Intersecting families of permutations

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U: Universe of objects;
P: Property of subsets of U;

Extremal question: How large can a subset of U be if it has the property P?

What are the largest subsets of U with the property P (the '*extremal*' subsets with property P)?

What can we say about the structure of 'large' subsets with property P?

Stability question: If a subset with property P has size 'close' to the maximum, must it be 'close' to one of extremal families, in some appropriate sense?

For example...

Let X be an n-element set;

Let $X^{(k)}$ denote the set of all k-element subsets of X.

Definition

We say a family $\mathcal{A} \subset X^{(k)}$ is *intersecting* if $x \cap y \neq \emptyset \ \forall x, y \in \mathcal{A}$.

Theorem (Erdős-Ko-Rado)

Let X be an n-element set. If $k \leq n/2$, any intersecting family $\mathcal{A} \subset X^{(k)}$ has size at most

$$\binom{n-1}{k-1}$$

If $k < n/2$, equality holds iff $\mathcal{A} = \{x \in X^{(k)} : \ i \in x\}$ for some $i \in [n]$.

Definition

We say an intersecting family $\mathcal{A} \subset X^{(k)}$ is *trivial* if there is some element of X contained in all of its members.

Question: What are the largest *non-trivial* intersecting families in $X^{(k)}$?

Theorem (Hilton-Milner, 1967)

For $4 \leq k < n/2$, if $\mathcal{A} \subset X^{(k)}$ is a non-trivial intersecting family of maximum size, then

$$\mathcal{A} = \{x \in X^{(k)} : i \in x,\ x \cap y \neq \emptyset\} \cup \{y\}$$

for some $i \in [n]$ and some k-set y not containing i, so it can be made into a trivial family by removing just one k-set.

The following 'rough stability' result was obtained by Dinur and Friedgut:

Theorem (Dinur-Friedgut, 2008)

For any $\eta > 0$, there exists some $c = c(\eta)$ such that for $k < (\frac{1}{2} - \eta)n$, if $\mathcal{A} \subset X^{(k)}$ is an intersecting family with

$$|\mathcal{A}| \geq (1 - \delta)\binom{n-1}{k-1}$$

then there is some $i \in X$ contained in all but at most $c\delta\binom{n}{k}$ members of $\mathcal{A}$.

Other results in this direction were obtained by Keevash and Mubayi.

In our talk: $U = G$, a fixed finite group.
First:

$$\begin{aligned} G &= S_n, \text{ the symmetric group} \\ &= \{\text{permutations of } \{1, 2, \ldots, n\}\} \\ &= \{\text{bijections from } \{1, 2, \ldots, n\} \text{ to itself}\} \end{aligned}$$

We start by considering analogues of the Erdős-Ko-Rado Theorem, the Dinur-Friedgut Theorem and the Hilton-Milner Theorem in the symmetric group.

Definition

A family of permutations $\mathcal{A} \subset S_n$ is said to be *intersecting* if any two permutations in $\mathcal{A}$ agree at some point —i.e. for any $\sigma, \pi \in \mathcal{A}$, $\exists i \in [n]$ such that $\sigma(i) = \pi(i)$.

E.g. this family of permutations in S_4 is intersecting:

2	1	3	4
2	1	4	3
2	3	1	4
2	3	4	1
2	4	1	3
2	4	3	1

If all permutations in $\mathcal{A}$ map i to j for some fixed $i, j \in [n]$, we say $\mathcal{A}$ is *trivial*.
For any fixed $i, j \in [n]$, the family $\mathcal{C}_{i \mapsto j} := \{\sigma \in S_n : \sigma(i) = j\}$ has size $(n-1)!$.
We call these families the '1-cosets' of S_n.

An example of a non-trivial intersecting family:

$$\{\sigma \in S_n : \sigma \text{ fixes at least two points in } \{1, 2, 3\}\}$$

— has size $3(n-2)! - 2(n-3)!$

e.g. for $n = 4$:

1	2	3	4
1	2	4	3
4	2	3	1
1	4	3	2

An example of a non-trivial intersecting family of size $\Theta((n-1)!)$:

$$\{\sigma : \sigma(1) = 1,\ \sigma(i) = i \text{ for some } i > 2\} \cup \{(1\ 2)\}$$

— has size $(1 - 1/e + o(1))(n-1)!$, but can be made trivial just by removing $(1\ 2)$.

