# A proof of the Cameron-Ku conjecture

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**Abstract**

A family of permutations $\mathcal{A} \subset S_n$ is said to be *intersecting* if any two permutations in $\mathcal{A}$ agree at some point, i.e. for any $\sigma, \pi \in \mathcal{A}$, there is some $i$ such that $\sigma(i) = \pi(i)$. Deza and Frankl [5] showed that if $\mathcal{A} \subset S_n$ is intersecting, then $|\mathcal{A}| \leq (n-1)!$. Cameron and Ku [4] showed that if equality holds, then $\mathcal{A} = \{\sigma \in S_n : \sigma(i) = j\}$ for some $i$ and $j$. They conjectured a 'stability' version of this result, namely that there exists a constant $c < 1$ such that if $\mathcal{A} \subset S_n$ is an intersecting family of size at least $c(n-1)!$, then there exist $i$ and $j$ such that every permutation in $\mathcal{A}$ maps $i$ to $j$ (we call such a family 'centred'). They also made a stronger 'Hilton-Milner' type conjecture, namely, that for $n \geq 6$, if $\mathcal{A} \subset S_n$ is a non-centred intersecting family, then $\mathcal{A}$ cannot be larger than the family $\mathcal{C} = \{\sigma \in S_n : \sigma(1) = 1, \sigma(i) = i \text{ for some } i > 2\} \cup \{(12)\}$, which has size $(1 - 1/e + o(1))(n-1)!$.

We prove the stability conjecture, and also the Hilton-Milner type conjecture for $n$ sufficiently large. Our proof makes use of the classical representation theory of $S_n$. One of our key tools will be an extremal result on cross-intersecting families of permutations, namely that for $n \geq 4$, if $\mathcal{A}, \mathcal{B} \subset S_n$ are cross-intersecting, then $|\mathcal{A}||\mathcal{B}| \leq ((n-1)!)^2$. This was a conjecture of Leader [15]; it was proved for $n$ sufficiently large by Friedgut, Pilpel and the author in [7].

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## 1 Introduction

We work in the symmetric group $S_n$, the group of all permutations of $\{1, 2, \ldots, n\} = [n]$. A family of permutations $\mathcal{A} \subset S_n$ is said to be *intersecting* if any two permutations in $\mathcal{A}$ agree at some point, i.e. for any $\sigma, \pi \in \mathcal{A}$, there is some $i \in [n]$ such that $\sigma(i) = \pi(i)$.

It is natural to ask: how large can an intersecting family be? The family of all permutations fixing 1 is an obvious example of a large intersecting family of permutations; it has size $(n-1)!$. More generally, for any $i, j \in [n]$, the collection of all permutations mapping $i$ to $j$ is clearly an intersecting family of the same size; we call these the '*1-cosets*' of $S_n$, since they are the cosets of the point-stabilizers.

Deza and Frankl [5] showed that if $\mathcal{A} \subset S_n$ is intersecting, then $|\mathcal{A}| \leq (n-1)!$; this is known as the Deza-Frankl theorem. They gave a short, direct 'partitioning' proof: take any $n$-cycle $\rho$, and let $H$ be the cyclic group of order $n$ generated by $\rho$. For any left coset $\sigma H$ of $H$, any two distinct permutations in $\sigma H$ disagree at every point, and therefore $\sigma H$ contains at most 1 member of $\mathcal{A}$. Since the left cosets of $H$ partition $S_n$, it follows that $|\mathcal{A}| \leq (n-1)!$.

Deza and Frankl conjectured that equality holds only for the 1-cosets of $S_n$. This turned out to be harder than expected; it was eventually proved independently by Cameron and Ku [4] and Larose and Malvenuto [14]; Wang and Zhang [19] have recently given a shorter proof.

We say that an intersecting family $\mathcal{A} \subset S_n$ is *centred* if there exist $i, j \in [n]$ such that every permutation in $\mathcal{A}$ maps $i$ to $j$, i.e. $\mathcal{A}$ is contained within a 1-coset of $S_n$. Cameron and Ku asked how large a *non-centred* intersecting family can be. Experimentation suggests that the further an intersecting family is from being centred, the smaller it must be. The following are natural examples of large non-centred intersecting families:

- $\mathcal{B} = \{\sigma \in S_n : \sigma \text{ fixes at least two points in } [3]\}$.

  This has size $3(n-2)! - 2(n-3)!$.
  It requires the removal of $(n-2)! - (n-3)!$ permutations to make it centred.

- $\mathcal{C} = \{\sigma : \sigma(1) = 1,\ \sigma \text{ intersects } (1\ 2)\} \cup \{(1\ 2)\}$.

  *Claim:* $|\mathcal{C}| = (1 - 1/e + o(1))(n-1)!$.
  *Proof of Claim:* Let $\mathcal{D}_n = \{\sigma \in S_n :\ \sigma(i) \neq i\ \forall i \in [n]\}$ denote the set of *derangements* of $[n]$ (permutations in $S_n$ without fixed points); let $d_n = |\mathcal{D}_n|$ denote the number of derangements of $[n]$. By the inclusion-exclusion formula,

  $$d_n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)! = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!} = n!(1/e + o(1)).$$

  Note that a permutation which fixes 1 intersects (1 2) if and only if it has a fixed point greater than 2. The number of permutations fixing 1 alone is clearly $d_{n-1}$; the number of permutations fixing 1 and 2 alone is clearly $d_{n-2}$, so the number of permutations fixing 1 and some other point $> 2$ is $(n-1)! - d_{n-1} - d_{n-2}$. Hence,

  $$|\mathcal{C}| = (n-1)! - d_{n-1} - d_{n-2} = (1 - 1/e + o(1))(n-1)!$$

as required.

Note that $\mathcal{C}$ can be made centred just by removing (1 2).

For $n \leq 5$, $\mathcal{B}$ and $\mathcal{C}$ have the same size; for $n \geq 6$, $\mathcal{C}$ is larger. Cameron and Ku [4] conjectured that for $n \geq 6$, $\mathcal{C}$ has the largest possible size of any non-centred intersecting family. Further, they conjectured that if $\mathcal{A} \subset S_n$ is a non-centred intersecting family of the same size as $\mathcal{C}$, then $\mathcal{A}$ must be a 'double translate' of $\mathcal{C}$, meaning that there exist $\pi, \tau \in S_n$ such that $\mathcal{A} = \pi\mathcal{C}\tau$. Note that if $\mathcal{F} \subset S_n$, any double translate of $\mathcal{F}$ has the same size as $\mathcal{F}$, is intersecting if and only if $\mathcal{F}$ is, and is centred if and only if $\mathcal{F}$ is. Double-translation will be our notion of 'isomorphism' for intersecting families of permutations.

We prove the Cameron-Ku conjecture for all sufficiently large $n$. This implies the weaker 'stability' conjecture of Cameron and Ku [4] — namely, that there exists a constant $c > 0$ such that any intersecting family $\mathcal{A} \subset S_n$ of size at least $(1-c)(n-1)!$ is centred. We prove the latter using a slightly shorter argument.

Our proof makes use of the classical representation theory of $S_n$. One of our key tools will be an extremal result on cross-intersecting families of permutations. A pair of families of permutations $\mathcal{A}, \mathcal{B} \subset S_n$ is said to be *cross-intersecting* if for any $\sigma \in \mathcal{A}, \tau \in \mathcal{B}$, $\sigma$ and $\tau$ agree at some point, i.e. there is some $i \in [n]$ such that $\sigma(i) = \tau(i)$. Leader [15] conjectured that for $n \geq 4$, if $\mathcal{A}, \mathcal{B}$ are cross-intersecting, then $|\mathcal{A}||\mathcal{B}| \leq ((n-1)!)^2$, with equality if and only if $\mathcal{A} = \mathcal{B} = \{\sigma \in S_n : \sigma(i) = j\}$ for some $i, j \in [n]$. Note that the statement does not hold for $n = 3$, as the pair

$$\mathcal{A} = \{(1), (123), (321)\}, \ \mathcal{B} = \{(12), (23), (31)\}$$

is cross-intersecting with $|\mathcal{A}||\mathcal{B}| = 9$.

A $k$-cross-intersecting generalization of Leader's conjecture was proved by Friedgut, Pilpel and the author in [7], for $n$ sufficiently large depending on $k$. In order to prove the Cameron-Ku conjecture for $n$ sufficiently large, we could in fact make do with the $k = 1$ case of this result. For completeness, however, we sketch a simpler proof of Leader's conjecture for all $n \geq 4$, based on the eigenvalues of the derangement graph rather than those of the weighted graph constructed in [7]. Interestingly, no combinatorial proof of Leader's conjecture is known.

There is a close analogy between intersecting families of permutations and intersecting families of $r$-sets, which we now describe. As usual, let $[n]^{(r)}$ denote the set of all $r$-element subsets ('$r$-sets') of $[n]$. We say that a family $\mathcal{A} \subset [n]^{(r)}$ is *intersecting* if any two of its sets have nonempty intersection. The classical Erdős-Ko-Rado theorem states that if $r < n/2$, then the largest intersecting families of $r$-subsets of $[n]$ are the 'stars', meaning the families of the form $\{x \in [n]^{(r)} : \ i \in x\}$ for $i \in [n]$. This corresponds to the fact that the largest intersecting families of permutations in $S_n$ are the 1-cosets.

We say that an intersecting family of $r$-sets is *trivial* if there is an element in all of its sets. Hilton and Milner [9] proved that for $r \geq 4$ and $n > 2r$, if $\mathcal{A} \subset [n]^{(r)}$ is a non-trivial intersecting family of maximum size, then

$$\mathcal{A} = \{x \in [n]^{(r)} : i \in [n], \ x \cap y \neq \emptyset\} \cup \{y\}$$

for some $i \in [n]$ and some $r$-set $y$ not containing $i$, so it can be made into a trivial family by removing just one $r$-set. The Cameron-Ku conjecture is an exact analogue of this for permutations.

## 2 Cross-intersecting families of permutations

Our aim in this section is to prove Leader's conjecture: if $n \geq 4$, and $\mathcal{A}, \mathcal{B} \subset S_n$ are cross-intersecting, then $|\mathcal{A}||\mathcal{B}| \leq ((n-1)!)^2$. We do this by applying a 'cross-independent' analogue of Hoffman's eigenvalue bound to the derangement graph; the eigenvalues of the derangement graph are analysed using the representation theory of $S_n$.

The *derangement graph* is the graph $\Gamma$ with vertex-set $S_n$, where two permutations are joined if they disagree everywhere, i.e.

$$V(\Gamma) = S_n, \quad E(\Gamma) = \{\sigma\tau : \ \sigma, \tau \in S_n, \ \sigma(i) \neq \tau(i) \ \forall i \in [n]\}.$$

Recall that if $G$ is a finite group, and $S \subset G$ is inverse-closed ($S^{-1} = S$), the *Cayley graph on G generated by S* is the graph with vertex-set $G$, where we join $g$ to $gs$ for each $g \in G$ and each $s \in S$. Clearly, the derangement graph $\Gamma$ is the Cayley graph on $S_n$ generated by $\mathcal{D}_n$, the set of derangments of $[n]$, so it is $d_n$-regular. Of course, an intersecting family of permutations in $S_n$ is precisely an independent set in $\Gamma$, and $\mathcal{A}, \mathcal{B} \subset S_n$ are cross-intersecting if and only if there are no edges of $\Gamma$ between them.

