mathcal{A}: x \in C} \binom{|w_C \Delta x| + n - k}{n - k}^{-1} \leq 1. \tag{3.5}$$

Proof of Claim:
This is an immediate consequence of Bollobás' Inequality. By symmetry, we may assume that $x = \mathbf{0}$. Let $\{C_1, \ldots, C_N\}$ be the collection of subcubes in $\mathcal{A}$ containing $\mathbf{0}$. Each C_i is of the form $\mathbb{P}v_i$ for some k-set v_i. Let $w_i = w_{C_i}$ be the private vertex chosen for C_i. Notice that $w_i \subset v_j$ if and only if $i = j$, i.e. $w_i \cap v_j^c = \emptyset$ if and only if $i = j$, so applying Bollobás' Inequality gives:

$$\sum_{i=1}^{N} \binom{|w_i| + |v_i^c|}{|v_i^c|}^{-1} \leq 1,$$

i.e.

$$\sum_{i=1}^{N} \binom{|w_i| + n - k}{n - k}^{-1} \leq 1, \tag{3.6}$$

proving the claim.

The inequality (3.5) expresses the fact that the private vertices chosen for the subcubes containing x cannot be too densely packed around x. Summing (3.5) over all $x \in \{0,1\}^n$, and interchanging the order of summation, we obtain:

$$\begin{aligned}
2^n &\geq \sum_{x\in\{0,1\}^n} \sum_{\substack{C\in\mathcal{A}:\\ x\in C}} \binom{|w_C \Delta x| + n - k}{n-k}^{-1} \\
&= \sum_{C\in\mathcal{A}} \sum_{x\in C} \binom{|w_C \Delta x| + n - k}{n-k}^{-1} \\
&= |\mathcal{A}| \sum_{l=0}^{k} \frac{\binom{k}{l}}{\binom{l+n-k}{n-k}} \\
&= |\mathcal{A}| \sum_{l=0}^{k} \frac{k!(n-k)!l!}{l!(k-l)!(l+n-k)!} \\
&= |\mathcal{A}| \frac{k!(n-k)!}{n!} \sum_{l=0}^{k} \frac{n!}{(k-l)!(n-(k-l))!} \\
&= \frac{|\mathcal{A}|}{\binom{n}{k}} \sum_{l=0}^{k} \binom{n}{k-l} \\
&= \frac{|\mathcal{A}|}{\binom{n}{k}} \sum_{l=0}^{k} \binom{n}{l}
\end{aligned}$$

Hence,

$$|\mathcal{A}| \leq \frac{2^n}{\sum_{l=0}^{k} \binom{n}{l}} \binom{n}{k}$$

as required. □

As observed by Meshulam, for $k \geq \frac{9}{10}n$, by standard estimates, the bound above is $< \binom{n}{k} + 1$, implying Conjecture 3.1 in this case. More precisely, let

$$H_2(\gamma) = \gamma \log_2(1/\gamma) + (1-\gamma)\log_2(1/(1-\gamma))$$

denote the binary entropy function, and let γ_0 be the unique solution of $H_2(\gamma_0) = \frac{1}{2}$ in $(\frac{1}{2}, 1)$, so that $\gamma_0 = 0.8900$ (to 4 d.p.); then we have the following

Corollary 3.5. *For n sufficiently large, and $k \geq \gamma_0 n$, any irredundant family of k-subcubes of $\{0,1\}^n$ has size at most $\binom{n}{k}$.*

A slight modification of our method yields a result which gives us more 'geometrical' insight into the problem:

Theorem 3.6. *Let B be a Hamming ball of radius k in $\{0,1\}^n$. If $\mathcal{A}$ is an irredundant family of k-subcubes of $\{0,1\}^n$, each with a private vertex in B, then $|\mathcal{A}| \le \binom{n}{k}$.*

Proof. By symmetry, we may assume that $B = [n]^{(\le k)}$. Let $\mathcal{A}$ be an irredundant family of k-subcubes, each with a private vertex in $[n]^{(\le k)}$. For each subcube $C \in \mathcal{A}$, choose a private vertex $w_C \in [n]^{(\le k)}$. Write $C = \{y \in Q_n : v_C \subset y \subset u_C\}$; we will call v_C the 'start vertex' of C and u_C its 'end vertex'. Let $C' = \{y \in Q_n : w_C \subset y \subset u_C\}$ be the $(k - |w_C| + |v_C|)$-dimensional sub-subcube of C between the private vertex and the end vertex of C.

Claim: For any vertex $x \in [n]^{(k)}$,

$$\sum_{C \in \mathcal{A}:\ x \in C'} \binom{|v_C| + k - |w_C|}{k - |w_C|}^{-1} \le 1 \tag{3.7}$$

Proof of Claim:
As before, this is an immediate consequence of Bollobás' Inequality. By symmetry, we may assume that $x = [k]$. Write $\{C \in \mathcal{A} :\ x \in C'\} = \{C_1, \ldots, C_N\}$. Let $v_i = v_{C_i}$ be the start vertex of C_i and $w_i = w_{C_i}$ its private vertex. Clearly, $v_i, w_i \subset [k]$ for every $i \in [N]$. Notice that $v_i \subset w_j$ if and only if $i = j$, i.e. $v_i \cap ([k] \setminus w_j) = \emptyset$ if and only if $i = j$. Hence, Bollobás' Inequality gives:

$$\sum_{i=1}^{N} \binom{|v_i| + k - |w_i|}{k - |w_i|}^{-1} \le 1$$

and the claim is proved.

Summing (3.7) over all $x \in [n]^{(k)}$, and interchanging the order of summation, we obtain:

$$\begin{aligned} \binom{n}{k} &\ge \sum_{x \in [n]^{(k)}} \sum_{\substack{C \in \mathcal{A}:\\ x \in C'}} \binom{|v_C| + k - |w_C|}{k - |w_C|}^{-1} \\ &= \sum_{C \in \mathcal{A}} \sum_{x \in C' \cap [n]^{(k)}} \binom{|v_C| + k - |w_C|}{k - |w_C|}^{-1} \end{aligned}$$

For each subcube $C \in \mathcal{A}$, the $(k - |w_C| + |v_C|)$-dimensional subcube C' contains $\binom{k - |w_C| + |v_C|}{k - |w_C|}$ vertices $x \in [n]^{(k)}$, and for each of them contributes

$\binom{|v_C|+k-|w_C|}{k-|w_C|}^{-1}$ to the above sum, i.e. a total of 1. Hence,

$$|\mathcal{A}| = \sum_{C \in \mathcal{A}} \sum_{x \in C' \cap [n]^{(k)}:\ x \in C'} \binom{|v_C| + k - |w_C|}{k - |w_C|}^{-1} \leq \binom{n}{k},$$

proving the theorem. $\square$

We have equality in Theorem 3.6 if $\mathcal{A}$ is the family of all k-subcubes through the centre of B. Notice that by fixing some choice of private vertices and averaging over all Hamming balls B of radius k, Theorem 3.6 immediately implies Theorem 3.4.

When $n = 2k+1$, the irredundant family of all k-subcubes containing either $\mathbf{0}$ or $\mathbf{1}$ has size $2\binom{n}{k}$, so we have equality in Theorem 3.4 when $n = 2k+1$.