The basic extremal question: What are the largest intersecting families in S_n?

Answer: The 1-cosets $\mathcal{C}_{i \mapsto j} = \{\sigma \in S_n : \ \sigma(i) = j\}$.

Theorem (Deza, Frankl 1977)

If $\mathcal{A} \subset S_n$ is intersecting, then $|\mathcal{A}| \leq (n-1)!$

Proof:

Take any n-cycle ρ, and let $H = \langle \rho \rangle$ be the cyclic group of order n generated by ρ.

e.g. for $n = 4, \rho = (1\ 2\ 3\ 4)$, $H =$

2 3 4 1
3 4 1 2
4 1 2 3
1 2 3 4

Notice that any two distinct permutations in H disagree at every point.

The same is true for any left coset σH of H.

So any left coset of H contains at most one permutation in $\mathcal{A}$.

The $(n-1)!$ left cosets of H partition S_n, e.g.

2 3 4 1	3 2 4 1	2 4 3 1	3 4 2 1	4 2 3 1	4 3 2 1
3 4 1 2	2 4 1 3	4 3 1 2	4 2 1 3	2 3 1 4	3 2 1 4
4 1 2 3	4 1 3 2	3 1 2 4	2 1 3 4	3 1 4 2	2 1 4 3
1 2 3 4	1 3 2 4	1 2 4 3	1 3 4 2	1 4 2 3	1 4 3 2

Hence, $|\mathcal{A}| \leq (n-1)!$ as required.

It turned out to be surprisingly hard to characterize the case of equality:

Theorem (Cameron-Ku / Larose-Malvenuto, 2003)

The intersecting families in S_n of size $(n-1)!$ are precisely the 1-cosets $\{\sigma \in S_n : \sigma(i) = j\}$.

Definition

A family $\mathcal{A} \subset S_n$ is *t-intersecting* if any two permutations in it agree on at least t points, i.e.
$\forall \sigma, \pi \in \mathcal{A}, \ |\{i \in [n] : \sigma(i) = \pi(i)\}| \geq t.$

For example, the set $\mathcal{C}$ of all permutations fixing $1, 2, \ldots,$ and t is a t-intersecting family of size $(n - t)!$.

e.g. for $n = 5, \ t = 2$:

1	2	3	4	5
1	2	3	5	4
1	2	4	3	5
1	2	4	5	3
1	2	5	3	4
1	2	5	4	3

Definition

The t-cosets of S_n are the 'double translates' $\{\pi \mathcal{C} \tau : \ \pi, \tau \in S_n\}$, i.e. the families $\mathcal{C}_{i_1 \mapsto j_1, \ldots, i_t \mapsto j_t} = \{\sigma \in S_n : \sigma(i_k) = j_k \ \forall k \in [t]\}$ for distinct $i_1, \ldots, i_t$ and distinct $j_1, \ldots, j_t$.

If n is small compared to t, the t-cosets may not be the largest t-intersecting families in S_n...
e.g. for $n = 2t + 1$, the family

$$\mathcal{G}_1 = \{\sigma \in S_n : \sigma \text{ has at least } t{+}1 \text{ fixed points in } \{1, 2, \ldots, t{+}2\}\}$$

is a t-intersecting family of size

$$(t+2)(n-t-1)! - (t+1)(n-t-2)! > (n-t)!$$

However...

Conjecture (Deza, Frankl 1977)

If n is sufficiently large depending on t, the t-cosets are the largest t-intersecting families in S_n.

This was proved independently by E. and by Ehud Friedgut and Haran Pilpel in 2008, using very similar techniques. (We have submitted a joint paper which combines our versions.)

Deza and Frankl observed that to generalize their Katona-type proof for $t = 1$, one would need to find a **sharply t-transitive subset** H of S_n.