Hoffman's theorem provides an upper bound on the maximum size of an independent set in a regular graph in terms of the minimum eigenvalue of the adjacency matrix of the graph. Recall that if $H = (V, E)$ is an $N$-vertex graph, the *adjacency matrix* of $H$ is the 0-1 matrix with rows and columns indexed by $V$, and with

$$A_{x,y} = \begin{cases} 1 & \text{if } xy \in E(H); \\ 0 & \text{otherwise.} \end{cases}$$

Since $A$ is a real symmetric matrix, all its eigenvalues are real, and for any inner product on $\mathbb{R}^V$, we can find an orthonormal basis of $\mathbb{R}^V$ consisting of real eigenvectors of $A$. Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N = \lambda_{\min}$ be the eigenvalues of $A$, repeated with their multiplicities. It is easy to see that if $H$ is $d$-regular, then $\lambda_1 = d$. Hoffman's bound is as follows:

**Theorem 2.1.** *(Hoffman's bound)*
*Let $H = (V, E)$ be a d-regular graph, and let $A$ be the adjacency matrix of $H$. Let $\lambda_{\min}$ denote the least eigenvalue of $A$. If $X \subset V(H)$ is an independent set in $H$, then*

$$\frac{|X|}{|V|} \leq \frac{-\lambda_{\min}}{d - \lambda_{\min}}.$$

*If equality holds, then the characterstic vector $v_X$ of $X$ satisfies:*

$$v_X - \tfrac{|X|}{|V|}\mathbf{1} \in Ker(A - \lambda_{\min} I).$$

Ku conjectured that the minimum eigenvalue of the derangement graph is $-d_n/(n-1)$. This was first proved by Renteln [17], using symmetric functions. Substituting this value into Hoffman's bound implies that an intersecting family of permutations in $S_n$ has size at most $(n-1)!$, recovering the theorem of Deza and Frankl.

To deal with cross-intersecting families, we first prove an analogue of Hoffman's bound for 'cross-independent' sets; this is a variant of a result in [3].

**Theorem 2.2.** *(i) Let $H = (V, E)$ be a d-regular graph on $N$ vertices, whose adjacency matrix $A$ has eigenvalues $\lambda_1 = d \geq \lambda_2 \geq \ldots \geq \lambda_N$. Let $\nu = \max(|\lambda_2|, |\lambda_N|)$. Suppose $X$ and $Y$ are sets of vertices of $\Gamma$ with no edges between them, i.e. $xy \notin E(\Gamma)$ for every $x \in X$ and $y \in Y$. Then*

$$\sqrt{|X||Y|} \leq \frac{\nu}{d + \nu} N. \tag{1}$$

*(ii) Suppose further that $|\lambda_2| \neq |\lambda_N|$, and let $\lambda'$ be the larger in modulus of the two. Let $v_X, v_Y \in \mathbb{R}^V$ be the characteristic vectors of $X, Y$ and let $\mathbf{f}$ denote the all-1's vector in $\mathbb{R}^V$. If equality holds in (1), then $|X| = |Y|$, and we have*

$$v_X - (|X|/N)\mathbf{f},\ v_Y - (|Y|/N)\mathbf{f} \ \in \ Ker(A - \lambda' I).$$

*Proof.* This is a straightforward extension of the proof of Hoffman's theorem, with an application of the Cauchy-Schwarz inequality. Equip $\mathbb{R}^V$ with the inner product:

$$\langle u, v \rangle = \frac{1}{N} \sum_{i=1}^{N} u(i)v(i),$$

and let

$$||u|| = \sqrt{\frac{1}{N} \sum_{i=1}^{N} u(i)^2}$$

be the induced Euclidean norm. Let $u_1 = \mathbf{f}, u_2, \ldots, u_N$ be an orthonormal basis of real eigenvectors of $A$ corresponding to the eigenvalues $\lambda_1 =$

$d, \lambda_2, \ldots, \lambda_N$. Let $X, Y$ be as above; let $\alpha = |X|/N$, and let $\beta = |Y|/N$. Write

$$v_X = \sum_{i=1}^{N} \xi_i u_i, \quad v_Y = \sum_{i=1}^{N} \eta_i u_i$$

as linear combinations of the eigenvectors of $A$. We have $\xi_1 = \alpha$, $\eta_1 = \beta$, and

$$\sum_{i=1}^{N} \xi_i^2 = ||v_X||^2 = |X|/N = \alpha, \quad \sum_{i=1}^{N} \eta_i^2 = ||v_Y||^2 = |Y|/N = \beta.$$

Since there are no edges of $H$ between $X$ and $Y$, we have:

$$0 = \sum_{x \in X, y \in Y} A_{x,y} = v_Y^\top A v_X = \sum_{i=1}^{N} \lambda_i \xi_i \eta_i = d\alpha\beta + \sum_{i=2}^{N} \lambda_i \xi_i \eta_i \geq d\alpha\beta - \nu \left| \sum_{i=2}^{N} \xi_i \eta_i \right|. \tag{2}$$

Provided $|\lambda_2| \neq |\lambda_N|$, if we have equality above, then $\xi_i = \eta_i = 0$ unless $\lambda_i = d$ or $\lambda'$, so $v_X - \alpha \mathbf{f}$ and $v_Y - \beta \mathbf{f}$ are both $\lambda'$-eigenvectors of $A$.

The Cauchy-Schwarz inequality gives:

$$\left| \sum_{i=2}^{N} \xi_i \eta_i \right| \leq \sqrt{\sum_{i=2}^{N} \xi_i^2 \sum_{i=2}^{N} \eta_i^2} = \sqrt{(\alpha - \alpha^2)(\beta - \beta^2)}.$$

Substituting this into (2) gives:

$$d\alpha\beta \leq \nu \sqrt{(\alpha - \alpha^2)(\beta - \beta^2)},$$

so

$$\frac{\alpha\beta}{(1-\alpha)(1-\beta)} \leq (\nu/d)^2.$$

By the AM/GM inequality, $(\alpha + \beta)/2 \geq \sqrt{\alpha\beta}$ with equality if and only if $\alpha = \beta$, so

$$\frac{\alpha\beta}{(1 - \sqrt{\alpha\beta})^2} = \frac{\alpha\beta}{1 - 2\sqrt{\alpha\beta} + \alpha\beta} \leq \frac{\alpha\beta}{1 - \alpha - \beta + \alpha\beta} \leq (\nu/d)^2,$$

implying that

$$\sqrt{\alpha\beta} \leq \frac{\nu}{d + \nu}.$$

Hence, we have

$$\sqrt{|X||Y|} \leq \frac{\nu}{d + \nu} N.$$

Provided $|\lambda_2| \neq |\lambda_N|$, we have equality only if $|X| = |Y| = \frac{\nu}{d+\nu} N$ and $v_X - \alpha \mathbf{f}$ and $v_Y - \beta \mathbf{f}$ are both $\lambda'$-eigenvectors of $A$, completing the proof. □

We will show that for $n \geq 5$, the derangement graph satisfies the hypotheses of this result with $\nu = d_n/(n-1)$; in fact, $\lambda_N = -\frac{d_n}{n-1}$, and $\max_{i \neq 1,N} |\lambda_i| = O((n-2)!)$. The derangement graph is a *normal* Cayley graph, meaning that its generating set is a union of conjugacy-classes; as is well-known, there is a particularly nice correspondence between eigenspaces of normal Cayley graphs, and irreducible representations of the group.

Note that the least eigenvalue of the derangement graph was first calculated by Renteln [17], using symmetric functions, and somewhat later by Ku and Wales [13], and by Godsil and Meagher [8]. We analyse the eigenvalues of the derangement graph differently, employing a convenient trick known as the 'trace method' to bound all eigenvalues of high multiplicity. The idea of the trace method is simple: if $H$ is a graph on $N$ vertices, whose adjacency matrix $A$ has eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N$, then for any $k \in \mathbb{N}$,

$$\text{Trace}(A^k) = \sum_{i=1}^{N} \lambda_i^k.$$

On the other hand, $\text{Trace}(A^k)$ is also the number of closed walks of length $k$ in $H$. We will apply this with $k = 2$; $\text{Trace}(A^2)$ is simply twice the number of edges of $H$.

### Background on general representation theory

We now recall the concepts we need from general representation theory. Readers familiar with representation theory may wish to skip this section; others may wish to refer to [10] for additional information.

Let $G$ be a finite group, and let $F$ be a field. A *representation of $G$ over $F$* is a pair $(\rho, V)$, where $V$ is a finite-dimensional vector space over $F$, and $\rho : G \to GL(V)$ is a group homomorphism from $G$ to the group of all invertible linear endomorphisms of $V$. The vector space $V$, together with the linear action of $G$ defined by $gv = \rho(g)(v)$, is sometimes called an *$FG$-module.* A *homomorphism* between two representations $(\rho, V)$ and $(\rho', V')$ is a linear map $\phi : V \to V'$ such that $\phi(\rho(g)(v)) = \rho'(g)(\phi(v))$ for all $g \in G$ and $v \in V$. If there exists such a $\phi$ which is bijective, then the two representations are said to be *isomorphic*, or *equivalent*, and we write $(\rho, V) \cong (\rho', V')$. If $\dim(V) = n$, we say that $\rho$ *has dimension $n$.*

The representation $(\rho, V)$ is said to be *irreducible* if it has no proper subrepresentation, i.e. there is no proper subspace of $V$ which is $\rho(g)$-invariant for all $g \in G$. *Schur's lemma* states that if $(\rho, V)$ is an irreducible representation of $G$ over $\mathbb{C}$, then the only linear endomorphisms of $V$ which commute with $\rho$ are scalar multiples of the identity.[1]

[1]Recall that a linear endomorphism $\alpha$ of $V$ is said to *commute* with $\rho$ if $\alpha$ commutes with $\rho(g)$ for all $g \in G$, i.e. $\rho(g) \circ \alpha = \alpha \circ \rho(g)$ for all $g \in G$.

When $F = \mathbb{R}$ or $\mathbb{C}$, it turns out that there are only finitely many isomorphism classes of irreducible representations of $G$, and *any* representation of $G$ is isomorphic to a direct sum of irreducible representations of $G$.

If $(\rho, V)$ is a representation of $V$, the *character* $\chi_\rho$ of $\rho$ is the map

$$\begin{aligned}\chi_\rho : G &\rightarrow F;\\ \chi_\rho(g) &= \mathrm{Trace}(\rho(g)).\end{aligned}$$

The usefulness of characters lies in the fact that *two complex representations are isomorphic if and only if they have the same character.*

Given two representations $(\rho, V)$ and $(\rho', V')$ of $G$, we can form their direct sum, the representation $(\rho \oplus \rho', V \oplus V')$, and their tensor product, the representation $(\rho \otimes \rho', V \otimes V')$. We have $\chi_{\rho\oplus\rho'} = \chi_\rho + \chi_{\rho'}$, and $\chi_{\rho\otimes\rho'} = \chi_\rho \cdot \chi_{\rho'}$ (the pointwise product).