We have been unable to find a counterexample to Conjecture 3.1. Notice that by the same projection argument as in Corollary 6 (see later), if the conjecture holds for n, k then it holds for $n+1, k+1$, so it suffices to consider the case $n = 2k-1$. For $n = 5, k = 3$, the conjecture can be verified by hand, but there are exactly two extremal families up to isomorphism (permuting the coordinates and translating): $\mathcal{F}_{\mathbf{0}}$ and the following family of ten 3-subcubes of Q_5, five through $\mathbf{0}$ and five through $\mathbf{1}$. The (unique) private vertices are indicated above the moving coordinates:

$(\overset{1}{*}, \overset{0}{*}, \overset{1}{*}, 0, 0)$
$(0, \overset{1}{*}, \overset{0}{*}, \overset{1}{*}, 0)$
$(0, 0, \overset{1}{*}, \overset{0}{*}, \overset{1}{*})$
$(\overset{1}{*}, 0, 0, \overset{1}{*}, \overset{0}{*})$
$(\overset{0}{*}, \overset{1}{*}, 0, 0, \overset{1}{*})$
$(\overset{0}{*}, \overset{1}{*}, \overset{0}{*}, 1, 1)$
$(1, \overset{0}{*}, \overset{1}{*}, \overset{0}{*}, 1)$
$(1, 1, \overset{0}{*}, \overset{1}{*}, \overset{0}{*})$
$(\overset{0}{*}, 1, 1, \overset{0}{*}, \overset{1}{*})$
$(\overset{1}{*}, \overset{0}{*}, 1, 1, \overset{0}{*})$

Clearly, this family is not of the form $\mathcal{F}_x$ for any $x \in \{0,1\}^5$. However, we have been unable to find another such example, and we conjecture that for $n > 5$ and $k > n/2$, the only irredudant families of k-subcubes of $\{0,1\}^n$ with size $\binom{n}{k}$ are of the form $\mathcal{F}_x$ for $x \in \{0,1\}^n$.

The best upper bound for $n = 2k-1$ is still Meshulam's bound, which in this case is:

$$\begin{aligned}
M(2k-1,k) &\leq \frac{2^{2k-1}}{2^{2k-2}+\binom{2k-1}{k}}\binom{2k-1}{k} \\
&= \frac{2}{1+2^{-(2k-2)}\binom{2k-1}{k}}\binom{2k-1}{k} \\
&= \frac{2}{1+2(1+o(1))/\sqrt{(2k-1)\pi}}\binom{2k-1}{k} \\
&= 2(1-\Theta(1/\sqrt{k}))\binom{2k-1}{k}.
\end{aligned}$$

To construct a large irredundant family when $k \geq n/2$, one might try just using subcubes containing $\mathbf{0}$ or $\mathbf{1}$, so that the k-subcubes containing $\mathbf{0}$ have private vertices in $[n]^{(\leq k)}$, and the k-subcubes containing $\mathbf{1}$ have private vertices in $[n]^{(\geq n-k)}$. However, a surprising linear algebra argument shows that even when $n = 2k$, such a family has size at most $\binom{n}{k}$:

Theorem 3.7. *If $\mathcal{A}$ is an irredundant family of k-subcubes of $\{0,1\}^{2k}$ which contain $\mathbf{0}$ or $\mathbf{1}$, then $|\mathcal{A}| \leq \binom{2k}{k}$.*

Proof. Let $\mathcal{A}$ be an irredundant family of k-subcubes of $\{0,1\}^{2k}$ which all contain either $\mathbf{0}$ or $\mathbf{1}$. We may assume that $\mathcal{A}$ is maximal with respect to this condition. For $v \in [2k]^{(k)}$, we write

$$\mathbb{U}v := \{y : v \subset y \subset [2k]\}$$

for the k-subcube between v and $[2k]$.

We partition the vertices of the middle layer $[2k]^{(k)}$ into three sets:

$$\begin{aligned}
S &= \{v \in [2k]^{(k)} : \mathbb{P}v, \mathbb{U}v \in \mathcal{A}\}; \\
T &= \{v \in [2k]^{(k)} : \text{ exactly one of } \mathbb{P}v \text{ and } \mathbb{U}v \text{ is in } \mathcal{A}\}; \\
R &= \{v \in [2k]^{(k)} : \mathbb{P}v \notin \mathcal{A}, \mathbb{U}v \notin \mathcal{A}\}.
\end{aligned}$$

Notice that

$$|\mathcal{A}| = \binom{2k}{k} + |S| - |R|;$$

we must show that $|S| \leq |R|$.

Write $S = \{v_1, \ldots, v_N\}$. For each $v_i \in S$, $\mathbb{P}v_i$ must have a private vertex $w_i \in [2k]^{(\leq k-1)}$. If $|w_i| < k-2$, then we may choose $b_i \in [2k]^{(k-1)}$

such that $w_i \subset b_i \subset v_i$; b_i must also be a private vertex for $\mathbb{P}v_i$, since any subcube containing both $\mathbf{0}$ and b_i must contain w_i as well. Similarly, we may choose a private vertex $c_i \in [2k]^{(k+1)}$ for $\mathbb{U}v_i$. Each point of T is a private vertex for the subcube in $\mathcal{A}$ containing it. Let $\mathcal{B} = \{b_1, \ldots, b_N\}$, and let $\mathcal{C} = \{c_1, \ldots, c_N\}$. Then we can choose all the private vertices to lie in $T \cup \mathcal{B} \cup \mathcal{C}$. For each i, let

$$B_i = \{x \in [2k]^{(k)} : b_i \subset x\}, \quad C_i = \{x \in [2k]^k : x \subset c_i\}$$

be the neighbourhoods of b_i and c_i in $[2k]^{(k)}$. First, we claim that

$$\left(\bigcup_{i=1}^{N} B_i\right) \cap \left(\bigcup_{i=1}^{N} C_i\right) = S \cup R.$$

To see this, take $x \in (\cup_{i=1}^{N} B_i) \cap (\cup_{i=1}^{N} C_i)$; then $b_i \subset x \subset c_j$ for some i and j. Suppose $\mathbb{P}x \in \mathcal{A}$; then $b_i \in \mathbb{P}x$, so $x = v_i \in S$, i.e. $\mathbb{U}x \in \mathcal{A}$ as well. Similarly, if $\mathbb{U}x \in \mathcal{A}$, then $\mathbb{P}x \in \mathcal{A}$ as well. Hence, $(\cup_{i=1}^{N} B_i) \cap (\bigcup_{i=1}^{N} C_i) \subset S \cup R$.

Clearly, $S \subset (\cup_{i=1}^{N} B_i) \cap (\cup_{i=1}^{N} C_i)$, as $b_i \subset v_i \subset c_i$ for every i. If $x \in R$, then by the maximality of $\mathcal{A}$, $\mathbb{P}x$ must contain some b_i (otherwise it could be added to $\mathcal{A}$ to produce a larger irredundant family), and similarly $\mathbb{U}x$ must contain some c_j. Hence, $x \in (\cup_{i=1}^{N} B_i) \cap (\cup_{i=1}^{N} C_i)$. It follows that $R \subset (\cup_{i=1}^{N} B_i) \cap (\cup_{i=1}^{N} C_i)$ as well, proving the claim.

For each i, let $B_i' = B_i \cap R = B_i \setminus S$, and let $C_i' = C_i \cap R = C_i \setminus S$; then $B_i', C_i' \subset R$ for each i. We claim that

$$|B_i' \cap C_i'| = 1 \text{ for each } i, \text{ and } |B_i' \cap C_j'| = 0 \text{ or } 2 \text{ for each } i \neq j. \tag{3.8}$$

To see this, first observe that for each i,

$$B_i \cap C_i = \{x \in [2k]^{(k)} : \ b_i \subset x \subset c_i\} = \{v_i, y_i\}$$

for some $y_i \in R$, and therefore

$$B_i' \cap C_i' = \{y_i\}.$$

For each $i \neq j$, if $b_i \not\subset c_j$, then

$$B_i \cap C_j = \emptyset$$

and therefore

$$B_i' \cap C_j' = \emptyset.$$

If $b_i \subset c_j$, then $B_i \cap C_j = \{x \in [2k]^{(k)} : \ b_i \subset x \subset c_j\}$ has size 2, and cannot contain a point of S, since if $b_i \subset v_l \subset c_j$, then $i = j = l$. Hence, $B_i' \cap C_j'$ also has size 2, proving (3.8).