Definition

A subset $H \subset S_n$ is *sharply t-transitive* if for any distinct $i_1, \ldots, i_t \in [n]$ and any distinct $j_1, \ldots, j_t \in [n]$, there is a unique $\sigma \in H$ such that $\sigma(i_k) = j_k \; \forall k \in [t]$.

A sharply t-transitive subset of S_n necessarily has size $n!/(n-t)!$.

Suppose there exists a sharply t-transitive subset H of S_n.
Then any two distinct permutations in H agree in at most $t-1$ places,
and the same is true for any left translate σH of H.
Hence, any t-intersecting family $\mathcal{A} \subset S_n$ contains at most one permutation in any left translate of H.
Averaging over all left translates of H implies that $|\mathcal{A}| \leq (n-t)!$.

For $t = 2$ and $n = q$ a prime power, S_n has a sharply 2-transitive *subgroup* H: identify the ground set with the finite field $\mathbb{F}_q$ of order q, and take H to be the group of all affine maps

$$x \mapsto ax + b : \ (a \in \mathbb{F}_q \setminus \{0\}, \ b \in \mathbb{F}_q)$$

For $t = 3$ and $n = q + 1$ (where q is a prime power), S_n has a sharply 3-transitive subgroup: identify the ground set with $\mathbb{F}_q \cup \{\infty\}$ and take H to be the group of all Mobius transformations

$$x \mapsto \frac{ax + b}{cx + d} \quad (a, b, c, d \in \mathbb{F}_q, \ ad - bc \neq 0)$$

However, for $t \geq 4$, sharply t-transitive subgroups of S_n only exist when

- $t = 4, n = 11$ (the Matthieu group M_{11})
- $t = 5, n = 12$ (the Matthieu group M_{12})
- $t = n - 2$ (the alternating group A_{n-2})
- $t = n$ (S_n)

Moreover, sharply t-transitive subsets of S_n have not been found for any other values of n and t.
Thus, it seems that this approach is doomed.

Instead, we use an eigenvalue technique combined with representation theory of S_n, together with a combinatorial argument.

Open problem: Characterize the maximum-sized t-intersecting families in S_n for *each* value of n and t.
The following are natural candidates for the largest t-intersecting families in S_n:

$$\mathcal{G}_i = \{\sigma \in S_n : \sigma \text{ has } \geq t + i \text{ fixed points in } \{1, 2, \ldots, t + 2i\}\}$$

Conjecture

For any n and t, a t-intersecting family in S_n has size at most

$$\max_i |\mathcal{G}_i|$$

i.e., 'one of the $\mathcal{G}_i$'s always wins'.
A maximum-sized t-intersecting family $\mathcal{A} \subset S_n$ is isomorphic to some $\mathcal{G}_i$, meaning $\mathcal{A}$ is a double translate of $\mathcal{G}_i$: there exist $\pi, \tau \in S_n$ such that $\mathcal{A} = \pi \mathcal{G}_i \tau$.

This would be an analogue of the Ahlswede-Khachatrian Theorem, which characterizes the maximum-sized t-intersecting families in $[n]^{(k)}$ for every value of n, k and t.
Let

$$\mathcal{F}_i = \{x \in [n]^{(k)} : |x \cap \{1, 2, \ldots, t + 2i\}| \geq t + i\}$$

Theorem (Ahlswede-Khachatrian, 1997)

For $n > 2k - t$, if $\mathcal{A} \subset [n]^{(k)}$ is t-intersecting, then

$$|\mathcal{A}| \leq \max_i |\mathcal{F}_i|$$

A maximum-sized t-intersecting family in $X^{(k)}$ is isomorphic to one of the $\mathcal{F}_i$'s.

Definition

We say a t-intersecting family is *trivial* if it is contained within a t-coset of S_n.

Theorem (E, 2008)

If n is sufficiently large depending on t, the non-trivial t-intersecting families in S_n of maximum size are isomorphic to the family

$$\begin{aligned}\mathcal{B} &= \{\sigma : \sigma(1) = 1, \ldots, \sigma(t) = t, \sigma(j) = j \textit{ for some } j > t+1\} \\ &\quad \cup \{(1\ t+1), (2\ t+1), \ldots, (t\ t+1)\}\end{aligned}$$

(which has size $(1 - 1/e + o(1))(n-t)!$).

