ON TERMINATION OF LOG FLIPS IN DIMENSION FOUR

CAUCHER BIRKAR

Abstract. We prove the termination of 4-fold log flips for klt pairs of Kodaira dimension $\kappa \geq 2$.

1. Introduction

We work over an algebraically closed field k of characteristic zero. See section 2 for notations.

Let $(X/Z, B)$ be a klt pair. We say that $(X/Z, B)$ has *log canonical dimension* $\lambda(X/Z, B)$ if $(X/Z, B)$ has a log minimal model $(\overline{X}/Z, \overline{B})$ such that $K_{\overline{X}} + \overline{B}$ is semi-ample/Z and its log canonical model has dimension $\lambda(X/Z, B)$. If $(X/Z, B)$ does not have such a log minimal model, we let $\lambda(X/Z, B) = -\infty$. On the other hand, when B is a $\mathbb{Q}$-divisor, we define the Kodaira dimension

$$\kappa(X/Z, B) := \kappa((K_X + B)|_F) + \dim Z$$

where F is the generic fibre of the surjective morphism $X \to Z$ and $\kappa((K_X+B)|_F)$ is the usual Kodaira dimension of the divisor $(K_X+B)|_F$ on F. If the minimal model conjecture and the abundance conjecture hold, then of course $\lambda(X/Z, B) = \kappa(X/Z, B)$. Here we recall the minimal model conjecture.

Conjecture 1.1 (Minimal model). *Any lc pair $(X/Z, B)$ has a log minimal model or a Mori fibre space.*

Though the above conjecture is settled in dimension 4 [10][3][4] but the termination problem is known only in some special cases. The following theorem is another step in this direction.

Theorem 1.2. *Let $(X/Z, B)$ be a klt pair of dimension 4 such that $\lambda(X/Z, B) \geq 2$. Then, any sequence of log flips starting with $(X/Z, B)$ terminates.*

Corollary 1.3. *Let $(X/Z, B)$ be a klt pair of dimension 4 such that B is a $\mathbb{Q}$-divisor and $\kappa(X/Z, B) \geq 2$. Then, any sequence of log flips starting with $(X/Z, B)$ terminates.*

Date: June 9, 2009.

Proof. Let F be the generic fibre of $X \to Z$. From the assumptions it is evident that $\kappa((K_X+B)|_F) \geq 0$. Thus, $(X/Z, B)$ has a log minimal model by [3] or [4] or [10]. Now apply [6, Theorem 3.8] to derive $\lambda(X/Z, B) \geq 2$ and use Theorem 1.2. □

It should not be difficult to generalise these results to the lc case but for simplicity we only treat the klt case. Finally, we remark that both the ACC conjecture for log canonical thresholds and the ACC conjecture for minimal log discrepancies in dimension 4 imply stronger versions of Theorem 1.2 [2][9].

Acknowledgements

I would like to thank the referee for reading this paper carefully and for the many constructive suggestions.

2. Preliminaries

We work over an algebraically closed field k of characteristic zero.

Definition 2.1. *A pair $(X/Z, B)$ consists of normal quasi-projective varieties X, Z over k, an $\mathbb{R}$-divisor B on X with coefficients in $[0,1]$ such that K_X+B is $\mathbb{R}$-Cartier, and a surjective projective morphism $X \to Z$. For a prime divisor D on some birational model of X with a nonempty centre on X, $C_X D$ denotes its centre and $a(D, X, B)$ the log discrepancy.*

Definition 2.2. *Let $(X/Z, B)$ be a klt pair. A klt pair $(\overline{X}/Z, \overline{B})$ is called a log minimal model of $(X/Z, B)$ if there is a birational map $\phi\colon X \dashrightarrow \overline{X}/Z$, such that ϕ^{-1} does not contract divisors, $\overline{B} = \phi_* B$, $\overline{X}$ is $\mathbb{Q}$-factorial, $K_{\overline{X}} + \overline{B}$ is nef/Z, and finally for any prime divisor D on X which is exceptional/$\overline{X}$, we have*

$$a(D, X, B) < a(D, \overline{X}, \overline{B})$$

If $K_{\overline{X}} + \overline{B}$ is semi-ample/Z, then there is a contraction $\psi\colon \overline{X} \to S/Z$ and an ample/Z $\mathbb{R}$-divisor H on the normal variety S such that $K_{\overline{X}} + \overline{B} \sim_{\mathbb{R}} \psi^* H/Z$. We call S the log canonical model of $(X/Z, B)$.