We recall the following easy lemma, the $p = 2$ case of which appears in [4]:

Lemma 3.8. *Let p be prime. If $F_1, \ldots, F_N, G_1, \ldots, G_N \subset [m]$ are such that*

$$\begin{aligned} |F_i \cap G_j| &\equiv 0 \mod p \quad \forall i \neq j \\ \text{and} \quad |F_i \cap G_i| &\not\equiv 0 \mod p \quad \forall i, \end{aligned}$$

then

$$N \leq m.$$

Proof. Let χ_F be the characteristic function of $F \subset [m]$. Consider it as an element of the m-dimensional vector space $\mathbb{F}_p^m$ over $\mathbb{F}_p$. Observe that $\{\chi_{F_1}, \ldots, \chi_{F_N}\}$ is linearly independent over $\mathbb{F}_p$. To see this, suppose

$$\sum_{i=1}^{N} r_i \chi_{F_i} = 0$$

for some $r_1, \ldots, r_N \in \mathbb{F}_p$. Taking the inner product of the above with χ_{G_j} gives $r_j = 0$. Hence, $N \leq m$ as required. □

Applying the $p = 2$ case of this lemma to the sets $B_1', \ldots, B_N', C_1', \ldots, C_N' \subset R$ shows that $|S| \leq |R|$, proving the theorem. □

We immediately obtain the same result for all $n \leq 2k$, by induction on n for fixed codimension $c = n - k$, using a projection argument:

Corollary 3.9. *Let $n \leq 2k$. If $\mathcal{A}$ is an irredundant family of k-subcubes of $\{0,1\}^n$ which contain $\mathbf{0}$ or $\mathbf{1}$, then $|\mathcal{A}| \leq \binom{n}{k}$.*

Proof. Suppose the result is true for some n and k such that $n \geq 2k$; we will prove it for $n+1, k+1$. Let $\mathcal{A}$ be an irredundant family of $(k+1)$-subcubes of $\{0,1\}^{n+1}$ which contain $\mathbf{0}$ or $\mathbf{1}$. Let $\mathcal{A}_i = \{C \in \mathcal{A} : C_i = *\}$ be the collection of subcubes in $\mathcal{A}$ with coordinate i moving; since each subcube has $k+1$ moving coordinates,

$$\sum_{i=0}^{n+1} |\mathcal{A}_i| = (k+1)|\mathcal{A}|.$$

We will show that $|\mathcal{A}_i| \leq \binom{n}{k}$ for each $i \in [n+1]$, giving $|\mathcal{A}| \leq \frac{n+1}{k+1}\binom{n}{k} = \binom{n+1}{k+1}$. Without loss of generality, $i = n+1$. We project the family $\mathcal{A}_{n+1}$ of $(k+1)$-subcubes onto $\{0,1\}^n$: let $\mathcal{A}'_{n+1} = \{C' : C \in \mathcal{A}_{n+1}\}$, where C' is the k-subcube of $\{0,1\}^n$ produced by projecting C onto $\{0,1\}^n$, i.e. deleting the $(n+1)$-coordinate of C (which is a $*$). Clearly, $\mathcal{A}'_{n+1}$ is a collection of $|\mathcal{A}_{n+1}|$ k-subcubes of $\{0,1\}^n$ through $\mathbf{0}$ or $\mathbf{1}$. It is also irredundant, as the projection of a private vertex of C in $\mathcal{A}_{n+1}$ is clearly a private vertex for C' in $\mathcal{A}'_{n+1}$. Hence, by the induction hypothesis, $|\mathcal{A}'_{n+1}| \leq \binom{n}{k}$, giving the result. □

Notice that we do not have uniqueness of the extremal families in Theorem 3.7 for any value of k: as well as taking $\mathcal{A} = \mathcal{F}_{\mathbf{0}}$ or $\mathcal{F}_{\mathbf{1}}$, any family $\mathcal{A}$ containing exactly one of $\mathbb{P}x, \mathbb{U}x$ for each $x \in [2k]^{(k)}$ is extremal. Slightly more surprisingly, we do not have uniqueness (in Corollary 3.9) for $n = 5, k = 3$ either: consider the irredundant family of ten 3-subcubes of $\{0,1\}^5$, five through $\mathbf{0}$ and five through $\mathbf{1}$, exhibited earlier.

3.3 Lower bounds

The case $n = 2k$.

Now, returning to general irredundant families, what can we say about the case $n = 2k$? Meshulam's bound gives:

$$\begin{aligned} M(2k,k) &\leq \frac{2}{1+2^{-2k}\binom{2k}{k}}\binom{2k}{k} \\ &= \frac{2}{1+(1+o(1))/\sqrt{2\pi k}}\binom{2k}{k} \\ &= 2(1-\Theta(1/\sqrt{k}))\binom{2k}{k} \end{aligned}$$

Our lower bound (3.3) no longer beats $\mathcal{F}_{\mathbf{0}}$, since it only gives

$$M(2k,k) \geq \beta(1-\beta)^{(1-\beta)/\beta}2^{2k} = (1+o(1))\frac{\beta}{e(1-\beta)}2^{2k} = (1+o(1))\frac{2}{e}\binom{2k}{k}.$$

Notice that $\mathcal{F}_{\mathbf{0}}$ is a maximal irredundant family. We know from Theorem 3.7 that any irredundant family of k-subcubes in which each goes through either $\mathbf{0}$ or $\mathbf{1}$ has size at most $\binom{2k}{k}$; we now exhibit a maximal such family $\mathcal{B}$ which is not maximal irredundant.

Let $\mathcal{B}_0 = \{\mathbb{P}x : 1 \in x\}$ be the collection of k-subcubes containing the line $(*, 0, 0, ..., 0)$, and $\mathcal{B}_1 = \{\mathbb{U}x : n \notin x\}$ the collection containing $(1, 1, ..., 1, *)$. Consider the family $\mathcal{B} = \mathcal{B}_0 \cup \mathcal{B}_1$; it has size $|\mathcal{B}| = 2\binom{2k-1}{k-1} = \binom{2k}{k}$; we will show that it is irredundant and not maximal. What are the $\mathcal{B}$-private vertices of each subcube $C \in \mathcal{B}$? Write C_i for the symbol ($0, 1$ or $*$) in the i-coordinate of the subcube C. There are 4 different types of subcubes in $\mathcal{B}$ to consider:

- $C \in \mathcal{B}_0$ with $C_n = 0$, e.g. $C = (*, *, \ldots, *, *, 0, \ldots, 0)$ has $\mathcal{B}_0$-private vertices $(*, 1, \ldots, 1, 1, 0, \ldots, 0)$; $(1, 1, \ldots, 1, 1, 0, \ldots, 0) \in (1, 1, \ldots, 1, 1, *, \ldots, *) \in \mathcal{B}_1$, but $(0, 1, \ldots, 1, 1, 0, \ldots, 0) \in [n]^{(k-1)}$ so is not in any $D \in \mathcal{B}_1$, so is the unique $\mathcal{B}$-private vertex of C.