The *group algebra* $FG$ denotes the $F$-vector space with basis $G$ and multiplication defined by extending the group multiplication linearly. In other words,

$$FG = \left\{ \sum_{g\in G} x_g g : \ x_g \in F \ \forall g \in G \right\},$$

and

$$\left(\sum_{g\in G} x_g g\right) \left(\sum_{h\in G} y_h h\right) = \sum_{g,h\in G} x_g y_h (gh).$$

As a vector space, $FG$ may be identified with $F[G]$, the vector-space of all $F$-valued functions on $G$, by identifying $\sum_{g\in G} x_g g$ with the function $g \mapsto x_g$. The representation defined by

$$\rho(g)(x) = gx \quad (g \in G, \ x \in FG)$$

is called the *left regular representation* of $G$; the corresponding $FG$-module is called the *group module.*

Let $\Gamma$ be a graph on $G$, and let $A$ be the adjacency matrix of $\Gamma$. We may consider $A$ as a linear operator on either $\mathbb{R}[G]$ or $\mathbb{C}[G]$; it makes no difference to the eigenvalues, but the latter is more convenient in general. We have

$$Af(g) = \sum_{\substack{h\in G:\\ gh\in E(\Gamma)}} f(h) \quad \forall f \in \mathbb{C}[G], \ g \in G.$$

If $\Gamma$ is a Cayley graph on $G$ with (inverse-closed) generating set $S$, then idenfying $\mathbb{C}G$ with $\mathbb{C}[G]$, the adjacency matrix of $\Gamma$ acts on $\mathbb{C}G$ by right-multiplication by $\sum_{s\in S} s$:

$$\begin{aligned} A\left(\sum_{g\in G} x_g g\right) &= \sum_{g\in G}\sum_{s\in S} x_{gs} g \\ &= \sum_{h\in G}\sum_{s\in S} x_h (hs^{-1}) \\ &= \sum_{h\in G}\sum_{s\in S} x_h (hs) \\ &= \left(\sum_{g\in G} x_g g\right)\left(\sum_{s\in S} s\right). \end{aligned}$$

If $\Gamma$ is a normal Cayley graph, then $\sum_{s\in S} s$ lies in the *centre* of $\mathbb{C}G$ — i.e., it commutes with every $x \in \mathbb{C}G$. This leads, via Schur's lemma, to an explicit 1-1 correspondence between the eigenvalues of $\Gamma$ and the isomorphism classes of irreducible representations of $G$:

**Theorem 2.3.** *(Schur; Babai; Diaconis, Shahshahani; Roichman.)*
*Let $G$ be a finite group, let $S \subset G$ be an inverse-closed, conjugation-invariant subset of $G$, and let $\Gamma$ be the Cayley graph on $G$ with generating set $S$. Let $A$ denote the adjacency matrix of $\Gamma$. Let $(\rho_1, V_1), \ldots, (\rho_k, V_k)$ be a complete set of non-isomorphic irreducible representations of $G$ — i.e., containing one representative from each isomorphism class of irreducible representations of $G$. Let $U_i$ be the sum of all submodules of the group module $\mathbb{C}G$ which are isomorphic to $V_i$. We have*

$$\mathbb{C}G = \bigoplus_{i=1}^{k} U_i,$$

*and each $U_i$ is an eigenspace of $A$ with dimension* $\dim(V_i)^2$ *and eigenvalue*

$$\lambda_{V_i} = \frac{1}{\dim(V_i)} \sum_{s\in S} \chi_i(s),$$

*where $\chi_i(g) = Trace(\rho_i(g))$ denotes the* character *of the irreducible representation $(\rho_i, V_i)$.*

Given $x \in \mathbb{C}G$, its projection onto the eigenspace $U_i$ can be found as follows. Write $\mathrm{Id} = \sum_{i=1}^{k} e_i$ where $e_i \in U_i$ for each $i \in [k]$. The $e_i$'s are called the *primitive central idempotents* of $\mathbb{C}G$; $U_i$ is the two-sided ideal of $\mathbb{C}G$ generated by $e_i$, and $e_i$ is given by the following formula:

$$e_i = \frac{\dim(V_i)}{|G|} \sum_{g\in G} \chi_i(g^{-1}) g. \tag{3}$$

For any $x \in \mathbb{C}G$, $x = \sum_{i=1}^{k} e_i x$ is the unique decomposition of $x$ into a sum of elements of the $U_i$'s; in other words, the projection of $x$ onto $U_i$ is $e_i x$.

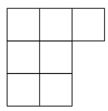
## Background on the representation theory of the symmetric group

We now collect the results we need from the representation theory of $S_n$; as in [7], our treatment follows [11] and [18]. Readers who are familiar with the representation theory of $S_n$ may wish to skip this section.

A *partition* of $n$ is a non-increasing sequence of positive integers summing to $n$, i.e. a sequence $\alpha = (\alpha_1, \ldots, \alpha_k)$ with $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_k \geq 1$ and $\sum_{i=1}^{k} \alpha_i = n$; we write $\alpha \vdash n$. For example, $(3, 2, 2) \vdash 7$; we sometimes use the shorthand $(3, 2, 2) = (3, 2^2)$.

The *cycle-type* of a permutation $\sigma \in S_n$ is the partition of $n$ obtained by expressing $\sigma$ as a product of disjoint cycles and listing its cycle-lengths in non-increasing order. Two permutations in $S_n$ are conjugate if and only if they have the same cycle-type, so the conjugacy classes of $S_n$ are in explicit 1-1 correspondence with the partitions of $n$. Moreover, there is an explicit 1-1 correspondence between partitions of $n$ and isomorphism classes of irreducible representations of $S_n$, which we now describe.

Let $\alpha = (\alpha_1, \ldots, \alpha_k)$ be a partiton of $n$. The *Young diagram* of $\alpha$ is an array of $n$ cells, having $k$ left-justified rows where row $i$ contains $\alpha_i$ cells. For example, the Young diagram of the partition $(3, 2^2)$ is



An $\alpha$-*tableau* is produced by placing the numbers $1, 2, \ldots, n$ into the cells of the Young diagram of $\alpha$ in some order; for example,

| | | |
|---|---|---|
| 6 | 1 | 7 |
| 5 | 4 | |
| 3 | 2 | |

is a $(3, 2^2)$-tableau. Two $\alpha$-tableaux are said to be *row-equivalent* if for each row, they have the same numbers in that row. If an $\alpha$-tableau $t$ has rows $R_1, \ldots, R_k \subset [n]$ and columns $C_1, \ldots, C_l \subset [n]$, we let $R_t = S_{R_1} \times S_{R_2} \times \ldots \times S_{R_k}$ be the row-stablizer of $t$ and $C_t = S_{C_1} \times S_{C_2} \times \ldots \times S_{C_l}$ be the column-stabilizer.

An $\alpha$-*tabloid* is an $\alpha$-tableau with unordered row entries (or formally, a row-equivalence class of $\alpha$-tableaux); given a tableau $t$, we write $[t]$ for the tabloid it produces. For example, the $(3, 2^2)$-tableau above produces the following $(3, 2^2)$-tabloid

$$\begin{array}{lll} \{1 & 6 & 7\} \\ \{4 & 5\} & \\ \{2 & 3\} & \end{array}$$

Consider the natural left action of $S_n$ on the set $X^\alpha$ of all $\alpha$-tabloids; let $M^\alpha = \mathbb{C}[X^\alpha]$ be the corresponding permutation module, i.e. the complex vector space with basis $X^\alpha$ and $S_n$ action given by extending this action linearly. Given an $\alpha$-tableau $t$, we define the corresponding *$\alpha$-polytabloid*

$$e_t := \sum_{\pi \in C_t} \epsilon(\pi)\pi[t].$$

We define the *Specht module* $S^\alpha$ to be the submodule of $M^\alpha$ spanned by the $\alpha$-polytabloids:

$$S^\alpha = \mathrm{Span}\{e_t : \ t \text{ is an } \alpha\text{-tableau}\}.$$

A central observation in the representation theory of $S_n$ is that *the Specht modules are a complete set of pairwise non-isomorphic, irreducible representations of $S_n$*. Hence, any irreducible representation $\rho$ of $S_n$ is isomorphic to some $S^\alpha$. For example, $S^{(n)} = M^{(n)}$ is the trivial representation; $M^{(1^n)}$ is the left-regular representation, and $S^{(1^n)}$ is the sign representation $S$.

We say that a tableau is *standard* if the numbers strictly increase along each row and down each column. It turns out that for any partition $\alpha$ of $n$,

$$\{e_t : t \text{ is a standard } \alpha\text{-tableau}\}$$

is a basis for the Specht module $S^\alpha$.

Given a partition $\alpha$ of $n$, for each cell $(i, j)$ in its Young diagram, we define the *hook length* $(h^\alpha_{i,j})$ to be the number of cells in its 'hook' (the set of cells in the same row to the right of it or in the same column below it, including itself) — for example, the hook lengths of $(3, 2^2)$ are as follows:

| | | |
|---|---|---|
| 5 | 4 | 1 |
| 3 | 2 | |
| 2 | 1 | |

The dimension $f^\alpha$ of the Specht module $S^\alpha$ is given by the following formula

$$f^\alpha = n!/\prod(\text{hook lengths of } \alpha). \tag{4}$$

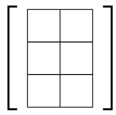
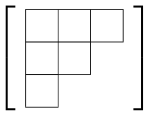
From now on we will write $[\alpha]$ for the equivalence class of the irreducible representation $S^\alpha$, $\chi_\alpha$ for the irreducible character $\chi_{S^\alpha}$, and $\xi_\alpha$ for the character of the permutation representation $M^\alpha$. Notice that the set of $\alpha$-tabloids form a basis for $M^\alpha$, and therefore $\xi_\alpha(\sigma)$, the trace of the corresponding permutation representation at $\sigma$, is precisely the number of $\alpha$-tabloids fixed by $\sigma$.

If $U \in [\alpha]$, $V \in [\beta]$, we define $[\alpha] + [\beta]$ to be the equivalence class of $U \oplus V$, and $[\alpha] \otimes [\beta]$ to be the equivalence class of $U \otimes V$.

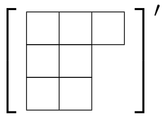
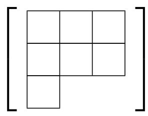
The Branching Theorem (see [11] §2.4) states that for any partition $\alpha$ of $n$, the restriction $[\alpha] \downarrow S_{n-1}$ is isomorphic to a direct sum of those irreducible representations $[\beta]$ of $S_{n-1}$ such that the Young diagram of $\beta$ can be obtained from that of $\alpha$ by deleting a single cell, i.e., if $\alpha^{i-}$ is the partition whose Young diagram is obtained by deleting the cell at the end of the $i$th row of that of $\alpha$, then

$$[\alpha] \downarrow S_{n-1} = \sum_{i:\alpha_i > \alpha_{i-1}} [\alpha^{i-}]. \tag{5}$$

For example, if $\alpha = (3, 2^2)$, we obtain:

$[3, 2^2] \downarrow S_6 =$  $+$  $= [2^3] + [3, 2, 1].$

For any partition $\alpha$ of $n$, we have $S^{(1^n)} \otimes S^\alpha \cong S^{\alpha'}$, where $\alpha'$ is the transpose of $\alpha$, the partition of $n$ with Young diagram obtained by interchanging rows with columns in the Young diagram of $\alpha$. Hence, $[1^n] \otimes [\alpha] = [\alpha']$, and $\chi_{\alpha'} = \epsilon \cdot \chi_\alpha$. For example, we obtain:

$[3, 2, 2] \otimes [1^7] =$  $=$  $= [3, 3, 1].$

We now explain how the permutation modules $M^\beta$ decompose into irreducibles.

**Definition.** *Let $\alpha, \beta$ be partitions of $n$. A* generalized $\alpha$-tableau *is produced by replacing each dot in the Young diagram of $\alpha$ with a number between 1 and $n$; if a generalized $\alpha$-tableau has $\beta_i$ $i$'s $(1 \le i \le n)$ it is said to have* content $\beta$. *A generalized $\alpha$-tableau is said to be* semistandard *if the numbers are non-decreasing along each row and strictly increasing down each column.*

**Definition.** *Let $\alpha, \beta$ be partitions of $n$. The* Kostka number $K_{\alpha,\beta}$ *is the number of semistandard generalized $\alpha$-tableaux with content $\beta$.*

*Young's Rule* states that for any partition $\beta$ of $n$, the permutation module $M^\beta$ decomposes into irreducibles as follows:

$$M^\beta \cong \oplus_{\alpha \vdash n} K_{\alpha,\beta} S^\alpha.$$

For example, $M^{(n-1,1)}$, which corresponds to the natural permutation action of $S_n$ on $[n]$, decomposes as

$$M^{(n-1,1)} \cong S^{(n-1,1)} \oplus S^{(n)},$$

and therefore

$$\xi_{(n-1,1)} = \chi_{(n-1,1)} + 1. \tag{6}$$

We now return to considering the derangement graph. Write $U_\alpha$ for the sum of all copies of $S^\alpha$ in $\mathbb{C}S_n$. Note that $U_{(n)} = \text{Span}\{\boldsymbol{f}\}$ is the subspace of constant vectors in $\mathbb{C}S_n$. Applying Theorem 2.3 to the derangement graph $\Gamma$, we have

$$\mathbb{C}S_n = \bigoplus_{\alpha \vdash n} U_\alpha,$$

and each $U_\alpha$ is an eigenspace of the derangement graph, with dimension $\dim(U_\alpha) = (f^\alpha)^2$ and corresponding eigenvalue

$$\lambda_\alpha = \frac{1}{f^\alpha} \sum_{\sigma \in \mathcal{D}_n} \chi_\alpha(\sigma). \tag{7}$$

We will use the following result, a variant of a result in [11]; for the reader's convenience, we include a proof using the Branching Theorem and the Hook Formula.