Note that this definition of log minimal models is equivalent to that of [3, §2] and [4, §2] in the klt case. We recall the definition of log flips.

Definition 2.3 (Cf. [2, §2])**.** *Let $(X/Z, B)$ be a pair and D an $\mathbb{R}$-Cartier divisor on X. A D-flip/Z is a diagram*

$$\begin{tikzcd} X \arrow[rr, dashed] \arrow[dr, "f"'] & & X^+ \arrow[dl, "f^+"] \\ & Z' & \end{tikzcd}$$

such that

- *X^+ and Z' are normal varieties $/Z$.*
- *f and f^+ are small projective birational contractions, where small means that they contract subvarieties of codimension ≥ 2.*
- *f is extremal, i.e. it is the contraction of an extremal ray.*
- *$-D$ is ample$/Z'$ and D^+ is $\mathbb{R}$-Cartier and ample$/Z'$ where D^+ is the birational transform of D.*

Note that if X is $\mathbb{Q}$-factorial, then X^+ is also $\mathbb{Q}$-factorial. A log flip$/Z$ *is a D-flip$/Z$ where $D = K_X + B$ and (X, B) is lc. We also call such a log flip, a $K_X + B$-flip.*

A log flip is called of type (a, b) if the flipping locus has dimension a and the flipped locus has dimension b. It is well-known that a log flip in dimension 4 is of type $(1, 2), (2, 1)$ or $(2, 2)$ [8, Lemma 5-1-17].

A sequence of log flips$/Z$ starting with $(X/Z, B)$ is a sequence $X_i \dashrightarrow X_{i+1}/Z_i$ where $i \in \mathbb{N}$, $(X_1/Z, B_1) = (X/Z, B)$ and $X_i \to Z_i \leftarrow X_{i+1}$ is a $K_{X_i} + B_i$-flip$/Z$ where B_i is the birational transform of B.

3. Proof of the theorem

Proof. Step 1. Let $X_i \dashrightarrow X_{i+1}/Z_i$ be a sequence of log flips starting with a klt pair $(X_1/Z, B_1) = (X/Z, B)$ of dimension 4 such that $\lambda(X/Z, B) \geq 2$. We can assume that X and so all the X_i are $\mathbb{Q}$-factorial by lifting the sequence using a $\mathbb{Q}$-factorialisation of $(X/Z, B)$ (see [2, Construction 3.1] or [4, Remark 2.4]).

Step 2. Since $\lambda(X/Z, B) \geq 2$, there is an $\mathbb{R}$-divisor $M > 0$ such that $K_X + B \sim_{\mathbb{R}} M/Z$. For a sufficiently small $\epsilon > 0$, $(X/Z, B + \epsilon M)$ is still klt so by replacing $(X/Z, B)$ and M with $(X/Z, B + \epsilon M)$ and $(1 + \epsilon)M$ respectively, we may assume that $\operatorname{Supp} M \subseteq \operatorname{Supp} B$. By [1, Theorem 2.15], for $i \gg 0$, the support of B_i does not contain any 2-dimensional component of the flipping locus or the flipped locus of the log flip $X_i \dashrightarrow X_{i+1}/Z_i$. But the support of B_i contains the entire flipping locus because the support of M_i, the birational transform of M, does so and because $\operatorname{Supp} M_i \subseteq \operatorname{Supp} B_i$. So, by truncating the sequence we can assume that for any i the support of B_i does not contain any 2-dimensional component of the flipping locus or the flipped locus of $X_i \dashrightarrow X_{i+1}/Z_i$ hence all the log flips are of type $(1, 2)$. For $i > 1$, let $V_i \subset X_i$ be a 2-dimensional component of the flipped locus of $X_{i-1} \dashrightarrow X_i/Z_{i-1}$.