Open Problem: Characterize the maximum-sized 'non-trivial' t-intersecting families in S_n for every value of n and t.

This has been done for t-intersecting families in $[n]^{(k)}$ by Ahlswede and Khachatrian (1996).

A crucial tool in the proof is the following 'rough stability' result:

Theorem (E, 2008)

Let $c > 0$ be any positive constant. If $\mathcal{A} \subset S_n$ is a t-intersecting family of permutations of size $|\mathcal{A}| \geq c(n-t)!$, then there exists a t-coset $\mathcal{C}$ of S_n such that all but a $O(1/n)$-fraction of $\mathcal{A}$ lies in $\mathcal{C}$, i.e.

$$|\mathcal{A} \setminus \mathcal{C}| \leq O(1/n)|\mathcal{A}|$$

This is proved using non-Abelian Fourier Analysis, together with an extremal result on cross-t-intersecting families of permutations.

A stability argument may then be used to deduce our Hilton-Milner type result.

In fact, more may be true...

Conjecture (E, 2009)

If $\mathcal{A} \subset S_n$ is **any** *t-intersecting family of permutations, then there exists some t-coset $\mathcal{C}$ containing all but at most*

$$O((n-t-1)!)$$

permutations in $\mathcal{A}$.

The $t = 1$ case of our Hilton-Milner type result was a conjecture of Cameron and Ku (2003).

Theorem (E, 2008)

If n is sufficiently large, the non-trivial intersecting families in S_n of maximum size are isomorphic to the 'Hilton-Milner' family

$$\mathcal{B} = \{\sigma : \sigma(1) = 1, \sigma(j) = j \textit{ for some } j > 2\} \cup \{(1\ 2)\}$$

— *which has size* $(1 - 1/e + o(1))(n - 1)!$.

Consider the graph Γ_t on S_n where we join two permutations iff they agree on less than t points.
$=$ the Cayley graph on S_n generated by the set X_t of all permutations with less than t fixed points.
A t-intersecting family $\equiv$ an independent set in Γ_t.
Task: Find the independent sets of maximum size in Γ_t.

Theorem (Hoffman, 1969)

Let Γ be a d-regular graph whose adjacency matrix A has eigenvalues $d = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N$; then any independent set $\mathcal{A}$ in $V(\Gamma)$ has

$$|\mathcal{A}| \leq \frac{-\lambda_N}{\lambda_1 - \lambda_N} N$$

and equality implies that the characteristic vector of $\mathcal{A}$ lies in the direct sum of the λ_1 and λ_N eigenspaces of A.

$t = 1 : \Gamma_1 =$ the *derangement graph*: join two permutations iff they disagree everywhere
$=$ the Cayley graph generated by the set X_1 of all *derangements* (permutations without fixed points).
It is d_n-regular, where $d_n =$ number of derangments
$= (1/e + o(1))n!$

Question (Ku, 2007): What is the least eigenvalue of the derangement graph?

Theorem (Renteln, 2007)

The derangement graph has least eigenvalue

$$\lambda_N = -d_n/(n-1)$$

Substituting this into Hoffman's bound immediately implies that any intersecting family $\mathcal{A} \subset S_n$ has size at most $(n-1)!$, giving an alternative proof of the theorem of Deza and Frankl.

The direct sum of the d_n and $-d_n/(n-1)$-eigenspaces of the derangement graph is precisely the subspace

$$U_1 = \mathrm{Span}\{v_{i\mapsto j} : \ i, j \in [n]\}$$

spanned by the characteristic vectors $v_{i\mapsto j}$ of the 1-cosets of S_n.
If $\mathcal{A}$ is an intersecting family of permutations of the maximum size $(n-1)!$, then equality must hold in Hoffman's bound, so the characteristic vector of $\mathcal{A}$ must be a linear combination of the characteristic vectors of the 1-cosets.
It turns out that any family whose characteristic vector is a linear combination of those of the 1-cosets of S_n must be a disjoint union of 1-cosets of S_n. (The most elegant proof of this is due to Friedgut and Pilpel.)
Hence, an intersecting family of size $(n-1)!$ must consist of a single 1-coset:

$$\mathcal{A} = \{\sigma \in S_n : \sigma(i) = j\}$$

for some $i, j \in [n]$.