**Lemma 2.4.** *For $n \geq 9$, the only Specht modules $S^\alpha$ of dimension $f^\alpha < \binom{n-1}{2} - 1$ are as follows:*

- $S^{(n)}$ *(the trivial representation), dimension 1;*
- $S^{(1^n)}$ *(the sign representation $S$), dimension 1;*
- $S^{(n-1,1)}$*, dimension $n-1$;*
- $S^{(2,1^{n-2})}$ *($\cong S \otimes S^{(n-1,1)}$), dimension $n-1$.*

$(*)$

This is well-known, but for completeness we include a proof using the Branching Theorem and the Hook Formula.

*Proof.* By direct calculation using (4), the lemma can be verified for $n = 9, 10$. We proceed by induction. Assume the lemma holds for $n-2, n-1$; we will prove it for $n$. Let $\alpha$ be a partition of $n$ such that $f^\alpha < \binom{n-1}{2} - 1$. Consider the restriction $[\alpha] \downarrow S_{n-1}$, which has the same dimension. First suppose $[\alpha] \downarrow S_{n-1}$ is reducible. If it has one of our 4 irreducible representations $(*)$ as a constituent, then by (5), the possibilies for $\alpha$ are as follows:

| constituent | possibilies for $\alpha$ |
|---|---|
| $[n-1]$ | $(n), (n-1,1)$ |
| $[1^{n-1}]$ | $(1^n), (2, 1^{n-1})$ |
| $[n-2,1]$ | $(n-1,1), (n-2,2), (n-2,1,1)$ |
| $[2,1^{n-3}]$ | $(2,1^{n-2}), (2,2,1^{n-4}), (3,1^{n-3})$ |

But using (4), we see that the new irreducible representations above all have dimension at least $\binom{n-1}{2} - 1$:

| $\alpha$ | $f^\alpha$ |
|---|---|
| $(n-2,2),(2,2,1^{n-4})$ | $\binom{n-1}{2}-1$ |
| $(n-2,1,1),(3,1^{n-3})$ | $\binom{n-1}{2}$ |

Hence, none of these are constituents of $[\alpha]\downarrow S_{n-1}$. So we may assume that the irreducible constituents of $[\alpha]\downarrow S_{n-1}$ do not include any of our 4 irreducible representations $(*)$, so by the induction hypothesis for $n-1$, each has dimension at least $\binom{n-2}{2}-1$. But $2(\binom{n-2}{2}-1)\geq\binom{n-1}{2}-1$ provided $n\geq 11$, so there is just one, i.e. $[\alpha]\downarrow S_{n-1}$ is irreducible. Therefore $[\alpha]=[s^t]$ for some $s,t\in\mathbb{N}$ with $st=n$, i.e. it has square Young diagram. Now consider

$$[\alpha]\downarrow S_{n-2}=[s^{t-1},s-2]+[s^{t-2},s-1,s-1].$$

Note that neither of these 2 irreducible constituents are any of our 4 irreducible representations $(*)$, so by the induction hypothesis for $n-2$, each has dimension at least $\binom{n-3}{2}-1$. But $2(\binom{n-3}{2}-1)\geq\binom{n-1}{2}-1$ for $n\geq 11$, contradicting $\dim([\alpha]\downarrow S_{n-2})<\binom{n-1}{2}-1$. □

If $\alpha$ is any partition of $n$ whose Specht module has high dimension $f^\alpha\geq\binom{n-1}{2}-1$, we may bound $|\lambda_\alpha|$ using the 'trace method' — specifically, we consider the trace of $A^2$:

**Lemma 2.5.** *Let $H$ be a graph on $N$ vertices whose adjancency matrix $A$ has eigenvalues $\lambda_1\geq\lambda_2\geq\ldots\geq\lambda_N$; then*

$$\sum_{i=1}^{N}\lambda_i^2=\mathit{Trace}(A^2)=2e(H).$$

This is well-known; we include a proof for completeness.

*Proof.* Diagonalize $A$: there exists a real invertible matrix $P$ such that $A=P^{-1}DP$, where $D$ is the diagonal matrix

$$D=\begin{pmatrix}\lambda_1 & 0 & \ldots & 0\\ 0 & \lambda_2 & & 0\\ \vdots & & \ddots & \vdots\\ 0 & & \ldots & \lambda_N\end{pmatrix}.$$

We have $A^2=P^{-1}D^2P$, and therefore

$$2e(H)=\sum_{i,j=1}^{N}A_{i,j}=\sum_{i,j=1}^{N}A_{i,j}^2=\mathrm{Trace}(A^2)=\mathrm{Trace}(P^{-1}D^2P)=\mathrm{Trace}(D^2)=\sum_{i=1}^{N}\lambda_i^2,$$

as required. □

Hence, the eigenvalues of the derangement graph satisfy:

$$\sum_{\alpha \vdash n} (f^{\alpha}\lambda_{\alpha})^2 = 2e(\Gamma) = n!d_n = (n!)^2(1/e + o(1)),$$

so for each partition $\alpha$ of $n$,

$$|\lambda_{\alpha}| \leq \frac{\sqrt{n!d_n}}{f^{\alpha}} = \frac{n!}{f^{\alpha}}\sqrt{1/e + o(1)}.$$

Let

$$\mathcal{M} = \left\{\alpha \vdash n : \ f^{\alpha} \geq \binom{n-1}{2} + 1\right\};$$

we have

$$\max_{\alpha \in \mathcal{M}} |\lambda_{\alpha}| \leq O((n-2)!).$$

For each of the other Specht modules $(*)$, we now explicitly calculate the corresponding eigenvalue using (7).

For the trivial module, $\chi_{(n)} \equiv 1$, so

$$\lambda_{(n)} = d_n.$$

For the sign module $S^{(1^n)}$, $\chi_{(1^n)} = \epsilon$, so

$$\lambda_{(1^n)} = \sum_{\sigma \in \mathcal{D}_n} \epsilon(\sigma) = e_n - o_n$$

where $e_n, o_n$ are the number of even and odd derangements of $[n]$, respectively. It is well known that for any $n \in \mathbb{N}$,

$$e_n - o_n = (-1)^{n-1}(n-1). \tag{8}$$

To see this, note that an odd permutation $\sigma \in S_n$ without fixed points can be written as $(i\ n)\rho$, where $\sigma(n) = i$, and $\rho$ is either an even permutation of $[n-1] \setminus \{i\}$ with no fixed points (if $\sigma(i) = n$), or an even permutation of $[n-1]$ with no fixed points (if $\sigma(i) \neq n$). Conversely, for any $i \neq n$, if $\rho$ is any even permutation of $[n-1]$ with no fixed points or any even permutation of $[n-1] \setminus \{i\}$ with no fixed points, then $(i\ n)\rho$ is a permutation of $[n]$ with no fixed points taking $n \mapsto i$. Hence, for all $n \geq 3$,

$$o_n = (n-1)(e_{n-1} + e_{n-2}).$$

Similarly,

$$e_n = (n-1)(o_{n-1} + o_{n-2}).$$

Equation (8) follows by induction on $n$.

Hence, we have:

$$\lambda_{(1^n)} = (-1)^{n-1}(n-1).$$

For the partition $(n-1,1)$, from (6) we have:

$$\chi_{(n-1,1)}(\sigma) = \xi_{(n-1,1)}(\sigma) - 1 = \#\{\text{fixed points of } \sigma\} - 1,$$

so we obtain

$$\lambda_{(n-1,1)} = \frac{1}{n-1} \sum_{\sigma \in \mathcal{D}_n} (-1) = -\frac{d_n}{n-1}.$$

For $S^{(2,1^{n-2})} \cong S^{(1^n)} \otimes S^{(n-1,1)}$, we have $\chi_{(2,1^{n-2})} = \epsilon \cdot \chi_{(n-1,1)}$, so

$$\chi_{(2,1^{n-2})}(\sigma) = \epsilon(\sigma)(\#\{\text{fixed points of } \sigma\} - 1),$$

and therefore

$$\lambda_{(2,1^{n-2})} = \frac{1}{n-1} \sum_{\sigma \in \mathcal{D}_n} -\epsilon(\sigma) = -\frac{e_n - o_n}{n-1} = (-1)^n.$$

To summarize, we obtain:

| $\alpha$ | $\lambda_\alpha$ |
|---|---|
| $(n)$ | $d_n$ |
| $(1^n)$ | $(-1)^{n-1}(n-1)$ |
| $(n-1,1)$ | $-d_n/(n-1)$ |
| $(2,1^{n-2})$ | $(-1)^n$ |

Hence, $U_{(n)}$ is the $d_n$-eigenspace, $U_{(n-1,1)}$ is the $-d_n/(n-1)$-eigenspace, and all other eigenvalues are $O((n-2)!)$. Hence, Leader's conjecture follows (for $n$ sufficiently large) by applying Theorem 2.2 to the derangement graph. It is easy to check that $\nu = d_n/(n-1)$ for all $n \geq 4$, giving

**Theorem 2.6.** *For $n \geq 4$, if $\mathcal{A}, \mathcal{B} \subset S_n$ are cross-intersecting, then*

$$|\mathcal{A}||\mathcal{B}| \leq ((n-1)!)^2.$$

If equality holds, then by Theorem 2.2 part *(ii)*, the characteristic vectors $v_{\mathcal{A}}, v_{\mathcal{B}}$ must lie in the direct sum of the $d_n$ and $-d_n/(n-1)$-eigenspaces. It can be checked that for $n \geq 5$, $|\lambda_\alpha| < d_n/(n-1)\ \forall \alpha \neq (n), (n-1,1)$, so the $d_n$ eigenspace is precisely $U_{(n)}$ and the $-d/(n-1)$-eigenspace is precisely $U_{(n-1,1)}$. But we have:

**Lemma 2.7.** *For $i, j \in [n]$, let $v_{i \mapsto j} = v_{\{\sigma \in S_n :\ \sigma(i) = j\}} \in \mathbb{C}S_n$ be the characteristic vector of the 1-coset $\{\sigma \in S_n :\ \sigma(i) = j\}$. Then*

$$U_{(n)} \oplus U_{(n-1,1)} = \mathit{Span}\{v_{i \mapsto j}\ : i, j \in [n]\}.$$

This is a special case of Theorem 7 in [7]. We give a short proof for completeness.

*Proof.* Let

$$U = \text{Span}\{v_{i \mapsto j} \ : i, j \in [n]\}.$$

For each $i \in [n]$, $\{v_{i,j} \ : j \in [n]\}$ is a basis for a copy $W_i$ of the permutation module $M^{(n-1,1)}$ in $\mathbb{C}S_n$. Since

$$M^{(n-1,1)} \cong S^{(n)} \oplus S^{(n-1,1)},$$

we have the decomposition

$$W_i = \text{Span}\{\boldsymbol{f}\} \oplus V_i,$$

where $V_i$ is some copy of $S^{(n-1,1)}$ in $\mathbb{C}S_n$, so

$$\text{Span}\{v_{i \mapsto j} \ : j \in [n]\} = W_i \leq U_{(n)} \oplus U_{(n-1,1)}$$

for each $i \in [n]$, and therefore $U \leq U_{(n)} \oplus U_{(n-1,1)}$.