Step 3. By [4] or [10], $(X_i/Z, B_i)$ has a log minimal model $(\overline{X}_i/Z, \overline{B}_i)$, equipped with a birational map $\phi_i \colon X_i \dashrightarrow \overline{X}_i$, which is obtained from $(X_i/Z, B_i)$ by a sequence of log flips and divisorial contractions. More precisely, the log minimal model is obtained by running the LMMP$/Z$ with scaling. In particular, the restriction of all the induced birational maps $X_1 \dashrightarrow X_i$ and $X_1 \dashrightarrow \overline{X}_i$ to $U = X_1 - \operatorname{Supp} M_1$ is an isomorphism onto the image of U. This follows from the fact that any extremal ray contracted in the process of $X_1 \dashrightarrow X_i$ or $X_i \dashrightarrow \overline{X}_i$ should be inside the support of the birational transform of M_1. Moreover, since V_i is not inside $\operatorname{Supp} M_i$, V_i is not inside the exceptional locus of $\phi_i \colon X_i \dashrightarrow \overline{X}_i$ hence it has a birational transform on $\overline{X}_i$ which we denote by $\overline{V}_i$.

Step 4. Since all the $(\overline{X}_i/Z, \overline{B}_i)$ are log minimal models of $(X/Z, B)$, and since $\lambda(X/Z, B) \geq 2$, each $K_{\overline{X}_i} + \overline{B}_i$ is semi-ample$/Z$ and it has a log canonical model S/Z of dimension ≥ 2 and a contraction $\psi_i \colon \overline{X}_i \to S$ as in Definition 2.2. In fact, S does not depend on i and there is an ample$/Z$ $\mathbb{R}$-divisor H, independent of i, such that $K_{\overline{X}_i} + \overline{B}_i \sim_{\mathbb{R}} \psi_i^* H/Z$.

Step 5. By admitting part (4) of the claim below, we finish the proof. Indeed, one can choose the ample$/Z$ $\mathbb{R}$-divisor $H \geq 0$ on S such that $W_i \subseteq \operatorname{Supp} H$ for infinitely many i where $W_i = \psi_i(\overline{V}_i)$. Let $\overline{N}_i = \psi_i^* H$. Then, there is an effective $\mathbb{R}$-divisor $N_1 \sim_{\mathbb{R}} K_{X_1} + B_1/Z$ on X_1 such that $\overline{N}_i$ is the birational transform of N_1 for every i. Let N_i on X_i be the birational transform of N_1. Now by applying [1] to $(X_1/Z, B_1 + \delta N_1)$, for some small $\delta > 0$, we deduce that $\operatorname{Supp} N_i$ does not contain V_i for $i \gg 0$. This in turn implies that $\operatorname{Supp} \overline{N}_i$ does not contain $\overline{V}_i$ for $i \gg 0$. This is a contradiction because by construction $\overline{V}_i \subseteq \operatorname{Supp} \overline{N}_i$ for infinitely many i.

From now on we deal with stating and proving the claim below. By truncating the sequence we may assume that X_i is terminal at the generic point of V_i, i.e. X_i is smooth at the generic point of V_i [1, Lemma 2.14]. Let E_i be an exceptional$/X_i$ prime divisor such that $a(E_i, X_i, B_i) = 2$ and such that its centre $C_{X_i} E_i = V_i$. Such E_i can be obtained by the blow up along V_i.