- $C \in \mathcal{B}_0$ with $C_n = *$: e.g. $C = (*, *, \ldots, *, 0, \ldots, 0, *)$ has $\mathcal{B}_0$-private vertices $(*, 1, \ldots, 1, 0, \ldots, 0, 1)$; this line has k fixed 0's in coordinates $\{2, \ldots, n-1\}$ whereas each $D \in \mathcal{B}_1$ has at most $k-1$ $*$'s in this range, hence this line is disjoint from $\mathcal{B}_1$ and both its vertices are the unique $\mathcal{B}$-private vertices of C.

- $C \in \mathcal{B}_1$ with $C_1 = 1$: e.g. $C = (1, *, \ldots, *, 1, \ldots, 1, *)$ has $\mathcal{B}$-private vertex $(1, 0, \ldots, 0, 1, \ldots, 1, 1)$

- $C \in \mathcal{B}_1$ with $C_1 = *$: e.g. $C = (*, *, \ldots, *, 1, \ldots, 1, *)$ has $\mathcal{B}$-private vertices $(0, 0, \ldots, 0, 1, \ldots, 1, *)$

Notice that

$$\cup_{D \in \mathcal{B}_0} D = [n]^{(\leq k-1)} \cup \{x \in [n]^{(k)} : 1 \in x\}$$

and

$$\cup_{D \in \mathcal{B}_1} D = [n]^{(\geq k+1)} \cup \{x \in [n]^{(k)} : n \notin x\}$$

Hence,

$$\{0, 1\}^n \setminus \cup_{D \in \mathcal{B}} D = \{x \in [n]^{(k)} : 1 \notin x, n \in x\}$$

Now let E be any k-subcube with $E_1 = 0, E_n = 1$.
Claim: $\mathcal{B} \cup \{E\}$ is also irredundant.
Proof of Claim: If E has s 0's and t 1's in coordinates $\{2, \ldots, n-1\}$, where $s + t = k - 2$, then setting $k-1-t$ $*$'s $= 1$ and the other $t+1$ $*$'s $= 0$, we find

an $x \in E \cap [n]^{(k)} : 1 \notin x, n \in x$, i.e. a $\mathcal{B}$-private vertex for E. We must now check that each of the above types of subcube in $\mathcal{B}$ has a $\mathcal{B}$-private vertex not in E:

- $C \in \mathcal{B}_0$ with $C_n = 0$: disjoint from E, so the $\mathcal{B}$-private vertex will do.
- $C \in \mathcal{B}_0$ with $C_n = *$: choose the $\mathcal{B}$-private vertex with 1-coordinate 1.
- $C \in \mathcal{B}_1$ with $C_1 = 1$: disjoint from E, so the $\mathcal{B}$-private vertex will do.
- $C \in \mathcal{B}_1$ with $C_1 = *$: choose the $\mathcal{B}$-private vertex with n-coordinate 0.

This proves the claim. How many such subcubes can we add on? We can certainly add on the family:

$$\mathcal{E} = \{E : E_1 = 0, E_n = 1, E_2 = *, E_i = 0 \text{ or } * \,\forall i \neq 1, 2 \text{ or } n\}$$

e.g. the subcube
$(0, *, 0, \ldots, 0, *, \ldots, *, 1)$ has private vertex
$(0, 1, 0, \ldots, 0, 1, \ldots, 1, 1)$.
Hence,

$$M(2k, k) \geq \binom{2k}{k} + \binom{2k-3}{k-1} = (1 + \tfrac{1}{8} + o(1))\binom{2k}{k}$$

but we still have a gap of $\frac{7}{8}$ between the constants in our lower and upper bounds.

Notice the sharp drop by a factor of order $\sqrt{n}$ from $M(n, \lfloor \gamma n \rfloor) = \Theta_\gamma(2^n)$ for $\gamma \in (0, \frac{1}{2})$ to

$$M(n, \lfloor n/2 \rfloor) \leq 2\binom{n}{\lfloor n/2 \rfloor} = 2(1 + o(1))\frac{2^n}{\sqrt{\pi n}}$$

The case $k < \frac{1}{2}n$

When $k < \frac{1}{2}n$, we can construct an irredundant family by taking a union of $\mathcal{F}_v$'s: choose a maximum $(2k+1)$-separated subset $S \subset \{0,1\}^n$ (i.e. a maximum k-error correcting code) and let

$$\mathcal{F}_S = \cup_{v \in S} \mathcal{F}_v$$

be the family of all k-subcubes containing a point of S; then

$$|\mathcal{F}_S| = |S|\binom{n}{k}.$$

When there is a subset $S \subset \{0,1\}^n$ such that the Hamming balls of radius k centred on the vertices of S *partition* $\{0,1\}^n$ (i.e. a perfect k-error correcting code),

$$|\mathcal{F}_S| = \frac{2^n}{\sum_{i=0}^{k} \binom{n}{i}} \binom{n}{k}$$

which exactly matches Meshulam's bound.

It is known that there is a perfect k-error correcting code in $\{0,1\}^n$ precisely in the following cases (see [31]):

- $k = 1$, $n+1$ is a power of 2 (take any Hamming code)
- $k = 3$, $n = 23$ (take the Golay code)
- $n = 2k+1$ (take a 'trivial' code, two vertices of distance n apart)

so in these cases, we have equality in Meshulam's bound:

$$M(n,k) = \frac{2^n}{\sum_{l=0}^{k} \binom{n}{l}} \binom{n}{k}.$$

First, consider the case $k = 1$; a 1-subcube is simply an edge of $\{0,1\}^n$. Meshulam's bound is

$$M(n,1) \leq \tfrac{n}{n+1} 2^n.$$

Kabatyanskii and Panchenko [21] proved the existence of asymptotically perfect packings of 1-balls into $\{0,1\}^n$, namely that there is a packing of

$$\frac{2^n}{n+1}(1 - O(\ln\ln n / \ln n))$$

1-balls into $\{0,1\}^n$. Taking all edges through the centre of each ball gives an irredundant family of size

$$\frac{n}{n+1} 2^n (1 - O(\ln\ln n/\ln n)) = 2^n(1 - O(\ln\ln n/\ln n))$$

We can in fact improve on this with the following 'product' construction. Let $s \in \mathbb{N}$ be maximal such that $2^s - 1 \leq n$; write $n = m + r$ where $m = 2^s - 1$. Take a perfect packing of 1-balls into $\{0,1\}^m$ and take all edges through the centre of each ball, producing an irredundant family $\mathcal{B}$ in $\{0,1\}^m$ of size $\frac{m}{m+1} 2^m$. Writing $\{0,1\}^n = \{0,1\}^m \times \{0,1\}^r$, let $\mathcal{A}$ be the family consisting of a copy of $\mathcal{B}$ in each of the 2^r disjoint copies of $\{0,1\}^r$; $|\mathcal{A}| = \frac{m}{m+1} 2^n$.

Notice that $m = 2^s - 1 \geq \frac{1}{2}n$, since otherwise $2^{s+1} - 1 \leq n$, contradicting the maximality of s. Hence, $|\mathcal{A}| \geq \frac{n}{n+2}2^n$, and we have

$$M(n,1) \geq \frac{n}{n+2}2^n \quad \forall n \in \mathbb{N},$$

so

$$M(n,1) = 2^n(1 - \Theta(1/n)).$$

What about for k fixed and n growing? It is a longstanding open problem in coding theory to determine whether, for k fixed, there is an asymptotically perfect packing of k-balls into $\{0,1\}^n$, i.e. a packing of

$$\frac{2^n}{\sum_{i=0}^{k}\binom{n}{i}}(1 - o(1))$$

k-balls into $\{0,1\}^n$; given such, by taking all k-subcubes through the centre of each ball, we would immediately obtain an irredundant family of size

$$\frac{\binom{n}{k}}{\sum_{l=0}^{k}\binom{n}{l}}2^n(1 - o(1)) = 2^n(1 - o(1))$$

However, this conjecture remains unsolved for all $k > 1$.