Definition

A pair of families $\mathcal{A}, \mathcal{B} \subset S_n$ is said to be *cross-intersecting* if for any $\sigma \in \mathcal{A}$ and any $\pi \in \mathcal{B}$, σ and π agree at some point, i.e. $\sigma(i) = \pi(i)$ for some $i \in [n]$.

Conjecture (Leader, 2005)

For $n \geq 4$, if $\mathcal{A}, \mathcal{B} \subset S_n$ are cross-intersecting, then $|\mathcal{A}||\mathcal{B}| \leq ((n-1)!)^2$, with equality iff $\mathcal{A} = \mathcal{B} = \{\sigma \in S_n : \sigma(i) = j\}$ for some $i, j \in [n]$.

No combinatorial proof of this conjecture is known, but given our knowledge about the derangement graph, we get it for free using the following 'cross-independent' version of Hoffman's bound:

Theorem (Alon, Kaplan, Krivelevich, Malkhi, Stern)

Let Γ be a d-regular graph on N vertices, whose adjacency matrix A has eigenvalues $d = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N$. Let $\nu = \max(|\lambda_2|, |\lambda_N|)$. If $\mathcal{A}, \mathcal{B} \subset V(\Gamma)$ have no edges of Γ between them, i.e. $e(\mathcal{A}, \mathcal{B}) = 0$, then

$$|\mathcal{A}||\mathcal{B}| \leq \left(\frac{\nu}{d+\nu}N\right)^2$$

If $|\lambda_2| < |\lambda_N|$, equality holds only if the characteristic vectors of $\mathcal{A}$ and $\mathcal{B}$ lie in the direct sum of the λ_1 and λ_N eigenspaces of A.

The derangement graph has $\lambda_2 < |\lambda_N|$ for $n \geq 5$, so plugging in $\nu = d_n/(n-1)$ proves Leader's conjecture for $n \geq 5$.

Armed with this result, a stability analysis can now be performed to prove the Cameron-Ku conjecture.
Since every eigenvalue of the derangement graph except for λ_1 and λ_N is $O(|\lambda_N|/n)$, it follows from the proof of Hoffman's Theorem that if $\mathcal{A} \subset S_n$ is an intersecting family of size at least $c(n-1)!$, then the characteristic vector of $\mathcal{A}$ is 'close' to the subspace

$$U_1 = \mathrm{Span}\{v_{i \mapsto j} : i, j \in [n]\}$$

This can be used to show that one of the sets

$$\mathcal{A}_{i \mapsto j} := \{\sigma \in \mathcal{A} : \sigma(i) = j\}$$

is large, of size $\omega((n-2)!)$.
Miraculously, this relatively weak statement is all we need...

We want to compare $|\mathcal{A}_{i\mapsto j}|$ with $|\mathcal{A}_{i\mapsto k}|$ for $j \neq k$.
Observe that

$$\mathcal{A}_{i\mapsto j}, \quad (j\ k)\mathcal{A}_{i\mapsto k}$$

are families of permutations which map i to j and cross-intersect on the domain $[n] \setminus \{i\}$. Hence,

$$(i\ j)\mathcal{A}_{i\mapsto j}, \quad (i\ j)(j\ k)\mathcal{A}_{i\mapsto k}$$

are families of permutations which fix i and cross-intersect on the domain $[n] \setminus \{i\}$.
Deleting i and applying Leader's conjecture,

$$|\mathcal{A}_{i\mapsto j}||\mathcal{A}_{i\mapsto k}| \leq ((n-2)!)^2$$

and so $|\mathcal{A}_{i\mapsto k}| = o((n-2)!)\ \forall k \neq j$.