It is well known that if $G$ is any finite group, and $T, T'$ are two isomorphic submodules of $\mathbb{C}G$, then there exists $s \in \mathbb{C}G$ such that the right multiplication map $x \mapsto xs$ is an isomorphism from $T$ to $T'$ (see for example [12]). Hence, for any $i \in [n]$, the sum of all right translates of $W_i$ contains $\text{Span}\{\boldsymbol{f}\}$ and all submodules of $\mathbb{C}S_n$ isomorphic to $S^{(n-1,1)}$, so $U_{(n)} \oplus U_{(n-1,1)} \leq U$. Hence, $U = U_{(n)} \oplus U_{(n-1,1)}$ as required. □

Hence, for $n \geq 5$, if equality holds in Theorem 2.6, then the characteristic vectors of $\mathcal{A}$ and $\mathcal{B}$ are linear combinations of the characteristic vectors of the 1-cosets. It was proved in [7] that if the characteristic vector of $\mathcal{A} \subset S_n$ is a linear combination of the characteristic vectors of the 1-cosets, then $\mathcal{A}$ is a disjoint union of 1-cosets. It follows that for $n \geq 5$, if equality holds in Theorem 2.6, then $\mathcal{A}$ and $\mathcal{B}$ are both disjoint unions of 1-cosets. Since they are cross-intersecting, they must both be equal to the same 1-coset, i.e.

$$\mathcal{A} = \mathcal{B} = \{\sigma \in S_n : \ \sigma(i) = j\}$$

for some $i, j \in [n]$. It is easily checked that the same conclusion holds when $n = 4$, so we have the following characterization of the case of equality in Leader's conjecture:

**Theorem 2.8.** *For $n \geq 4$, if $\mathcal{A}, \mathcal{B} \subset S_n$ are cross-intersecting and satisfy*

$$|\mathcal{A}||\mathcal{B}| = ((n-1)!)^2,$$

*then*

$$\mathcal{A} = \mathcal{B} = \{\sigma \in S_n : \ \sigma(i) = j\}$$

*for some $i, j \in [n]$.*

## 3 Stability

We will now perform a stability analysis for intersecting families of permutations. First, we prove a 'rough' stability result: for any positive constant $c > 0$, if $\mathcal{A} \subset S_n$ is an intersecting family of permutations with $|\mathcal{A}| \geq c(n-1)!$, then there exist $i$ and $j$ such that all but $O((n-2)!)$ permutations in $\mathcal{A}$ map $i$ to $j$, i.e. $\mathcal{A}$ is 'almost' centred. In other words, writing $\mathcal{A}_{i \mapsto j}$ for the collection of all permutations in $\mathcal{A}$ mapping $i$ to $j$, we have $|\mathcal{A} \setminus \mathcal{A}_{i \mapsto j}| \leq O((n-2)!)$. To prove this, our first step will be to show that if $\mathcal{A} \subset S_n$ is an intersecting family with $|\mathcal{A}| \geq c(n-1)!$, then the characteristic vector $v_{\mathcal{A}}$ of $\mathcal{A}$ cannot be too far from the subspace $U$ spanned by the characteristic vectors of the 1-cosets, the intersecting families of maximum size. Secondly, we will use this to show that there exist $i, j \in [n]$ such that $|\mathcal{A}_{i \mapsto j}| \geq \omega((n-2)!)$. Clearly, for any fixed $i \in [n]$,

$$\sum_{k=1}^{n} |\mathcal{A}_{i \mapsto k}| = |\mathcal{A}|,$$

and therefore the average size of an $|\mathcal{A}_{i \mapsto k}|$ is $|\mathcal{A}|/n$; we have found $i$ and $j$ such that $|\mathcal{A}_{i \mapsto j}|$ is $\omega$ of the average size. This statement would at first seem too weak to help us, but using the fact that $\mathcal{A}$ is intersecting, we will 'bootstrap' it to the much stronger statement $|\mathcal{A}_{i \mapsto j}| \geq (1 - o(1))|\mathcal{A}|$. In detail, we will deduce from Theorem 2.6 that for any $k \neq j$,

$$|\mathcal{A}_{i \mapsto j}||\mathcal{A}_{i \mapsto k}| \leq ((n-2)!)^2,$$

giving $|\mathcal{A}_{i \mapsto k}| \leq o((n-2)!)$ for any $k \neq j$. Summing over all $k \neq j$ will give $|\mathcal{A} \setminus \mathcal{A}_{i \mapsto j}| \leq o((n-1)!)$, enabling us to complete the proof.

Note that this is enough to prove the stability conjecture of Cameron and Ku: if $\mathcal{A}$ is non-centred, it must contain some permutation $\tau$ such that $\tau(i) \neq j$. This immediately forces $|\mathcal{A}_{i \mapsto j}|$ to be less than $(1 - 1/e + o(1))(n-1)!$, yielding a contradiction if $c > 1 - 1/e$, and $n$ is sufficiently large depending on $c$.

Here, then, is our rough stability result:

**Theorem 3.1.** *For any $c > 0$, there exists $K > 0$ such that the following holds. If $\mathcal{A} \subset S_n$ is an intersecting family of permutations with $|\mathcal{A}| \geq c(n-1)!$, then there exist $i, j \in [n]$ such that*

$$|\mathcal{A} \setminus \mathcal{A}_{i \mapsto j}| \leq K(n-2)!.$$

To carry out the first of the above steps, we will need a 'stability' version of Hoffman's theorem:

**Lemma 3.2.** *Let $H = (V, E)$ be a $d$-regular graph on $N$ vertices, and let $A$ denote the adjacency matrix of $H$. Let $\lambda_N$ denote the least eigenvalue of*

*A, and let* $U = Span(\mathbf{f}) \oplus Ker(A - \lambda_N I)$. *Let* $\lambda_M = \min\{\lambda_i : \lambda_i \neq \lambda_N\}$. *Let* $X \subset V(H)$ *be an independent set in* $H$, *and let* $\alpha = |X|/N$ *denote its measure. Equip* $\mathbb{R}^V$ *with the inner product:*

$$\langle u, v\rangle = \frac{1}{N}\sum_{i=1}^{N} u(i)v(i),$$

*and let*

$$||u|| = \sqrt{\frac{1}{N}\sum_{i=1}^{N} u(i)^2}$$

*be the induced Euclidean norm. Let* $D$ *denote the Euclidean distance from the characteristic vector* $v_X$ *of* $X$ *to the subspace* $U$, *i.e. the norm* $||P_{U^\perp}(v_X)||$ *of the projection of* $v_X$ *onto* $U^\perp$. *Then*

$$D^2 \leq \frac{(1-\alpha)|\lambda_N| - \lambda_1\alpha}{|\lambda_N| - |\lambda_M|}\alpha.$$

*Proof.* This is a straightforward adaptation of the proof of Hoffman's theorem. Let $u_1 = \mathbf{f}, u_2, \ldots, u_N$ be an orthonormal basis of real eigenvectors of $A$ corresponding to the eigenvalues $\lambda_1 = d, \lambda_2, \ldots, \lambda_N$. Write

$$v_X = \sum_{i=1}^{N} \xi_i u_i$$

as a linear combination of these eigenvectors. We have $\xi_1 = \alpha$ and

$$\sum_{i=1}^{N} \xi_i^2 = ||v_X||^2 = \alpha.$$

Since $X$ is an independent set in $H$, we have:

$$0 = \sum_{x,y\in X} A_{x,y} = v_X^\top A v_X = \sum_{i=1}^{N} \lambda_i \xi_i^2 \geq d\xi_1^2 + \lambda_N \sum_{i:\lambda_i=\lambda_N} \xi_i^2 + \lambda_M \sum_{i>1:\lambda_i\neq\lambda_N} \xi_i^2.$$

Note that

$$\sum_{i>1:\lambda_i\neq\lambda_N} \xi_i^2 = D^2$$

and

$$\sum_{i:\lambda_i=\lambda_N} \xi_i^2 = \alpha - \alpha^2 - D^2,$$

so we have

$$0 \geq d\alpha^2 + \lambda_N(\alpha - \alpha^2 - D^2) + \lambda_M D^2.$$

Rearranging, we obtain:

$$D^2 \leq \frac{(1-\alpha)|\lambda_N| - d\alpha}{|\lambda_N| - |\lambda_M|}\alpha,$$

as required. ∎

For the second step, we will need an isoperimetric inequality for the *transposition graph* on $S_n$. If $H = (V, E)$ is a graph, and $x, y \in V$, we define the *graph distance* $d_H(x, y)$ to be the length of the shortest path in $H$ between $x$ and $y$. If $X \subset V(H)$, and $h > 0$, we define the *h-neighbourhood* $N_h(X)$ to be the set of vertices of $H$ which are at distance at most $h$ from $X$, i.e.

$$N_h(X) = \{y \in V : \ d_H(x, y) \leq h \text{ for some } x \in X\}.$$

The *transposition graph* $T$ is the Cayley graph on $S_n$ generated by the transpositions, i.e. $V(T) = S_n$ and $\sigma\pi \in E(T)$ if and only if $\sigma^{-1}\pi$ is a transposition. We will use the following isoperimetric inequality for $T$, essentially the martingale inequality of Maurey:

**Theorem 3.3.** *Let $0 < a < 1$, and let $X \subset V(T)$ with $|X| \geq an!$. Then for any $h \geq h_0 := \sqrt{\frac{1}{2}(n-1)\log\frac{1}{a}}$,*

$$|N_h(X)| \geq \left(1 - e^{-\frac{2(h-h_0)^2}{n-1}}\right) n!.$$

(For a proof, see for example [16].) We will deduce from this that for any two sets $X, Y \subset S_n$ which are not too small, there exist permutations $\sigma \in X$ and $\tau \in Y$ which are 'close' to one another in $T$.

Finally, we need the following simple consequence of Theorem 2.6:

**Lemma 3.4.** *Let $\mathcal{A} \subset S_n$ be an intersecting family; then for all $i, j$ and $k$ with $k \neq j$,*

$$|\mathcal{A}_{i\mapsto j}||\mathcal{A}_{i\mapsto k}| \leq ((n-2)!)^2.$$

*Proof.* By double translation, we may assume that $i = j = 1$ and $k = 2$. Let $\sigma \in \mathcal{A}_{1\mapsto 1}$ and $\pi \in \mathcal{A}_{1\mapsto 2}$; then there exists $p \neq 1$ such that $\sigma(p) = \pi(p) > 2$. Hence, the translates $\mathcal{E} = \mathcal{A}_{1\mapsto 1}$ and $\mathcal{F} = (1\ 2)\mathcal{A}_{1\mapsto 2}$ are families of permutations fixing 1 and cross-intersecting on the domain $\{2, 3, \ldots, n\}$. Deleting 1 from each permutation in the two families gives a cross-intersecting pair $\mathcal{E}', \mathcal{F}'$ of families of permutations of $\{2, 3, \ldots, n\}$; applying Theorem 2.6 gives:

$$|\mathcal{A}_{1\mapsto 1}||\mathcal{A}_{1\mapsto 2}| = |\mathcal{E}'||\mathcal{F}'| \leq ((n-2)!)^2.$$

∎

*of Theorem 3.1.* Let $c > 0$ be a positive constant, and let $\mathcal{A} \subset S_n$ be an intersecting family of permutations with $|\mathcal{A}| \geq c(n-1)!$. Write $\alpha = |\mathcal{A}|/n!$. Since $\mathcal{A}$ is an independent set in the derangement graph $\Gamma$, which has $|\lambda_M| = O((n-2)!)$, Lemma 3.2 yields:

$$\begin{aligned} D^2 &\leq \frac{(1-\alpha)d_n/(n-1) - d_n\alpha}{d_n/(n-1) - |\lambda_M|}\frac{|\mathcal{A}|}{n!} \\ &= \frac{1-\alpha-\alpha(n-1)}{1-(n-1)|\lambda_M|/d_n}\frac{|\mathcal{A}|}{n!} \\ &= \frac{1-\alpha n}{1-O(1/n)}\frac{|\mathcal{A}|}{n!} \\ &= (1-\alpha n)(1+O(1/n))|\mathcal{A}|/n!, \end{aligned}$$

where $D = ||P_{U^\perp}(v_\mathcal{A})||$ denotes the Euclidean distance from $v_\mathcal{A}$ to the subspace

$$U = U_{(n)} \oplus U_{(n-1,1)} = \text{Span}\{v_{i\mapsto j}\ : i,j \in [n]\}.$$

Write $|\mathcal{A}| = (1-\delta)(n-1)!$, where $\delta < 1$. Then

$$||P_{U^\perp}(v_\mathcal{A})||^2 = D^2 \leq \delta(1+O(1/n))|\mathcal{A}|/n!. \tag{9}$$