Moreover, for $k = \Omega(n)$, the approach outlined above can only give a relatively small irredundant family. Corrádi and Katai [7] proved the following:

Theorem 3.10 (Corrádi-Katai, 1969). *Let $S \subset \{0,1\}^n$ be an $(n/2)$-separated set; then*

- $|S| \leq n+1$ *if n is odd*
- $|S| \leq n+2$ *if* $n \equiv 2 \mod 4$
- $|S| \leq 2n$ *if* $n \equiv 0 \mod 4$

(For a proof of this, we refer the reader for example to [5] §10.)

So we see that, for example, any $(2k+1)$-separated family S of vertices in Q_{4k} must have $|S| \leq 8k$, and so taking all k-subcubes through each of these vertices only gives

$$|\mathcal{F}_S| \leq 8k\binom{4k}{k} \leq 8k \exp\left(-\frac{4k}{32}\right)2^{4k}.$$

We now improve on this using a probabilistic method. The idea is to take a random subset $S \subset \{0,1\}^n$ where each vertex is present independently with some fixed probability p; for each vertex $w \in \{0,1\}^n$ of (Hamming) distance k from S, we choose a k-subcube C_w between w and some vertex of S, giving a random irredundant family of k-subcubes $\mathcal{A} = \{C_w : d(w,S) = k\}$; the expected size of this family is then a lower bound for $M(n,k)$.

Theorem 3.11. *For any $k \leq n$, there exists an irredundant family of k-subcubes of $\{0,1\}^n$ of size at least*

$$\beta(1-\beta)^{(1-\beta)/\beta}2^n,$$

where

$$\beta = \beta_{n,k} := \frac{\binom{n}{k}}{\sum_{i=0}^{k}\binom{n}{i}}.$$

Proof. Let S be a random set of vertices in $\{0,1\}^n$ where each vertex is present independently with probability p (to be chosen later). Consider the random set of vertices

$$W = \{x \in \{0,1\}^n : d(x,S) = k\},$$

where $d(x,y) = |x\Delta y|$ denotes the Hamming distance between x and y. For each $w \in W$, choose any $x_w \in S$ such that $|w\Delta x_w| = k$, and let C_w be the k-subcube between x_w and w, i.e.

$$C_w = \{y \in \{0,1\}^n : \ y\Delta w \subset x_w \Delta w\}.$$

Consider the random family of k-subcubes

$$\mathcal{A} = \{C_w : w \in W\}.$$

Note that the subcubes C_w are pairwise distinct: x_w is the unique point of S in C_w, and w is the 'opposite' point, so C_w determines w. Moreover, $\mathcal{A}$ is irredundant, since w is a private vertex of C_w. (If $w \in C_{w'}$, then $|x_{w'}\Delta w| \leq k$, so $|x_{w'}\Delta w| = k$, so w is the unique vertex in $C_{w'}$ of distance k from $x_{w'}$, so $w = w'$.) We now calculate the expectation of the random variable $|\mathcal{A}| = |W|$. A vertex $v \in \{0,1\}^n$ is in W if and only if the $(k-1)$-ball around v contains no vertices of S but the k-ball around v does contain a vertex of S; the probability of this event is

$$(1-p)^{\sum_{i=0}^{k-1}\binom{n}{i}} - (1-p)^{\sum_{i=0}^{k}\binom{n}{i}}.$$

Hence, the expected size of $\mathcal{A}$ is

$$\mathbb{E}|\mathcal{A}| = 2^n\left((1-p)^{\sum_{i=0}^{k-1}\binom{n}{i}} - (1-p)^{\sum_{i=0}^{k}\binom{n}{i}}\right).$$

Let

$$\beta = \beta_{n,k} := \frac{\binom{n}{k}}{\sum_{i=0}^{k}\binom{n}{i}}, \quad t := (1-p)^{\sum_{i=0}^{k}\binom{n}{i}};$$

then

$$\mathbb{E}|\mathcal{A}| = 2^n(t^{1-\beta} - t).$$

The function

$$\begin{array}{rcl} f : [0,1] & \to & \mathbb{R}; \\ t & \mapsto & t^{1-\beta} - t \end{array}$$

attains its maximum of

$$\beta(1-\beta)^{(1-\beta)/\beta}$$

at

$$t = (1-\beta)^{1/\beta}.$$

Hence, choosing p such that

$$(1-p)^{\sum_{i=0}^{k}\binom{n}{i}} = (1-\beta)^{1/\beta},$$

our random irredudant family has expected size

$$\mathbb{E}|\mathcal{A}| = \beta(1-\beta)^{(1-\beta)/\beta}2^n.$$

Hence, there exists an irredundant family of size at least this, proving the theorem. □

Combining this with Meshulam's bound, we see that

$$\beta(1-\beta)^{(1-\beta)/\beta}2^n \leq M(n,k) \leq \beta 2^n. \tag{3.9}$$

The ratio between the lower and upper bound above is

$$g(\beta) := (1-\beta)^{(1-\beta)/\beta}.$$

Observe that $g'(\beta) > 0 \ \forall \beta \in (0,1)$, so g is strictly increasing on $(0,1)$. Note that

$$\ln(g(\beta)) = \frac{1-\beta}{\beta}\ln(1-\beta) \to -1 \quad \text{as } \beta \to 0,$$

so $g(\beta) \to 1/e$ as $\beta \to 0$; $\ln(g(\beta)) \to 0$ as $\beta \to 1$, so $g(\beta) \to 1$ as $\beta \to 1$. Hence, $1/e \leq g(\beta) \leq 1 \ \forall \beta \in (0,1)$, so the ratio between the upper and lower bounds above never exceeds e. We believe that the upper bound is closer to the true value, but we have been unable to improve our lower bound.

If $k = o(n)$, then $\beta = 1 - o(1)$. Let

$$\eta = 1 - \beta = \frac{\sum_{i=0}^{k-1} \binom{n}{i}}{\sum_{i=0}^{k} \binom{n}{i}};$$

then $\eta = o(1)$.

Theorem 3.11 implies that

$$M(n,k) \geq (1-\eta)\eta^{\eta/(1-\eta)} 2^n = (1 - O(\eta \ln(1/\eta)))2^n;$$

which asymptotically matches the upper bound from Meshulam's theorem,

$$M(n,k) \leq \beta 2^n = (1-\eta)2^n.$$

If $k = \lfloor \gamma n \rfloor$ for some $\gamma \in (0, \frac{1}{2})$, using the fact that as l decreases from $k-1$ to 0, $\binom{n}{l}$ decreases geometrically, we obtain

$$\beta_{n,\lfloor \gamma n \rfloor} = (1 + o(1)) \frac{1-2\gamma}{1-\gamma};$$

substituting this into (3.9) gives:

$$(1+o(1)) \left(\frac{\gamma}{1-\gamma}\right)^{\frac{\gamma}{1-2\gamma}} \left(\frac{1-2\gamma}{1-\gamma}\right) 2^n \leq M(n, \lfloor \gamma n \rfloor) \leq (1+o(1)) \left(\frac{1-2\gamma}{1-\gamma}\right) 2^n.$$

Hence, we see that

$$M(n, \lfloor \gamma n \rfloor) = \Theta_\gamma(2^n).$$

Comparing this with

$$M(n, \lfloor n/2 \rfloor) = \Theta\left(\binom{n}{\lfloor n/2 \rfloor}\right) = \Theta(2^n/\sqrt{n}),$$

we see that $M(n, \lfloor \gamma n \rfloor)$ experiences a drop in its order of magnitude at $\gamma = 1/2$.