Claim 3.1. *Under the above notation and assumptions (in Step 1-5), we have the following properties:*
(1) $C_{\overline{X}_j}E_i \neq C_{\overline{X}_j}E_j$ *if* $i<j$*,*
(2) $C_{\overline{X}_1}E_i \nsubseteq \operatorname{Supp}\overline{M}_1$ *for any* i*: we use* $\overline{M}_1$ *to denote the birational transform of* M_1 *(similarly for every* i*),*
(3) the closure of $\Theta := \bigcup C_{\overline{X}_1}E_i$ *in* $\overline{X}_1$ *is of dimension* ≤ 2*,*
(4) the ample/Z $\mathbb{R}$*-divisor* $H \geq 0$ *can be chosen so that* $W_i \subseteq \operatorname{Supp} H$ *for infinitely many* i *where* $W_i = \psi_i(\overline{V}_i)$*.*

Proof. (1) This follows from the fact that $C_{X_j}E_i \neq C_{X_j}E_j$ on X_j if $i<j$.

(2) By construction, $C_{\overline{X}_i}E_i \nsubseteq \operatorname{Supp}\overline{M}_i$ because $C_{X_i}E_i \nsubseteq \operatorname{Supp} M_i$. On the other hand,

$$a(E_i,\overline{X}_1,\overline{B}_1+\tau\overline{M}_1)=a(E_i,\overline{X}_i,\overline{B}_i+\tau\overline{M}_i)=a(E_i,\overline{X}_i,\overline{B}_i)$$

for any $0\leq\tau\ll 1$ where the first equality follows from the fact that both $(\overline{X}_1,\overline{B}_1+\tau\overline{M}_1)$ and $(\overline{X}_i,\overline{B}_i+\tau\overline{M}_i)$ are log minimal models of $(X_1,B_1+\tau M_1)$, and the second equality follows from $C_{\overline{X}_i}E_i \nsubseteq \operatorname{Supp}\overline{M}_i$. But if $C_{\overline{X}_1}E_i \subseteq \operatorname{Supp}\overline{M}_1$, then

$$a(E_i,\overline{X}_1,\overline{B}_1+\tau\overline{M}_1)<a(E_i,\overline{X}_1,\overline{B}_1)=a(E_i,\overline{X}_i,\overline{B}_i)$$

for $0<\tau\ll 1$ which is a contradiction.

(3) By step 3 above, the log minimal model $(\overline{X}_1/Z,\overline{B}_1)$ is obtained from $(X_1/Z,B_1)$ by running the LMMP/Z with scaling on $K_{X_1}+B_1\sim_{\mathbb{R}} M_1/Z$, which consists of finitely many divisorial contractions and log flips. Again by step 3, the rational map $\phi_1\colon X_1 \dashrightarrow \overline{X}_1$ gives an isomorphism $U=X_1-\operatorname{Supp}M_1\to\phi_1(U)$. Since $\phi_1(U)\subseteq\overline{X}_1-\operatorname{Supp}\overline{M}_1$ and since $\phi_1^{-1}\colon\overline{X}_1\dashrightarrow X_1$ does not contract any divisor, the complement of $\phi_1(U)$ in $\overline{X}_1-\operatorname{Supp}\overline{M}_1$, say Π, does not contain any divisor. On the other hand, for any i, the generic point of $C_{\overline{X}_1}E_i$ is not in $\operatorname{Supp}\overline{M}_1$ by (2) nor in $\phi_1(U)$ as $C_{X_1}E_i\cap U=\emptyset$, so $C_{\overline{X}_1}E_i$ is contained in the closure of Π which has dimension at most 2. Therefore, the closure of Θ also has dimension at most 2 as it is contained in the closure of Π.