We now derive a formula for $P_U(v_\mathcal{A})$. The projection of $v_\mathcal{A}$ onto $U_{(n)} = \text{Span}\{\mathbf{f}\}$ is clearly $(|\mathcal{A}|/n!)\mathbf{f}$. By (3), the primitive central idempotent generating $U_{(n-1,1)}$ is

$$\frac{n-1}{n!}\sum_{\pi\in S_n}\chi_{(n-1,1)}(\pi^{-1})\pi,$$

and therefore the projection of $v_\mathcal{A}$ onto $U_{(n-1,1)}$ is given by

$$P_{U_{(n-1,1)}}(v_\mathcal{A}) = \frac{n-1}{n!}\sum_{\rho\in\mathcal{A}}\sum_{\pi\in S_n}\chi_{(n-1,1)}(\pi^{-1})\pi\rho,$$

which has $\sigma$-coordinate

$$\begin{aligned} P_{U_{(n-1,1)}}(v_\mathcal{A})_\sigma &= \frac{n-1}{n!}\sum_{\rho\in\mathcal{A}}\chi_{(n-1,1)}(\rho\sigma^{-1}) \\ &= \frac{n-1}{n!}\sum_{\rho\in\mathcal{A}}(\xi_{(n-1,1)}(\rho\sigma^{-1})-1) \\ &= \frac{n-1}{n!}\sum_{\rho\in\mathcal{A}}(\#\{\text{fixed points of } \rho\sigma^{-1}\}-1) \\ &= \frac{n-1}{n!}(\#\{(\rho,i) : \rho\in\mathcal{A},\ i\in[n],\ \rho(i)=\sigma(i)\} - |\mathcal{A}|) \\ &= \frac{n-1}{n!}\sum_{i=1}^{n}|\mathcal{A}_{i\mapsto\sigma(i)}| - \frac{n-1}{n!}|\mathcal{A}|. \end{aligned}$$

Hence, the $\sigma$-coordinate $P_\sigma$ of the projection of $v_{\mathcal{A}}$ onto $U = U_{(n)} \oplus U_{(n-1,1)}$ is given by

$$P_\sigma = \frac{n-1}{n!}\sum_{i=1}^{n} |\mathcal{A}_{i\mapsto\sigma(i)}| - \frac{(n-2)}{n!}|\mathcal{A}|,$$

which is a linear function of

$$\sum_{i=1}^{n} |\mathcal{A}_{i\mapsto\sigma(i)}| = \#\{(\rho, i) \in \mathcal{A} \times [n] : \ \rho(i) = \sigma(i)\},$$

the number of times $\sigma$ agrees with a permutation in $\mathcal{A}$.

From (9), we have

$$\sum_{\sigma\in\mathcal{A}} (1 - P_\sigma)^2 + \sum_{\sigma\notin\mathcal{A}} P_\sigma^2 \leq |\mathcal{A}|\delta(1 + O(1/n)).$$

Choose $C > 0$ such that $|\mathcal{A}|(1 - 1/n)\delta(1 + C/n)$ is at least the right-hand side; then

$$(1 - P_\sigma)^2 < \delta(1 + C/n)$$

for at least $|\mathcal{A}|/n$ permutations in $\mathcal{A}$, so the subset

$$\mathcal{A}' := \{\sigma \in \mathcal{A} : (1 - P_\sigma)^2 < \delta(1 + C/n)\}$$

has

$$|\mathcal{A}'| \geq |\mathcal{A}|/n. \tag{10}$$

Similarly, $P_\sigma^2 < 2\delta/n$ for all but at most

$$n|\mathcal{A}|(1 + O(1/n))/2 = (1 - \delta)n!(1 + O(1/n))/2$$

permutations $\sigma \notin \mathcal{A}$, so the subset $\mathcal{R} = \{\sigma \notin \mathcal{A} : P_\sigma^2 < 2\delta/n\}$ has

$$|\mathcal{R}| \geq n! - (1 - \delta)(n - 1)! - (1 - \delta)n!(1 + O(1/n))/2. \tag{11}$$

The permutations $\sigma \in \mathcal{A}'$ have $P_\sigma$ close to 1; the permutations $\pi \in \mathcal{R}$ have $P_\pi$ close to 0. Using only the lower bounds (10) and (11) on the sizes of $\mathcal{A}'$ and $\mathcal{R}$, we may prove the following:

*Claim:* There exist permutations $\sigma \in \mathcal{A}'$, $\pi \in \mathcal{R}$ such that $\sigma^{-1}\pi$ is a product of at most $h = h(n)$ transpositions, where $h = 2\sqrt{2(n-1)\log n}$.

*Proof of Claim:* Apply Theorem 3.3 to the set $\mathcal{A}'$, with $a = 1/n^4$ and $h = 2h_0$. Since $|\mathcal{A}'| \geq \frac{c(n-1)!}{n} \geq \frac{n!}{n^4}$, we have

$$|N_h(\mathcal{A}')| \geq (1 - n^{-4})n!,$$

so certainly $N_h(\mathcal{A}') \cap \mathcal{R} \neq \emptyset$, proving the claim.

We now have two permutations $\sigma \in \mathcal{A}$, $\pi \notin \mathcal{A}$ which are 'close' to one another in $T$ (differing in only $O(\sqrt{n \log n})$ transpositions) such that $P_\sigma > 1 - \sqrt{\delta(1 + C/n)}$ and $P_\pi < \sqrt{2\delta/n}$, and therefore $P_\sigma - P_\pi > 1 - \sqrt{\delta} - O(1/\sqrt{n})$, i.e. $\sigma$ agrees many more times than $\pi$ with permutations in $\mathcal{A}$:

$$\sum_{i=1}^{n} |\mathcal{A}_{i \mapsto \sigma(i)}| - \sum_{i=1}^{n} |\mathcal{A}_{i \mapsto \pi(i)}| \geq (n-1)!(1 - \sqrt{\delta} - O(1/\sqrt{n})).$$

Suppose for this pair we have $\pi = \sigma\tau_1\tau_2\ldots\tau_l$ for transpositions $\tau_1, \ldots, \tau_l$, where $l \leq t$. Let $I$ be the set of numbers appearing in these transpositions; then $|I| \leq 2l \leq 2t$, and $\sigma(i) = \pi(i)$ for each $i \notin I$. Hence,

$$\sum_{i \in I} |\mathcal{A}_{i \mapsto \sigma(i)}| - \sum_{i \in I} |\mathcal{A}_{i \mapsto \pi(i)}| \geq (n-1)!(1 - \sqrt{\delta} - O(1/\sqrt{n})),$$

so certainly,

$$\sum_{i \in I} |\mathcal{A}_{i \mapsto \sigma(i)}| \geq (n-1)!(1 - \sqrt{\delta} - O(1/\sqrt{n})).$$

By averaging,

$$\begin{aligned} |\mathcal{A}_{i \mapsto \sigma(i)}| &\geq \frac{1}{|I|}(n-1)!(1 - \sqrt{\delta} - O(1/\sqrt{n})) \\ &\geq \frac{(n-1)!}{4\sqrt{2(n-1)\log n}}(1 - \sqrt{\delta} - O(1/\sqrt{n})) \end{aligned}$$

for some $i \in I$. Let $\sigma(i) = j$; then

$$|\mathcal{A}_{i \mapsto j}| \geq \frac{(n-1)!}{4\sqrt{2(n-1)\log n}}(1 - \sqrt{1-c} - O(1/\sqrt{n})) = \omega((n-2)!).$$

It follows from Lemma 3.4 that $|\mathcal{A}_{i \mapsto k}| \leq o((n-2)!)$ for all $k \neq j$. Summing over all $k \neq j$ gives

$$|\mathcal{A} \setminus \mathcal{A}_{i \mapsto j}| = \sum_{k \neq j} |\mathcal{A}_{i \mapsto k}| \leq o((n-1)!),$$

and therefore

$$|\mathcal{A}_{i \mapsto j}| = |\mathcal{A}| - |\mathcal{A} \setminus \mathcal{A}_{i \mapsto j}| \geq (c - o(1))(n-1)!. \tag{12}$$

Applying Lemma 3.4 again gives

$$|\mathcal{A}_{i \mapsto k}| \leq O((n-3)!)$$

for all $k \neq j$; summing over all $k \neq j$ gives

$$|\mathcal{A} \setminus \mathcal{A}_{i \mapsto j}| \leq O((n-2)!),$$

proving Theorem 3.1. □

The stability conjecture of Cameron and Ku follows easily:

**Corollary 3.5.** *Let $c > 1 - 1/e$; then for $n$ sufficiently large depending on $c$, any intersecting family $\mathcal{A} \subset S_n$ with $|\mathcal{A}| \geq c(n-1)!$ is centred.*

*Proof.* Let $c > 1-1/e$, and let $\mathcal{A} \subset S_n$ be intersecting, with $|\mathcal{A}| \geq c(n-1)!$. By Theorem 3.1, there exist $i, j \in [n]$ such that $|\mathcal{A} \setminus \mathcal{A}_{i \mapsto j}| \leq O((n-2)!)$, and therefore

$$|\mathcal{A}_{i \mapsto j}| \geq (c - O(1/n))(n-1)!. \tag{13}$$

Suppose for a contradiction that $\mathcal{A}$ is non-centred. Then there exists a permutation $\tau \in \mathcal{A}$ such that $\tau(i) \neq j$. Any permutation in $\mathcal{A}_{i \mapsto j}$ must agree with $\tau$ at some point. But for any $i, j \in [n]$ and any $\tau \in S_n$ such that $\tau(i) \neq j$, the number of permutations in $S_n$ which map $i$ to $j$ and agree with $\tau$ at some point is

$$(n-1)! - d_{n-1} - d_{n-2} = (1 - 1/e - o(1))(n-1)!.$$

(By double translation, we may assume that $i = j = 1$ and $\tau = (1\ 2)$; we observed above that the number of permutations fixing 1 and intersecting $(1\ 2)$ is $(n-1)! - d_{n-1} - d_{n-2}$.) This contradicts (13) provided $n$ is sufficiently large depending on $c$. □

We now use our rough stability result to prove the Hilton-Milner type conjecture of Cameron and Ku, for $n$ sufficiently large. First, we introduce an extra notion which will be useful in the proof. Following Cameron and Ku [4], given a permutation $\pi \in S_n$ and $i \in [n]$, we define the *$i$-fix* of $\pi$ to be the permutation $\pi_i$ which fixes $i$, maps the preimage of $i$ to the image of $i$, and agrees with $\pi$ at all other points of $[n]$, i.e.

$$\pi_i(i) = i;\ \pi_i(\pi^{-1}(i)) = \pi(i);\ \pi_i(k) = \pi(k)\ \forall k \neq i, \pi^{-1}(i).$$

In other words, $\pi_i = \pi(\pi^{-1}(i)\ i)$. We inductively define

$$\pi_{i_1,\ldots,i_l} = (\pi_{i_1,\ldots,i_{l-1}})_{i_l}.$$

Notice that if $\sigma$ fixes $j$, then $\sigma$ agrees with $\pi_j$ wherever it agrees with $\pi$.