Chapter 4

Generating all subsets of a finite set with disjoint unions

4.1 Introduction

We call a family $\mathcal{G} \subset \mathbb{P}[n]$ a *k-generator* of $\mathbb{P}[n]$ if every $x \subset [n]$ can be expressed as a union of at most k disjoint sets in $\mathcal{G}$. Frein, Lévêque and Sebő [25] conjectured that for any $n \geq k$, such a family must be at least as large as the k-generator

$$\mathcal{F}_{n,k} := \bigcup_{i=1}^{k} \mathbb{P}V_i \setminus \{\emptyset\}, \tag{4.1}$$

where $(V_i)_{i=1}^{k}$ is a partition of $[n]$ into k classes of sizes $\lfloor n/k \rfloor$ and $\lceil n/k \rceil$. The $k = 2$ case is a weakening of a conjecture of Erdős, namely that if $\mathcal{G} \subset \mathbb{P}[n]$ is a family such that any subset of $[n]$ is a union (not necessarily disjoint) of at most two sets in $\mathcal{G}$, then $\mathcal{G}$ is at least as large as

$$\mathcal{F}_{n,2} = \mathbb{P}V_1 \cup \mathbb{P}V_2 \setminus \{\emptyset\}, \tag{4.2}$$

where (V_1, V_2) is a partition of $[n]$ into two classes of sizes $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$. We refer the reader to the paper [14] of Füredi and Katona for some results related to this conjecture. In fact, Frein, Lévêque and Sebő [25] made the analagous conjecture for all k. (We call a family $\mathcal{G} \subset \mathbb{P}[n]$ a *k-base* of $\mathbb{P}[n]$ if every $x \subset [n]$ can be expressed as a union of at most k sets in $\mathcal{G}$; they conjectured that for any $k \leq n$, a k-base of $\mathbb{P}[n]$ must be at least as large as $\mathcal{F}_{n,k}$.)

In this chapter, we show that for k fixed, a k-generator of $\mathbb{P}[n]$ must have size at least $k2^{n/k}(1 - o(1))$. When n is a multiple of k, this is asymptotic to $|\mathcal{F}_{n,k}| = k(2^{n/k} - 1)$.

As observed in [25], if $\mathcal{G}$ is a k-generator of $\mathbb{P}[n]$, then the number of subsets of $\mathcal{G}$ of size at most k must be at least the number of subsets of $[n]$, so if $|G| = m$, then

$$\sum_{i=0}^{k} \binom{m}{i} \geq 2^n.$$

Since the number of subsets of $\mathcal{G}$ of size at most $k - 1$ is

$$\sum_{i=0}^{k-1} \binom{m}{i} = \Theta(1/m)\binom{m}{k},$$

it follows that

$$\sum_{i=0}^{k} \binom{m}{i} = (1 + \Theta(1/m))\binom{m}{k} = (1 + \Theta(1/m))m^k/k!.$$

Hence,

$$|\mathcal{G}| \geq (k!)^{1/k} 2^{n/k}(1 - o(1)), \tag{4.3}$$

where the $o(1)$ term tends to 0 as $n \to \infty$, for fixed k. Notice that this argument ignores the requirement that the unions must be disjoint, so (4.3) is also a lower bound on the size of a k-base. We will improve the constant from $(k!)^{1/k} \approx k/e$ to k by taking into account the disjointness requirement. Our main result, Theorem 4.4, states that for any fixed $k \in \mathbb{N}$ and $\delta > 0$, if $m \geq 2^{(1/(k+1)+\delta)n}$, then any family $\mathcal{G} \subset \mathbb{P}[n]$ of size $|\mathcal{G}| = m$ contains at most

$$\left(\frac{k!}{k^k} + o(1)\right)\binom{m}{k}$$

unordered k-tuples $\{A_1, \ldots, A_k\}$ of pairwise disjoint sets, where the $o(1)$ term tends to 0 as $n \to \infty$ for fixed k and δ. In other words, if $\mathcal{H}_n$ denotes the 'Kneser graph' with vertex set $\mathbb{P}[n]$ and edge set $\{xy : x \cap y = \emptyset\}$, the density of K_k's in any sufficiently large induced subgraph of $\mathcal{H}_n$ is at most $k!/k^k + o(1)$.

The $k = 2$ case of Theorem 4.4 was proved by Alon and Frankl (Theorem 1.3 of [2]): for any fixed $\delta > 0$, if $m \geq 2^{(1/3+\delta)n}$, then any family $\mathcal{G} \subset \mathbb{P}[n]$ of size m contains at most

$$\left(\tfrac{1}{2} + o(1)\right)\binom{m}{2}$$

disjoint pairs, where the $o(1)$ term tends to 0 as $n \to \infty$. In other words, the edge-density in any sufficiently large induced subgraph of the 'Kneser graph' $\mathcal{H}_n$ is at most $\frac{1}{2} + o(1)$.

Our result on k-generators will follow quickly from Theorem 4.4. From the trivial bound (4.3), any k-generator $\mathcal{G} \subset \mathbb{P}[n]$ has size $m \geq 2^{n/k}$, so putting $\delta = 1/k(k+1)$, Theorem 4.4 implies that the number of unordered k-tuples of pairwise disjoint sets in $\mathcal{G}$ is at most

$$\left(\frac{k!}{k^k} + o(1)\right)\binom{m}{k}.$$

Hence,

$$2^n \leq \left(\frac{k!}{k^k} + o(1) + \Theta(1/m)\right)\binom{m}{k} = \left(\frac{m}{k}\right)^k (1 + o(1)),$$

and therefore

$$m \geq k2^{n/k}(1 - o(1)),$$

where the $o(1)$ term tends to 0 as $n \to \infty$ for fixed $k \in \mathbb{N}$.

For $k = 2$, this improves the estimate $m \geq \sqrt{2}2^{n/2} - 1$ in [25] (Theorem 5.3) by a factor of $\sqrt{2}$. For n even, it is asymptotically tight, but for n odd, the conjectured smallest 2-generator (4.2) has size $(3/\sqrt{2})2^{n/2} - 1$, so our constant is 'out' by a factor of $3/(2\sqrt{2}) = 1.061$ (to 3 d.p.).

For general k and $n = qk + r$, the conjectured smallest k-generator (4.1) has size

$$(k - r)2^q + r2^{q+1} - k = (k + r)2^{-r/k}2^{n/k} - k$$

so our constant is 'out' by a factor of $(1+r/k)2^{-r/k} \leq 2^{1-1/\ln 2}/\ln 2 = 1.061$ (to 3 d.p.).

It seems that different arguments will be required to improve the constant for $k \nmid n$, or to prove the exact result. Further, it seems likely that proving the same bounds for k-bases (i.e. without the assumption of disjoint unions) would be much harder, and require different techniques altogether.

4.2 Density of K_k's in large subsets of the Kneser graph

First for some terminology and notation. If G is a graph on n vertices, we write $K_r(G)$ for the number of K_r's in G; the *density* of K_r's in G is defined to be $K_r(G)/\binom{n}{r}$. The complete s-partite graph on n vertices with classes of sizes $\lfloor n/s \rfloor$ and $\lceil n/s \rceil$ is known as the *s-partite Turán graph on n vertices.* If S is a set of vertices of G, we write $G[S]$ for the subgraph of G induced on S.

Our starting-point for proving Theorem 4.4 is the following lemma of Alon and Frankl [2], which gives an upper bound on the number of $K_{k+1}(t)$'s in large induced subgraphs of the Kneser graph $\mathcal{H}_n$:

Lemma 4.1 (Alon, Frankl, 1985)**.** *If $\mathcal{G} \subset \mathbb{P}[n]$ with $|\mathcal{G}| = m \geq 2^{(1/(k+1)+\delta)n}$, then the subgraph $\mathcal{H}_n[\mathcal{G}]$ contains at most*

$$(k+1)2^{n(1-\delta t)}\binom{m}{t}^{k+1}\frac{1}{(k+1)!}$$

copies of $K_{k+1}(t)$.