Hence,

$$|\mathcal{A} \setminus \mathcal{A}_{i \mapsto j}| = \sum_{k \neq j} |\mathcal{A}_{i \mapsto k}| \leq o((n-1)!)$$

i.e. $\mathcal{A}$ is 'almost' contained within the 1-coset $\mathcal{C}_{i \mapsto j}$.
This enables us to complete the proof of the rough stability result.
We can then show that if $|\mathcal{A} \setminus \mathcal{A}_{i \mapsto j}| \geq 2$, then $\mathcal{A}$ is $\Omega(n-2)!$ smaller than the 'Hilton-Milner family'

$$\mathcal{B} = \{\sigma : \sigma(1) = 1, \sigma(j) = j \text{ for some } j > 2\} \cup \{(1\ 2)\}$$

proving the Cameron-Ku conjecture.

How does one calculate the eigenvalues of Γ_t?
Crucially, Γ_t is a Cayley graph generated by a union of conjugacy-classes of S_n, so we may use the following classical theorem to calculate its eigenvalues:

Theorem (Diaconis, Shahshahani)

Let G be a finite group, let $X \subset G$ be inverse-closed, and let $\Gamma = \Gamma(G, X)$ be the Cayley graph on G generated by X. Suppose X is conjugation-invariant. Let A be the adjacency matrix of Γ. Then for any irreducible representation V of G, the sum U_V of all copies of V in $\mathbb{C}[G]$ is an eigenspace of Γ with eigenvalue

$$\lambda_V = \frac{1}{\dim(V)} \sum_{x \in X} \chi_V(x)$$

where $\chi_V(x) = Trace_V(x)$ is the trace *of the representation V.*

Unfortunately, for $t > 1$, the adjacency matrix of Γ_t doesn't have the right eigenvalues to deduce our desired bound $|\mathcal{A}| \leq (n-t)!$ from Hoffman's bound. But...

Delsarte: Hoffman's bound still holds if we replace the adjacency matrix of our graph Γ with a *pseudo-adjacency matrix*, i.e. a real symmetric matrix M with $M_{x,y} = 0$ whenever $xy \notin E(\Gamma)$.

- **Conjugation-invariance:** Restrict to M's which are linear combinations of adjacency matrices of Cayley graphs generated by unions of conjugacy classes. This enables us to calculate eigenvalues. But we have $p(n) \sim (\sqrt{3}n/4)\exp(\pi\sqrt{2n/3})$ eigenvalues to deal with! Here, $p(n) =$ no. of partitions of n.
- **Boundedness:** Restrict to M's whose entries are uniformly bounded by some constant depending on t alone. This enables us to 'ignore' the eigenvalues corresponding to 'high-dimensional' representations of S_n. There are only $2p(t)$ constraints left to satisfy.

Using the representation theory of S_n combined with combinatorial arguments, we show that these can be satisfied for $n \geq n_0(t)$.

Applying the Delsarte-Hoffman bound then shows that any t-intersecting $\mathcal{A} \subset S_n$ satisfies $|\mathcal{A}| \leq (n-t)!$ provided $n \geq n_0(t)$, proving the Deza-Frankl conjecture.

Limitations of the 'conjugation-invariance' approach:
Our complete t-intersection conjecture would imply $n_0(t) = 2t + 2$, **but**

There does not exist a suitable matrix M satisfying the conjugation invariance condition when e.g. $t = 2$, $n = 6$.
Wilson has constructed a suitable pseudo-adjacency matrix in this case, but it is not clear how to generalize his construction for other values.

If $n \geq n_0(t)$ and $\mathcal{A}$ is a t-intersecting family of the maximum size $(n-t)!$, then the characteristic vector of $\mathcal{A}$ is in the subspace U_t spanned by the characteristic vectors of the t-cosets.

As in the $t = 1$ case, it can be shown that if $\mathcal{A} \subset S_n$ has characteristic vector in U_t, it must be a disjoint union of t-cosets. (Benabbas-Friedgut-Pilpel, 2008)

It follows that the only t-intersecting families in S_n of size $(n-t)!$ are the t-cosets.

A more complicated stability analysis can be used to prove the rough stability result for t-intersecting families and also the Hilton-Milner type result mentioned earlier. This gives an alternative proof that the t-intersecting families of size $(n-t)!$ are precisely the t-cosets of S_n for n sufficiently large.

Thank you!