**Theorem 3.6.** *For $n$ sufficiently large, if $\mathcal{A} \subset S_n$ is a non-centred intersecting family, then $\mathcal{A}$ is at most as large as the family*

$$\mathcal{C} = \{\sigma \in S_n : \sigma(1) = 1, \sigma(i) = i \text{ for some } i > 2\} \cup \{(12)\},$$

*which has size $(n-1)! - d_{n-1} - d_{n-2} + 1 = (1 - 1/e + o(1))(n-1)!$. Equality holds if and only if $\mathcal{A}$ is a double translate of $\mathcal{C}$, i.e. $\mathcal{A} = \pi\mathcal{C}\tau$ for some $\pi, \tau \in S_n$.*

*Proof.* Let $\mathcal{A} \subset S_n$ be a non-centred intersecting family with the same size as $\mathcal{C}$; we must show that $\mathcal{A}$ is a double translate of $\mathcal{C}$. By Theorem 3.1, there exist $i, j \in [n]$ such that $|\mathcal{A} \setminus \mathcal{A}_{i \mapsto j}| \leq O((n-2)!)$, and therefore

$$|\mathcal{A}_{i \mapsto j}| \geq (n-1)! - d_{n-1} - d_{n-2} + 1 - O(n-2)! = (1 - 1/e - o(1))(n-1)!.$$

Since $\mathcal{A}$ is non-centred, it must contain some permutation $\rho$ such that $\rho(i) \neq j$. By double translation, we may assume that $i = j = 1$ and $\rho = (1\ 2)$; we will show that under these hypotheses, $\mathcal{A} = \mathcal{C}$. We have

$$|\mathcal{A}_{1 \mapsto 1}| \geq (1 - 1/e - o(1))(n-1)! \tag{14}$$

and $(1\ 2) \in \mathcal{A}$. Note that every permutation in $\mathcal{A}$ must intersect $(1\ 2)$, and therefore

$$\mathcal{A}_{1 \mapsto 1} \cup \{(1\ 2)\} \subset \mathcal{C}.$$

We need to show that $(1\ 2)$ is the only permutation in $\mathcal{A}$ that does not fix 1. Suppose for a contradiction that $\mathcal{A}$ contains some other permutation $\pi$ not fixing 1. Then $\pi$ must shift some point $p > 2$. If $\sigma$ fixes both 1 and $p$, then $\sigma$ agrees with $\pi_{1,p} = (\pi_1)_p$ wherever it agrees with $\pi$. There are exactly $d_{n-2}$ permutations which fix 1 and $p$ and disagree with $\pi_{1,p}$ at every point of $\{2, \ldots, n\} \setminus \{p\}$; each disagrees everywhere with $\pi$, so none are in $\mathcal{A}$, and therefore

$$|\mathcal{A}_{1 \mapsto 1}| \leq (n-1)! - d_{n-1} - 2d_{n-2}.$$

Hence, by assumption,

$$|\mathcal{A} \setminus \mathcal{A}_{1 \mapsto 1}| \geq d_{n-2} + 1 = \Omega((n-2)!).$$

Notice that we have the following trivial bound on the size of a $t$-intersecting family[2] $\mathcal{F} \subset S_n$:

$$|\mathcal{F}| \leq \binom{n}{t}(n-t)! = n!/t!$$

since every permutation in $\mathcal{F}$ must agree with a fixed $\rho \in \mathcal{F}$ in at least $t$ places.

Hence, $\mathcal{A} \setminus \mathcal{A}_{1 \mapsto 1}$ cannot be $(\log n)$-intersecting and therefore contains two permutations $\rho, \tau$ agreeing on at most $\log n$ points. The number of permutations fixing 1 and agreeing with both $\tau_1$ and $\tau_2$ at one of these points is at most $(\log n)(n-2)!$. All other permutations in $\mathcal{A} \cap \mathcal{C}$ agree with $\rho$ and $\tau$ at two separate points of $\{2, \ldots, n\}$, and by the above argument, the same holds for the 1-fixes $\rho_1$ and $\tau_1$. The number of permutations fixing 1 that agree with $\rho_1$ and $\tau_1$ at two separate points of $\{2, \ldots, n\}$ is at most $((1 - 1/e)^2 + o(1))(n-1)!$ (it is easily checked that given two fixed

[2] We say that a family $\mathcal{F} \subset S_n$ is *$t$-intersecting* if any two permutations in $\mathcal{F}$ agree on at least $t$ points.

permutations, the probability that a uniform random permutation agrees with them at separate points is at most $(1-1/e)^2+o(1)$). Hence,

$$\begin{aligned}|\mathcal{A}_{1\mapsto 1}| &\leq ((1-1/e)^2+o(1))(n-1)!+(\log n)(n-2)!\\ &= ((1-1/e)^2+o(1))(n-1)!,\end{aligned}$$

contradicting (14) provided $n$ is sufficiently large.

Hence, (1 2) is the only permutation in $\mathcal{A}$ that does not fix 1, so $\mathcal{A} = \mathcal{A}_{1\mapsto 1} \cup \{(1\ 2)\} \subset \mathcal{C}$; since $|\mathcal{A}| = |\mathcal{C}|$, we have $\mathcal{A} = \mathcal{C}$ as required. □

We now perform a very similar stability analysis for cross-intersecting families. First, we prove a 'rough' stability result analogous to Theorem 3.1, namely that for any positive constant $c > 0$, if $\mathcal{A}, \mathcal{B} \subset S_n$ are cross-intersecting with $\sqrt{|\mathcal{A}||\mathcal{B}|} \geq c(n-1)!$, then there exist $i, j \in [n]$ such that all but at most $O((n-2)!)$ permutations in $\mathcal{A}$ and all but at most $O((n-2)!)$ permutations in $\mathcal{B}$ map $i$ to $j$.

**Theorem 3.7.** *Let $c > 0$ be a positive constant. If $\mathcal{A}, \mathcal{B} \subset S_n$ are cross-intersecting with $\sqrt{|\mathcal{A}||\mathcal{B}|} \geq c(n-1)!$, then there exist $i, j \in [n]$ such that all but at most $O((n-2)!)$ permutations in $\mathcal{A}$ and all but at most $O((n-2)!)$ permutations in $\mathcal{B}$ map $i$ to $j$.*

*Proof.* Let $|\mathcal{A}| \leq |\mathcal{B}|$. First, we adapt the proof of Theorem 2.2 to obtain information about the distances $D := ||P_{U^\perp}(v_X)||$ and $E := ||P_{U^\perp}(v_Y)||$. This time, we have

$$\sum_{i>1:\lambda_i\neq\lambda_N} \xi_i^2 = D^2;$$

$$\sum_{i>1:\lambda_i\neq\lambda_N} \eta_i^2 = E^2;$$

$$\sum_{i>1:\lambda_i=\lambda_N} \xi_i^2 = \alpha-\alpha^2-D^2;$$

$$\sum_{i>1:\lambda_i=\lambda_N} \eta_i^2 = \beta-\beta^2-E^2.$$

Substituting into (2) gives:

$$\begin{aligned}d\alpha\beta &= -\sum_{i>1:\lambda_i\neq\lambda_N} \lambda_i\xi_i\eta_i - \lambda_N \sum_{i>1:\lambda_i=\lambda_N} \xi_i\eta_i\\ &\leq \mu \sum_{i>1:\lambda_i\neq\lambda_N} |\xi_i||\eta_i| + |\lambda_N| \sum_{i>1:\lambda_i=\lambda_N} |\xi_i||\eta_i|\\ &\leq \mu \sqrt{\sum_{i>1:\lambda_i\neq\lambda_N} \xi_i^2}\sqrt{\sum_{i>1:\lambda_i\neq\lambda_N} \eta_i^2} + |\lambda_N|\sqrt{\sum_{i>1:\lambda_i=\lambda_N} \xi_i^2}\sqrt{\sum_{i>1:\lambda_i=\lambda_N} \eta_i^2}\\ &= \mu DE + |\lambda_N|\sqrt{\alpha-\alpha^2-D^2}\sqrt{\beta-\beta^2-E^2},\end{aligned}$$

where $\mu = \max_{i>1:\lambda_i \neq \lambda_N} |\lambda_i|$. Note that the derangement graph $\Gamma$ has $\mu \leq O((n-2)!)$. Hence, applying the above result to a cross-intersecting pair $\mathcal{A}, \mathcal{B} \subset S_n$ with $\sqrt{|\mathcal{A}||\mathcal{B}|} = (1-\delta)(n-1)!$, we obtain

$$\sqrt{1-\alpha-D^2/\alpha}\sqrt{1-\beta-E^2/\beta} \geq \frac{d_n\sqrt{\alpha\beta} - \mu(D/\sqrt{\alpha})(E/\sqrt{\beta})}{|\lambda_N|} \geq 1-\delta-O(1/n),$$

and therefore $1-\alpha-D^2/\alpha \geq (1-\delta)^2 - O(1/n)$, so $D^2 \leq \alpha(2\delta-\delta^2+O(1/n))$. Replacing $\delta$ with $2\delta - \delta^2 + O(1/n)$ in the proof of Theorem 3.1, we see that there exist $i, j \in [n]$ such that

$$|\mathcal{A}_{i\mapsto j}| \geq \frac{(n-1)!}{4\sqrt{2(n-1)\log n}}(1-\sqrt{2\delta-\delta^2} - O(1/\sqrt{n})) = \omega((n-2)!),$$

since $\delta < 1-c$. For each $k \neq j$, the pair $\mathcal{A}_{i\mapsto j}, \mathcal{B}_{i\mapsto k}$ is cross-intersecting, so as in Lemma 3.4, we have:

$$|\mathcal{A}_{i\mapsto j}||\mathcal{B}_{i\mapsto k}| \leq ((n-2)!)^2.$$

Hence, for all $k \neq j$,

$$|\mathcal{B}_{i\mapsto k}| \leq o((n-2)!),$$

so summing over all $j \neq k$ gives

$$|\mathcal{B} \setminus \mathcal{B}_{i\mapsto j}| \leq o((n-1)!).$$

Since $|\mathcal{B}| \geq |\mathcal{A}|$, $|\mathcal{B}| \geq c(n-1)!$, and therefore

$$|\mathcal{B}_{i\mapsto j}| \geq (c-o(1))(n-1)!.$$

For each $k \neq j$, the pair $\mathcal{A}_{i\mapsto k}, \mathcal{B}_{i\mapsto j}$ is cross-intersecting, so as before, we have:

$$|\mathcal{A}_{i\mapsto k}||\mathcal{B}_{i\mapsto j}| \leq ((n-2)!)^2.$$

Hence, for all $k \neq j$,

$$|\mathcal{A}_{i\mapsto k}| \leq O((n-3)!),$$

so summing over all $j \neq k$ gives

$$|\mathcal{A} \setminus \mathcal{A}_{i\mapsto j}| \leq O((n-2)!).$$

Also, $|\mathcal{B}| = |\mathcal{B}_{i\mapsto j}| + |\mathcal{B} \setminus \mathcal{B}_{i\mapsto j}| \leq (1+o(1))(n-1)!$, so $|\mathcal{A}| \geq c^2(1-o(1))(n-1)!$. Hence,

$$|\mathcal{A}_{i\mapsto j}| \geq c^2(1-o(1))(n-1)!,$$

so by the same argument as above,

$$|\mathcal{B}_{i\mapsto k}| \leq O((n-3)!)$$

for all $k \neq j$, and therefore

$$|\mathcal{B} \setminus \mathcal{B}_{i\mapsto j}| \leq O((n-2)!)$$

as well, proving Theorem 3.7. □

We may use Theorem 3.7 to deduce two Hilton-Milner type results on cross-intersecting families:

**Theorem 3.8.** *For $n$ sufficiently large, if $\mathcal{A}, \mathcal{B} \subset S_n$ are cross-intersecting but not both contained within the same 1-coset, then*

$$\min(|\mathcal{A}|, |\mathcal{B}|) \leq |\mathcal{C}| = (n-1)! - d_{n-1} - d_{n-2} + 1,$$

*with equality if and only if*

$$\begin{aligned} \mathcal{A} &= \{\sigma \in S_n : \sigma(i) = j,\ \sigma \text{ intersects } \tau\} \cup \{\rho\}, \\ \mathcal{B} &= \{\sigma \in S_n : \sigma(i) = j, \sigma \text{ intersects } \rho\} \cup \{\tau\} \end{aligned}$$

*for some $i, j \in [n]$ and some $\tau, \rho \in S_n$ which intersect and do not map $i$ to $j$.*

*Proof.* Let $\mathcal{A}, \mathcal{B} \subset S_n$ be cross-intersecting, and not both centred, with

$$\min(|\mathcal{A}|, |\mathcal{B}|) \geq |\mathcal{C}|.$$

Applying Theorem 3.7 with any $c < 1 - 1/e$, we see that there exist $i, j \in [n]$ such that

$$|\mathcal{A} \setminus \mathcal{A}_{i \mapsto j}|, |\mathcal{B} \setminus \mathcal{B}_{i \mapsto j}| \leq O((n-2)!).$$