We include their proof for completeness:

Proof. The probability that a t-subset $\{A_1, \ldots, A_t\}$ chosen uniformly at random from $\mathcal{G}$ has union of size at most $\frac{n}{k+1}$ is at most

$$\sum_{S \subset [n]:\ |S| \leq n/(k+1)} \binom{2^{|S|}}{t} \Big/ \binom{m}{t} \leq 2^n (2^{n/(k+1)}/m)^t \leq 2^{n(1-\delta t)}.$$

Choose $k+1$ such t-sets independently at random (with replacement); the probability that at least one has union of size at most $n/(k+1)$ is at most

$$(k+1)2^{n(1-\delta)t}.$$

But this condition holds if our $k+1$ t-sets are the vertex classes of a $K_{k+1}(t)$ in G. Hence, the number of copies of $K_{k+1}(t)$ in $\mathcal{H}_n[\mathcal{G}]$ is at most

$$(k+1)2^{n(1-\delta t)} \binom{m}{t}^{k+1} \frac{1}{(k+1)!},$$

as required. □

We will deduce from this that if $\mathcal{G}$ is a sufficiently large subset of $\mathbb{P}[n]$, then the $K_{k+1}(t)$-density in $\mathcal{H}_n(\mathcal{G})$ is $o(1)$. This in turn will imply that the K_{k+1}-density in $\mathcal{H}_n(\mathcal{G})$ is $o(1)$, via an argument of Erdős. More precisely, we will appeal to the following

Lemma 4.2. *Let $s, t \in \mathbb{N}$ be a fixed positive integers. If G is an n-vertex graph with $p\binom{n}{s}$ K_s's, then the number of $K_s(t)$'s in G is at least*

$$p^{t^s} \frac{\binom{n}{t}^s}{s!}(1 - O(1/n)).$$

Proof. Let $\mathcal{L}$ be the s-uniform hypergraph whose vertices are the vertices of G, and whose edges are the K_s's of G. Then $\mathcal{L}$ has at least $p\binom{n}{s}$ edges. Hence, by an argument of Erdős in [10], $\mathcal{L}$ has at least

$$p^{t^s}(1 - O(1/n)) \frac{\binom{n}{t}^s}{s!}$$

complete s-partite sub-hypergraphs with classes of size t. Each corresponds to a different copy of $K_s(t)$ in G, proving the lemma. □

Fix a sufficiently large positive integer l; we will see that since the K_{k+1}-density in $\mathcal{H}_n(\mathcal{G})$ is $o(1)$, all but a $o(1)$-fraction of the l-subsets of $\mathcal{G}$ are K_{k+1}-free. We will then appeal to another result of Erdős [11] to bound the K_k-density in each of these l-subsets:

Theorem 4.3 (Erdős, 1962)**.** *If G is a K_{s+1}-free graph on n vertices, then it contains at most as many K_r's as the s-partite Turán graph on n vertices.*

Provided l is sufficiently large depending on ϵ, the K_k-density in the k-partite Turán graph on l vertices is at most $k!/k^k + \epsilon$, so the K_k-density in all but at most a $o(1)$ fraction of the l-subsets of $\mathcal{G}$ is at most $k!/k^k + \epsilon$. An averaging argument will immediately imply that the K_k-density in the whole graph $\mathcal{H}_n(\mathcal{G})$ is at most $k!/k^k + \epsilon + o(1)$, proving the main theorem.

Here then is the formal statement and proof of the main theorem:

Theorem 4.4. *For any fixed $k \in \mathbb{N}$ and $\delta > 0$, if $m \geq 2^{(1/(k+1)+\delta)n}$, then any family $\mathcal{G} \subset \mathbb{P}[n]$ of size $|\mathcal{G}| = m$ contains at most*

$$\left(\frac{k!}{k^k} + o(1)\right)\binom{m}{k}$$

unordered k-tuples $\{A_1, \ldots, A_k\}$ of pairwise disjoint sets, where the $o(1)$ term tends to 0 as $n \to \infty$ for fixed k, δ.

Proof. Let $\mathcal{G} \subset \mathbb{P}[n]$ of size $|\mathcal{G}| = m \geq 2^{(1/(k+1)+\delta)n}$. Choosing $t = \lceil 2/\delta \rceil$, Lemma 4.1 implies that $\mathcal{H}_n(\mathcal{G})$ contains at most

$$(k+1)2^{-n}\frac{\binom{m}{t}^{k+1}}{(k+1)!}$$

copies of $K_{k+1}(t)$, i.e. the $K_{k+1}(t)$-density in $\mathcal{H}_n(\mathcal{G})$ is $o(1)$. Let p be the K_{k+1}-density in $\mathcal{H}_n(\mathcal{G})$. It follows from Lemma 4.2 that

$$p^{t^{k+1}}(1 - O(1/m)) \leq (k+1)2^{-n},$$

and therefore

$$p \leq c_k 2^{-n/t^{k+1}}$$

for some constant c_k depending on k alone, i.e. the K_{k+1}-density in $\mathcal{H}_n(\mathcal{G})$ is $o(1)$ as well.

Let $l \in \mathbb{N}$; observe that the K_k-density in the k-partite Turán graph on l vertices tends to $k!/k^k$ as $l \to \infty$. Let $\epsilon > 0$, and choose $l \in \mathbb{N}$ sufficiently large depending on ϵ such that this density is at most $k!/k^k + \epsilon$.

Let N be the number of l-subsets of $\mathcal{G}$ containing a K_{k+1}. Since each

K_{k+1} in $\mathcal{G}$ is contained in $\binom{m-k-1}{l-k-1}$ l-sets, we have

$$\begin{aligned} N &\leq p\binom{m}{k+1}\binom{m-k-1}{l-k-1} \\ &\leq c_k 2^{-n/t^{k+1}}\binom{m}{k+1}\binom{m-k-1}{l-k-1} \\ &= c_k 2^{-n/t^{k+1}}\binom{m}{l}\binom{l}{k+1} \\ &= o(1)\binom{m}{l}, \end{aligned}$$

—i.e. all but at most a $o(1)$-fraction of the l-subsets of $\mathcal{G}$ are K_{k+1}-free. Applying Theorem 4.3 with $r = s = k$ and m in place of n, we see that each of the K_{k+1}-free l-subsets of $\mathcal{G}$ has K_k-density at most the K_k-density in the k-partite Turán graph on l vertices, which is at most $k!/k^k + \epsilon$. It follows immediately by averaging over all l-subsets of $\mathcal{G}$ that the K_k-density in $\mathcal{H}_n(\mathcal{G})$ is at most $k!/k^k + \epsilon + o(1)$. In detail, let q be the K_k-density in $\mathcal{H}_n(\mathcal{G})$. Double-counting the number of times an l-subset of $\mathcal{G}$ contains a K_k, we obtain:

$$\begin{aligned} q\binom{m}{k}\binom{m-k}{l-k} &\leq N\binom{l}{k} + \left(\binom{m}{l} - N\right)\left(\frac{k!}{k^k} + \epsilon\right)\binom{l}{k} \\ &= (o(1) + (1-o(1))(k!/k^k + \epsilon))\binom{m}{l}\binom{l}{k} \\ &= (k!/k^k + \epsilon + o(1))\binom{m}{l}\binom{l}{k} \\ &= (k!/k^k + \epsilon + o(1))\binom{m}{k}\binom{m-k}{l-k}, \end{aligned}$$

and therefore

$$q \leq \frac{k!}{k^k} + \epsilon + o(1).$$

Since $\epsilon > 0$ was arbitrary, it follows that

$$q \leq \frac{k!}{k^k} + o(1),$$

proving the theorem. □

As explained above, our result on k-generators quickly follows:

Theorem 4.5. *Let $k \in \mathbb{N}$ be a fixed positive integer. Then any k-generator of $\mathbb{P}[n]$ must contain at least $k2^{n/k}(1-o(1))$ sets.*

Proof. Let $\mathcal{G}$ be a k-generator of $\mathbb{P}[n]$, with $|\mathcal{G}| = m$. The trivial counting argument in the Introduction implies that $m \geq 2^{n/k}$, so applying Theorem 4.4 with $\delta = 1/k(k+1)$, we see that the number of ways of choosing k pairwise disjoint sets in $\mathcal{G}$ is at most

$$\left(\frac{k!}{k^k} + o(1)\right)\binom{m}{k}.$$

The number of ways of choosing less than k pairwise disjoint sets is, very crudely, at most $\sum_{i=0}^{k-1}\binom{m}{i} = \Theta(1/m)\binom{m}{k}$; since every subset of $[n]$ is a disjoint union of at most k sets in $\mathcal{G}$, we obtain

$$2^n \leq \left(\frac{k!}{k^k} + o(1) + \Theta(1/m)\right)\binom{m}{k} = \left(\frac{m}{k}\right)^k (1+o(1))$$

(where the o(1) term tends to 0 as $n \to \infty$), and therefore

$$m \geq k2^{n/k}(1-o(1))$$

(where the $o(1)$ term tends to 0 as $n \to \infty$). □

Appendix A

The Strong Duality Theorem of Linear Programming

For the reader's convenience, we reproduce a short proof of the Strong Duality Theorem of Linear Programming. This relies on the closely related Farkas' Lemma, a consequence of the Separating Hyperplane Theorem:

Theorem A.1. *(Separating Hyperplane Theorem) Let $S \subset \mathbb{R}^n$ be a closed, convex set, and $b \in \mathbb{R}^n \setminus S$. Then there exists a vector $c \in \mathbb{R}^n$ such that*

$$c \cdot b < c \cdot s \quad \forall s \in S$$

Proof. By a compactness argument, there exists a $y \in S$ of minimum distance from b, i.e. such that

$$||y - b||^2 \leq ||x - b||^2 \quad \forall x \in S$$

Let $c = y - b$. We claim that $c \cdot b < c \cdot x \ \forall x \in S$. Take any $x \in S$. By convexity of S, for any $t \in [0, 1]$, $tx + (1 - t)y = y + t(x - y) \in S$. Hence, by our choice of y,

$$||y - b||^2 \leq ||y + t(x - y) - b||^2 = ||y - b||^2 + 2t(y - b) \cdot (x - y) + t^2||x - y||^2$$

so

$$2t(y - b) \cdot (x - y) + t^2||x - y||^2 \geq 0$$

Letting $t \downarrow 0$ gives

$$(y - b) \cdot (x - y) \geq 0$$

and therefore

$$\begin{aligned} c \cdot x &= (y-b)\cdot(x-y) + (y-b)\cdot y \\ &\geq (y-b)\cdot y \\ &= (y-b)\cdot b + (y-b)\cdot(y-b) \\ &> (y-b)\cdot y \\ &= c \cdot y \end{aligned}$$

as required. □

Lemma A.2. *(Farkas' Lemma) Let A be a real $m \times n$ matrix, and $b \in \mathbb{R}^m$. Then the system of inequalities*

$$(Ax = b, \quad x \geq 0)$$

is unsolvable iff there exists $p \in \mathbb{R}^m$ such that $p^\top A \geq 0$ and $p^\top b < 0$.

Proof. Let $S = \{Ax : x \geq 0\} \subset \mathbb{R}^m$. Note that S is a closed, convex set. Suppose the system of inequalities is unsolvable, i.e. $b \notin S$. Then by the hyperplane separation theorem, there exists a vector $p \in \mathbb{R}^m$ such that $p^\top b < p^\top s \quad \forall s \in S$. Since $0 \in S$, $p^\top b < 0$. Note also that for any $j \in [n]$ and any $\lambda \geq 0$, $\lambda A e_j \in S$. Hence, $p^\top(\lambda A e_j) > p^\top b$. Letting $\lambda \to \infty$ gives $p^\top A e_j \geq 0$. Hence, $p^\top A \geq 0$ as required.

Clearly, if there exists $p \in \mathbb{R}^m$ such that $p^\top A \geq 0$ and $p^\top b < 0$, then for any $x \geq 0$, $p^\top(Ax) = (p^\top A)x \geq 0$, and therefore the system of inequalities above is unsolvable. □

A restatement of Farkas' Lemma is as follows: if A is a real $m \times n$ matrix with columns $A_1, \ldots, A_n$, and $b \in \mathbb{R}^m$ is such that $p^\top b \geq 0$ whenever $p^\top A_i \geq 0 \; \forall i \in [n]$, then b is a non-negative linear combination of columns of A, i.e. there exists a vector $x \in \mathbb{R}^n$ such that $Ax = b$.

Theorem A.3. *(Strong Duality Theorem of Linear Programming) Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$. If the linear programming problem:*

$$\textit{minimize} \quad c^\top x \quad \textit{subject to} \quad Ax \geq b$$

has an optimal solution x^*, then the dual problem:

$$\textit{maximize} \quad \lambda^\top b \quad \textit{subject to} \quad \lambda^\top A = c^\top, \; \lambda \geq 0$$

has an optimal solution p with the same value, i.e. $p^\top b = c^\top x^*$.

Proof. For $i \in [m]$, let $a_i \in \mathbb{R}^n$ be the ith row of A; let

$$I = \{i \in [m] : a_i^\top x^* = b_i\}$$

be the set of constraints that are tight at x^*. Note that if $y \in \mathbb{R}^n$ is such that $y^\top a_i \geq 0 \ \forall i \in I$, then $y^\top c \geq 0$, otherwise for $\epsilon > 0$ sufficiently small, $A(x^* - \epsilon y) \geq b$ but $c^\top(x^* - \epsilon y) > c^\top x^*$, contradicting the optimality of x^*. Hence, by Farkas' Lemma, c is a non-negative linear combination of the vectors $(a_i : i \in I)$, i.e. there exist $(p_i \geq 0 : i \in I)$ such that

$$c = \sum_{i \in I} p_i a_i$$

Define $p_i = 0 \ \forall i \notin I$; then the vector $p \in \mathbb{R}^m$ satisfies $p^\top A = c^\top$ and

$$p^\top b = \sum_{i \in I} p_i b_i = \sum_{i \in I} p_i a_i^\top x^* = \left(\sum_{i \in I} p_i a_i \right)^\top x^* = c^\top x^*$$

Hence, p is a solution to the dual problem with the same value, $b^\top p = c^\top x^*$. Note that if x is any solution to the primal problem and λ is any solution to the dual problem, then

$$\lambda^\top b \leq \lambda^\top A x = c^\top x$$

and therefore any other solution λ to the dual problem satisfies $\lambda^\top b \leq p^\top b$, so p is an optimal solution to the dual problem. □

We apply this in the proof of Theorem 2.13 by noting that the system of inequalities (2.5) is solvable iff the problem

$$\text{minimize} \quad \sum_i x_i + \sum_j y_j \quad \text{subject to} \quad x_i + y_j \geq -a_{i,j} \ (1 \leq i, j \leq n)$$

has an optimal solution with value ≤ 0.

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