(4) If $\dim S\geq 3$, this follows from (3). So, we assume that $\dim S=2$. By construction, $a(E_i,\overline{X}_1,\overline{B}_1)=a(E_i,\overline{X}_i,\overline{B}_i)=2$ for all i. This could happen only if on a terminal crepant model $(\overline{X}_1^t/Z,\overline{B}_1^t)$ of $(\overline{X}_1/Z,\overline{B}_1)$, almost all $C_{\overline{X}_1^t}E_i$ are surfaces [1, Lemma 1.5]. Though [1, Lemma 1.5] deals only with log discrepancy <2 but we still can use it by taking a divisor $L\geq 0$ containing Θ and applying the lemma to $(\overline{X}_1/Z,\overline{B}_1+\tau L)$ for some small $\tau>0$.

For each i, the induced birational map $\overline{X}_1\dashrightarrow\overline{X}_i$ is decomposed into a sequence of flops as follows: take an ample divisor $A\geq 0$ on $\overline{X}_i$, and $\alpha > 0$ such that $(\overline{X}_1/Z, \overline{B}_1 + \alpha A')$ and $(\overline{X}_i/Z, \overline{B}_i + \alpha A)$ are klt where A' is the birational transform of A. Now since $K_{\overline{X}_i} + \overline{B}_i + \alpha A$ is ample/S, $K_{\overline{X}_1} + \overline{B}_1 + \alpha A'$ is big/S and by [5, Corollary 1.4.2] there is a sequence $Y_j \dashrightarrow Y_{j+1}/T_j$ of $K_{\overline{X}_1} + \overline{B}_1 + \alpha A'$-flips/$S$ such that $1 \leq j < m$, $Y_1 = \overline{X}_1$ and $Y_m = \overline{X}_i$ for some m (see also [5, Corollary 1.1.3]). Since $K_{\overline{X}_1} + \overline{B}_1 \equiv 0/S$, these flips are flops with respect to $K_{\overline{X}_1} + \overline{B}_1$. Note that here the sequence $Y_j \dashrightarrow Y_{j+1}/T_j$ depends on i.

Assume that the centre of E_i is not in the flopping locus anywhere in the sequence $Y_j \dashrightarrow Y_{j+1}/T_j$, that is, the centre of E_i is not inside the exceptional locus of $Y_j \to T_j$ for $j = 1, \dots, m-1$. Then $C_{\overline{X}_1}E_i$ is the birational transform of $\overline{V}_i = C_{\overline{X}_i}E_i$ and since $\dim \overline{V}_i = \dim V_i = 2$, $C_{\overline{X}_1}E_i$ is a 2-dimensional component of the closure of Θ. Moreover, if $1 < i' < i$ then by (1), $C_{\overline{X}_i}E_{i'} \neq C_{\overline{X}_i}E_i$ hence $C_{\overline{X}_1}E_{i'} \neq C_{\overline{X}_1}E_i$. On the other hand, by (4), the closure of Θ has only finitely many 2-dimensional components. Therefore, if i is sufficiently large, then the centre of E_i is in the flopping locus somewhere in the sequence $Y_j \dashrightarrow Y_{j+1}/T_j$.

Suppose that $i \gg 0$ and that $C_{Y_k}E_i$ is in the flopping locus of $Y_k \dashrightarrow Y_{k+1}/T_k$ for some k of the corresponding sequence $Y_j \dashrightarrow Y_{j+1}/T_j$. In particular, $\dim C_{T_k}E_i \leq 1$ and since T_k/S we deduce that $\dim W_i \leq 1$. Now let Σ be a minimal (in the sense of inclusion) subvariety of the closure of Θ which contains $C_{\overline{X}_1}E_i$ for infinitely many i. Let $\sigma = \{i \mid C_{\overline{X}_1}E_i \subseteq \Sigma\}$. Clearly, $\dim \Sigma \leq 2$. If $\dim \psi_1(\Sigma) \leq 1$, we are done, so we may assume that Σ is a surface and the restricted map $\psi_1 \colon \Sigma \to S$ is generically finite. Since $\dim W_i \leq 1$ for almost all i, $C_{\overline{X}_1}E_i \neq \Sigma$ for almost all $i \in \sigma$ which means that $\dim C_{\overline{X}_1}E_i \leq 1$ for almost all $i \in \sigma$. This in turn implies that $C_{\overline{X}_1^t}E_i$ is inside a prime exceptional divisor G of the morphism $\overline{X}_1^t \to \overline{X}_1$ for all $i \in \sigma'$ where $\sigma' \subseteq \sigma$ is some infinite subset. Since Σ was chosen to be minimal, G is mapped onto Σ, and so $\dim C_{\overline{X}_1}E_i = 1$ for almost all $i \in \sigma'$. Since $\Sigma \to S$ is generically finite, $\dim W_i = 1$ for almost all $i \in \sigma'$. In addition, $\{W_i\}_{i \in \sigma'}$ is not a finite set by minimality of Σ.