By double translation, we may assume that $i = j = 1$, so

$$|\mathcal{A} \setminus \mathcal{A}_{1 \mapsto 1}|, |\mathcal{B} \setminus \mathcal{B}_{1 \mapsto 1}| \leq O((n-2)!).$$

Assume $\mathcal{A}$ is not contained within the 1-coset $\{\sigma \in S_n : \ \sigma(1) = 1\}$; let $\rho$ be a permutation in $\mathcal{A}$ not fixing 1. Suppose for a contradiction that $\mathcal{A}$ contains another permutation $\pi$ not fixing 1. As in the proof of Theorem 3.6, this implies that

$$|\mathcal{B}_{1 \mapsto 1}| \leq (n-1)! - d_{n-1} - 2d_{n-2},$$

and so by assumption,

$$|\mathcal{B} \setminus \mathcal{B}_{1 \mapsto 1}| \geq d_{n-2} + 1,$$

so $\mathcal{B} \setminus \mathcal{B}_{1 \mapsto 1}$ cannot be $(\log n)$-intersecting. As in the proof of Theorem 3.6, this implies that

$$|\mathcal{A}_{1 \mapsto 1}| \leq ((1 - 1/e)^2 + o(1))(n-1)!,$$

giving

$$|\mathcal{A}| \leq ((1 - 1/e)^2 + o(1))(n-1)! < |\mathcal{C}|$$

—a contradiction. Hence,

$$\mathcal{A} = \mathcal{A}_{1 \mapsto 1} \cup \{\rho\}.$$

If $\mathcal{B}$ were centred, then every permutation in $\mathcal{B}$ would have to fix 1 and intersect $\rho$, and we would have $|\mathcal{B}| = |\mathcal{B}_{1\mapsto 1}| \leq (n-1)! - d_{n-1} - d_{n-2} < |\mathcal{C}|$, a contradiction. Hence, $\mathcal{B}$ is also non-centred. Repeating the above argument with $\mathcal{B}$ in place of $\mathcal{A}$, we see that $\mathcal{B}$ contains just one permutation not fixing 1, $\tau$ say. Hence,

$$\mathcal{B} = \mathcal{B}_{1\mapsto 1} \cup \{\tau\}.$$

Since $\min(|\mathcal{A}|, |\mathcal{B}|) \geq |\mathcal{C}|$, we have

$$\begin{aligned} \mathcal{A}_{1\mapsto 1} &= \{\sigma \in S_n : \sigma(1) = 1,\ \sigma \text{ intersects } \tau\}, \\ \mathcal{B}_{1\mapsto 1} &= \{\sigma \in S_n : \sigma(1) = 1,\ \sigma \text{ intersects } \rho\}, \end{aligned}$$

proving the theorem. □

Similarly, we may prove

**Theorem 3.9.** *For $n$ sufficiently large, if $\mathcal{A}, \mathcal{B} \subset S_n$ are cross-intersecting but not both contained within the same 1-coset, then*

$$|\mathcal{A}||\mathcal{B}| \leq ((n-1)! - d_{n-1} - d_{n-2})((n-1)! + 1),$$

*with equality if and only if*

$$\mathcal{A} = \{\sigma \in S_n : \sigma(i) = j,\ \sigma \text{ intersects } \rho\}, \quad \mathcal{B} = \{\sigma \in S_n : \sigma(i) = j\} \cup \{\rho\}$$

*for some $i, j \in [n]$ and some $\rho \in S_n$ with $\rho(i) \neq j$.*

*Proof.* Let $\mathcal{A}, \mathcal{B} \subset S_n$ be cross-intersecting, and not both centred, with

$$|\mathcal{A}||\mathcal{B}| \geq ((n-1)! - d_{n-1} - d_{n-2})((n-1)! + 1).$$

We have

$$\sqrt{|\mathcal{A}||\mathcal{B}|} \geq (\sqrt{1 - 1/e} - O(1/n))(n-1)!,$$

so applying Theorem 3.7 with any $c < \sqrt{1 - 1/e}$, we see that there exist $i, j \in [n]$ such that

$$|\mathcal{A} \setminus \mathcal{A}_{i\mapsto j}|, |\mathcal{B} \setminus \mathcal{B}_{i\mapsto j}| \leq O((n-2)!).$$

By double translation, we may assume that $i = j = 1$, so

$$|\mathcal{A} \setminus \mathcal{A}_{1\mapsto 1}|, |\mathcal{B} \setminus \mathcal{B}_{1\mapsto 1}| \leq O((n-2)!).$$

Therefore,

$$\sqrt{|\mathcal{A}_{1\mapsto 1}||\mathcal{B}_{1\mapsto 1}|} \geq (\sqrt{1 - 1/e} - O(1/n))(n-1)!. \tag{15}$$

If $\mathcal{B}$ contains some permutation $\rho$ not fixing 1, then

$$\mathcal{A}_{1\mapsto 1} \subset \{\sigma \in S_n : \sigma(1) = 1, \sigma \text{ intersects } \rho\},$$

and therefore

$$|\mathcal{A}_{1\mapsto 1}| \leq (n-1)! - d_{n-1} - d_{n-2} = (1 - 1/e + o(1))(n-1)!.$$

Similarly, if $\mathcal{A}$ contains a permutation not fixing 1, then

$$|\mathcal{B}_{1\mapsto 1}| \leq (1 - 1/e + o(1))(n-1)!.$$

By (15), both statements cannot hold (provided $n$ is large), so we may assume that every permutation in $\mathcal{A}$ fixes 1, and that $\mathcal{B}$ contains some permutation $\rho$ not fixing 1. Hence,

$$\mathcal{A} \subset \{\sigma \in S_n : \sigma(1) = 1, \sigma \text{ intersects } \rho\},$$

and

$$|\mathcal{A}| \leq (n-1)! - d_{n-1} - d_{n-2} = (1 - 1/e + o(1))(n-1)!. \tag{16}$$

So by assumption,

$$|\mathcal{B}| \geq (n-1)! + 1. \tag{17}$$

Suppose for a contradiction that $\mathcal{B}$ contains another permutation $\pi \neq \rho$ such that $\pi(1) \neq 1$. Then, by the same argument as in the proof of Theorem 3.6, we would have

$$|\mathcal{A}| = |\mathcal{A}_{1\mapsto 1}| \leq (n-1)! - d_{n-1} - 2d_{n-2},$$

so by assumption,

$$|\mathcal{B}| \geq \frac{((n-1)! - d_{n-1} - d_{n-2})((n-1)! + 1)}{(n-1)! - d_{n-1} - 2d_{n-2}} = (n-1)! + \Omega((n-2)!).$$

This implies that $|\mathcal{B} \setminus \mathcal{B}_{1\mapsto 1}| = \Omega((n-2)!)$, so $\mathcal{B} \setminus \mathcal{B}_{1\mapsto 1}$ cannot be $(\log n)$-intersecting. Hence, by the same argument as in the proof of Theorem 3.6,

$$|\mathcal{A}_{1\mapsto 1}| \leq ((1 - 1/e)^2 + o(1))(n-1)!.$$

Therefore,

$$\sqrt{|\mathcal{A}_{1\mapsto 1}||\mathcal{B}_{1\mapsto 1}|} \leq (1 - 1/e + o(1))(n-1)!$$

— contradicting (15). Hence, $\rho$ is the only permutation in $\mathcal{B}$ not fixing 1, i.e.

$$\mathcal{B} = \mathcal{B}_{1\mapsto 1} \cup \{\rho\}.$$

So we must have equality in (17), i.e.

$$\mathcal{B}_{1\mapsto 1} = \{\sigma \in S_n : \ \sigma(1) = 1\}.$$

But then we must also have equality in (16), i.e.

$$\mathcal{A} = \{\sigma \in S_n : \ \sigma(1) = 1, \ \sigma \text{ intersects } \rho\},$$

proving the theorem. □

## 4 Conclusion and open problems

Due to our use of the martingale inequality in Theorem 3.3, our proof of the Cameron-Ku conjecture requires $n > 10^4$, so it is obviously impracticable to check the remaining cases using a computer. It would be interesting to find a proof that works for all $n \geq 6$; we do not rule out the possibility of a purely combinatorial proof, although we have been unable to find one.

We now turn to the question of $k$-intersecting families of permutations. In [7], it is proved that for $n$ sufficiently large depending on $k$, if $\mathcal{A} \subset S_n$ is $k$-intersecting, then $|\mathcal{A}| \leq (n-k)!$, with equality only if $\mathcal{A}$ is a '$k$-coset', meaning a family of the form

$$\{\sigma \in S_n : \sigma(i_1) = j_1, \sigma(i_2) = j_2, \ldots, \sigma(i_k) = j_k\},$$

for some distinct $i_1, \ldots, i_k \in [n]$ and distinct $j_1, \ldots, j_k \in [n]$. One of the most natural open problems in the area is to obtain an analogue of the Ahlswede-Khachatrian theorem (see [1]) for $k$-intersecting families in $S_n$, i.e. to determine the maximum-sized $k$-intersecting families in $S_n$ for *every* value of $n$ and $k$. We make the following conjecture:

**Conjecture 1.** *A maximum-sized $k$-intersecting family in $S_n$ must be a double translate of one of the families*

$$\mathcal{F}_i = \{\sigma \in S_n : \sigma \text{ has at least } k{+}i \text{ fixed points in } [k{+}2i]\} \ (0 \leq i \leq (n{-}k)/2).$$

This would imply that the maximum size is $(n-k)!$ for $n > 2k$. We believe that new techniques will be required to prove the above conjecture.

In [6], the author proves an analogue of the Cameron-Ku conjecture for $k$-intersecting families of permutations:

**Theorem 4.1.** *For $n$ sufficiently large depending on $k$, if $\mathcal{A} \subset S_n$ is a $k$-intersecting family which is not contained within a $k$-coset, then $\mathcal{A}$ is no larger than the family*

$$\begin{aligned} \mathcal{D} &= \{\sigma \in S_n : \sigma(i) = i \ \forall i \leq k, \ \sigma(j) = j \text{ for some } j > k+1\} \\ &\quad \cup \{(1\ k+1), (2\ k+1), \ldots, (k\ k+1)\}, \end{aligned}$$

*which has size $(1 - 1/e + o(1))(n-k)!$. Moreover, if $\mathcal{A}$ has the same size as $\mathcal{D}$, then it must be a double translate of $\mathcal{D}$.*

The methods used are similar to those in this paper, but the representation-theoretic arguments are substantially more involved. It would also be interesting to obtain an analogue of the complete non-trivial $k$-intersection theorem of Ahlswede and Khachatrian in [2]. We make the following conjecture:

**Conjecture 2.** *For any $n$ and $k$, if $\mathcal{A} \subset S_n$ is a $k$-intersecting family which is not contained within a $k$-coset, and has the maximum size subject to these conditions, then it must be a double translate of the family $\mathcal{D}$ in Theorem 4.1, or of one of the $\mathcal{F}_i$'s.*

We now turn to the question of improving Theorem 3.1. We conjecture that the hypothesis $|\mathcal{A}| \geq \Omega((n-1)!)$ is unnecessary; in fact, we make the following:

**Conjecture 3.** *If $\mathcal{A} \subset S_n$ is intersecting, then it requires the removal of at most*

$$(n-2)! - (n-3)!$$

*permutations to make it centred. If $n \geq 6$, then equality holds only if $\mathcal{A}$ is a double translate of*

$$\{\sigma \in S_n : \sigma \text{ has at least 2 fixed points in } \{1, 2, 3\}\}.$$

We make the analogous conjecture for $k$-intersecting families:

**Conjecture 4.** *For $n$ sufficiently large depending on $k$, if $\mathcal{A} \subset S_n$ is $k$-intersecting, then there exists a $k$-coset containing all but at most*

$$k((n-k-1)! - (n-k-2)!)$$

*of the permutations in $\mathcal{A}$. This is sharp only when $\mathcal{A}$ is a double translate of $\mathcal{F}_1$.*

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