

ALGEBRAIC TOPOLOGY (PART II)
EXAMPLE SHEET 3

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- (1) Let X and Y be triangulable spaces. Show that $H_i(X \vee Y) \cong H_i(X) \oplus H_i(Y)$ for $i > 0$. Show that for a wedge of n circles,

$$H_i(S^1 \vee \cdots \vee S^1) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z}^n & \text{if } i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

- (2) Compute the homology groups of the two-holed torus, that is, the torus with two points removed.
- (3) Compute the homology groups and Betti numbers of the space obtained from the torus $S^1 \times S^1$ by identifying $S^1 \times \{p\}$ to a point and $S^1 \times \{q\}$ to a point, for two distinct points p and q in S^1 .
- (4) Let K and K' be simplicial complexes and $A, B \subset K$, $A', B' \subset K'$ subcomplexes. Let $f : K \rightarrow K'$ be a simplicial map that sends A into A' and B into B' .

(a) Use the definition of the Mayer-Vietoris homomorphism Δ_* to show that the following diagram commutes.

$$\begin{array}{ccc} H_{n+1}(K) & \xrightarrow{\Delta_*} & H_n(A \cap B) \\ f_* \downarrow & & \downarrow f_* \\ H_{n+1}(K') & \xrightarrow{\Delta_*} & H_n(A' \cap B') \end{array}$$

(b) Show that if the homomorphisms $H_n(A \cap B) \rightarrow H_n(A' \cap B')$, $H_n(A) \rightarrow H_n(A')$, and $H_n(B) \rightarrow H_n(B')$ are isomorphisms for all n , then the homomorphisms $H_n(K) \rightarrow H_n(K')$ are isomorphisms for all n .

- (5) The antipodal map $f: S^n \rightarrow S^n$, $f(x) = -x$.
- (a) Let K be a simplicial complex such that $|K|$ is homeomorphic to the sphere S^{n-1} . By adding two vertices make a new complex L such that $|L|$ is homeomorphic to the sphere S^n .
- (b) For $n \geq 1$, show that the antipodal map $S^n \rightarrow S^n$ induces on $H_n(S^n)$ the map multiplication by $(-1)^{n+1}$.
- (c) Show that if n is even, the antipodal map on S^n is not homotopic to the identity.
- (6) (a) Let $f: X \rightarrow Y$ be a finite covering map where finiteness means that $f^{-1}\{y\}$ is finite for $y \in Y$. Suppose that Y has a triangulation; this always induces a triangulation of X . Show that $\chi(X) = \deg(f)\chi(Y)$. Do the Betti numbers always satisfy $b_i(X) = \deg(f)b_i(Y)$?
- (b) Let G be a finite group acting on S^n freely, that is, each $x \in S^n$ has a neighborhood U such that $U \cap g(U) = \emptyset$ for any $1 \neq g \in G$. Show that G has order at most 2 if n is even. (Assume that the quotient space is triangulable.)
- (7) Let K be a simplicial complex and $f: K \rightarrow K$ be a simplicial self-map. Choose an order for the vertices of K (which determines an orientation of each simplex in K). Let C_i denote the vector space of i -chains with coefficients in \mathbb{R} (for each $i \geq 0$) and $C_i f: C_i \rightarrow C_i$ the induced linear transformation.
- (a) Use the standard basis for C_i (of i -simplices of K) to show that if the trace $\text{tr}(C_i f)$ is not zero, then $f: |K| \rightarrow |K|$ has a fixed point.
- (b) Show that $\text{tr}(C_i f) = \text{tr}(Z_i f) + \text{tr}(B_{i-1} f)$ and $\text{tr}(Z_i f) = \text{tr}(B_i f) + \text{tr}(H_i f)$. Here Z_i denotes the vector space of i -cycles (with coefficients in \mathbb{R}) and $Z_i f$ is the induced map on the cycles; $B_{i-1} \cong C_i/Z_i$ and $B_{i-1} f$ is the induced map on B_{i-1} ; and $H_i = H_i(K, \mathbb{R}) = Z_i/B_i$ and $H_i f$ is the induced map on H_i .
- (c) Show that

$$\sum (-1)^i \text{tr}(C_i f) = \sum (-1)^i \text{tr}(H_i f).$$

- (d) Conclude that if $\sum (-1)^i \text{tr}(H_i f)$ is not zero, then $f: |K| \rightarrow |K|$ has a fixed point.

This is a weak version of the Lefschetz fixed point theorem. It only takes uniform continuity, plus the simplicial approximation theorem, plus some bookkeeping to prove the full version:

Theorem (The Lefschetz fixed point theorem). Let X be a compact triangulable space and let $f : X \rightarrow X$ be a continuous map. If the Lefschetz number $\Lambda(f) = \sum (-1)^i \text{tr}(H_i f)$ is not zero, then f has a fixed point.

(See for example M. A. Armstrong, *Basic Topology*, section 9.4 for an argument.)

What makes this theorem particularly powerful is that the formula for $\Lambda(f)$ only depends on the homotopy class of the map. (Also note: the Euler characteristic is the Lefschetz number of the identity map.)

- (8) An n -dimensional pseudomanifold is a simplicial complex K with the following properties.
- (i) Every simplex is a subsimplex of an n -simplex.
 - (ii) Every $(n-1)$ -simplex is a face of exactly two n -simplices.
 - (iii) For any two n -simplices σ and τ , there is a sequence $\sigma = \sigma_0, \sigma_1, \dots, \sigma_r = \tau$ such that each σ_i and σ_{i+1} intersect along an $(n-1)$ -simplex.

We remark that any triangulation of a connected n -manifold is an n -dimensional pseudomanifold.

Let K be an n -dimensional pseudomanifold. Show that $H_n(K)$ is isomorphic to either 0 or \mathbb{Z} . (One can say that K is orientable in the latter case.) If $H_n(K) = \mathbb{Z}$, find an n -chain in $Z_n(K)$ generating $H_n(K)$.

- (9) (a) Compute the Euler characteristic of all closed surfaces.
 (b) Show that the Klein bottle is homeomorphic to the connected sum of two copies of the real projective plane.