Choose a large $i \in \sigma'$. So, $\dim C_{\overline{X}_1}E_i = \dim W_i = 1$ and as mentioned earlier, the centre of E_i is in the exceptional locus of some $Y_k \to T_k$. So, there is a surface $\Gamma_i \subseteq Y_k$ which is in the exceptional locus of $Y_k \to T_k$ and such that it is mapped onto $C_{T_k}E_i$ and also onto W_i. Now let $l \leq k$ be the smallest number such that there is a surface Γ_i' which is in the exceptional locus of $Y_l \to T_l$ and such that it is mapped onto W_i. The surface Γ_i' is the birational transform of a surface Γ_i'' on $\overline{X}_1$ by the minimality of l.

Note that since $\overline{M}_1$ is effective and numerically zero over S, it is not horizontal over S, that is, it is not mapped onto S. On the other hand, if $\Gamma''_i \subseteq \operatorname{Supp} \overline{M}_1$, then W_i is contained in the image of $\overline{M}_1$ on S hence $\Gamma''_i \nsubseteq \operatorname{Supp} \overline{M}_1$ for almost all $i \in \sigma'$.

Now $\{\Gamma''_i\}_{i\in\sigma'}$ is infinite because $\{W_i\}_{i\in\sigma'}$ is infinite. Since the birational map $\phi_1^{-1}\colon \overline{X}_1 \dashrightarrow X_1$ does not contract divisors by definition (2.2) and since each Γ''_i is a surface and there are infinitely many such surfaces, Γ''_i has a birational transform on X_1 for infinitely many $i \in \sigma'$. Moreover, since $\Gamma''_i \nsubseteq \operatorname{Supp} \overline{M}_1$ for almost all $i \in \sigma'$, the birational transform of Γ''_i on X_1 is not inside $\operatorname{Supp} M_1$ for infinitely many $i \in \sigma'$. So Γ''_i has birational transforms on X_i and on $\overline{X}_i$ for infinitely many $i \in \sigma'$ because $X_1 \dashrightarrow X_i$ and $X_1 \dashrightarrow \overline{X}_i$ are both obtained by running the LMMP on $K_{X_1} + B_1 \sim_{\mathbb{R}} M_1/Z$. This contradicts the fact that Γ''_i is flopped in the sequence $Y_j \dashrightarrow Y_{j+1}/T_j$ hence does not have a birational transform on $\overline{X}_i$. □

4. A REMARK

In the proof of Theorem 1.2, if there were only finitely many log minimal models $\overline{X}_i$, then the proof would work without the restriction $\lambda(X/Z, B) \geq 2$. However, in general the number of log minimal models is not finite but it is conjectured that they are finite up to isomorphism, that is, when we forget about the induced birational relations $\overline{X}_i \dashrightarrow \overline{X}_j$. This weak finiteness follows from a conjecture of Morrison, Kawamata and Totaro about Calabi-Yau fibre spaces [11, Conjecture 8.1], [7]. Most probably a modification of the proof of Theorem 1.2 works if one has this weak finiteness.

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DPMMS, Centre for Mathematical Sciences,
Cambridge University,
Wilberforce Road,
Cambridge, CB3 0WB,
UK
email: c.birkar@dpmms.cam.ac.uk