Boundedness of ϵ-Log Canonical Complements on Surfaces

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Abstract

We prove a conjecture of Shokurov about boundedness of complements, in dimesion 2. More precisely, we prove that for any $\delta > 0$ there exist a finite set $\mathcal{N}_\delta$ of positive integers and $\varepsilon > 0$ such that any 2-dimensional totally δ-lc weak log Fano pair $(X/P \in Z, B)$, where $B \in \{\frac{m-1}{m}\}_{m\in\mathbb{N}}$, is (ε, n)-complementary$/P \in Z$ for some $n \in \mathcal{N}_\delta$. As a corollary, we give a completely new proof of the Alexeev-Borisov conjecture in dimension two, that is, we prove the boundedness of totally δ-lc log del Pezzo surfaces.

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1 Introduction

The concept of *complement* was introduced and studied by Shokurov [14] [15]. He used complements as a tool in the construction of 3-fold log flips [14] and in the classification of singularities and contractions [15]. Roughly speaking, an n-complement is a "good member" of the linear system $|-nK_X|$ divided by n. The existence of such a good member and the behaviour of the index n are the most important problems in the theory of complements.

In this paper, we study (ε, n)-complements which are the same as n-complements when $\varepsilon = 0$. The notion of (ε, n)-complement was also defined by Shokurov to capture more subtle properties of singularities and to use as an important tool in the study of the Alexeev-Borisov conjecture. For notations and terminology see section 2.

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Definition 1.1 ((ε, n)-complement) Let $\big(X/Z, B = \sum b_i B_i\big)$ be a pair of dimension d. Then, $K_X + B^+$ is an (ε, n)-*complement*/$P \in Z$ for $K_X + B$ if $B^+ = \sum b_i^+ B_i$ has the following properties:

- ◇ $(X, K_X + B^+)$ is totally ε-lc/$P \in Z$ and $n(K_X + B^+) \sim 0/P \in Z$.
- ◇ $\llcorner (n+1) b_i \lrcorner \leq n b_i^+$ for all i.

We say that $(X/P \in Z, B)$ is (ε, n)-*complementary*/P if there exists an (ε, n)-complement/P for $K_X + B$.

Despite the somewhat tricky definition above, complements have very good birational and inductive properties which make the theory a powerful tool to apply to the log minimal model program (LMMP). Complements do not always exist even with strong conditions such as $-(K_X+B)$ nef [15, 1.1]. But they certainly do exist when $(X/Z, B)$ is a klt weak log Fano and B is a $\mathbb{Q}$-divisor. In this paper, we concentrate on the problem of boundedness of complements.

Definition 1.2 Let $\Gamma \subseteq \mathbb{R}$. For a divisor $B = \sum b_i B_i$, we write $B \in \Gamma$ if all nonzero $b_i \in \Gamma$. The set $\Phi_{\mathrm{sm}} = \{\frac{k-1}{k} | k \in \mathbb{N}\} \cup \{1\}$ is called the set of *standard boundary coefficients*.

We now state Shokurov's conjectures on the boundedness of complements.

Conjecture 1.3 (Weak (ε, n)-complements) *Let $\Gamma \subset [0,1]$, $\delta > 0$ be a real number and d a natural number. Then, there exist a finite set $\mathcal{N}_{\delta,d,\Gamma}$ of positive integers and $\varepsilon > 0$ such that any d-dimensional totally δ-lc weak log Fano pair $(X/P \in Z, B)$, where $B \in \Gamma$, is (ε, n)-complementary/$P \in Z$ for some $n \in \mathcal{N}_{\delta,d,\Gamma}$.*

In practise, Γ is equal to Φ_{sm} or generalisations of it. Also note that we can always assume that $\delta \in (0, 1)$.

Conjecture 1.4 (Strong (ε, n)-complements) *Conjecture 1.3 holds with $\varepsilon = \delta$.*

If we replace $\delta, \varepsilon > 0$ with $\delta = \varepsilon = 0$ in the above conjectures, we get the usual conjecture on the boundedness of lc complements which has been studied by Shokurov, Prokhorov and others [15][13][12]. It is proved in dimension 2 for certain Γ [15][7].

The following important conjecture due to Alexeev and the Borisov brothers, is related to the above conjectures.

Conjecture 1.5 (Alexeev-Borisov) *Let $\delta > 0$ be a real number and d a natural number. Then, projective varieties X for which (X, B) is a d-dimensional totally δ-lc weak log Fano pair for some boundary B, are bounded.*

Alexeev [1] proved this conjecture in dimension 2 but still open in dimension ≥ 3. This conjecture is also closely related to other major problems in the minimal model program such as the ascending chain condition (acc) for lc thresholds [11] and termmination of log flips [3][4].

Main Theorem 1.6 *Conjecture 1.3 holds in dimension* 2 *when* $\Gamma = \Phi_{\mathrm{sm}}$.

Corollary 1.7 *Alexeev-Borisov conjecture holds in dimension* 2.

Corollary 1.8 *Conjecture 1.3 holds in dimension* 2 *in the global case (i.e.,* $Z = \mathrm{pt.}$*) when* Γ *is a finite set of rational numbers.*

Our main theorem also implies that lc complements can be constructed in dimension 3 [13] using only the theory of complements.

2 Preliminaries

Throughout this paper, we assume that all the varieties involved are algebraic varieties over a fixed field of charcteristic zero. By a *pair* (X, B), we mean a normal variety X and an $\mathbb{R}$-boundary B with coefficients in $[0, 1]$ such that $K_X + B$ is $\mathbb{R}$-Cartier. Moreover, a pair $(X/Z, B)$ consists of a log pair (X, B) and a normal variety Z equipped with a projective morphism $f \colon X \to Z$. When we write $(X/P \in Z, B)$, we mean a pair $(X/Z, B)$ with a fixed point $P \in Z$; in this situation, we may shrink Z around P in the Zariski topology without mention. We denote the *log discrepancy* of (X, B) at a prime divisor E as $a(E, X, B)$. We use the usual definition of terminal, canonical, klt, dlt and lc singularities as in [16]. A pair $(X/Z, B)$ is *weak log Fano* if (X, B) is lc and $-(K_X + B)$ is nef and big/Z and X is $\mathbb{Q}$-factorial. A variety X/Z is *Fano type* if there is B' such that $(X/Z, B')$ is a klt weak log Fano.

When we say that a property holds/$P \in Z$, we mean that that property holds in some $f^{-1}U$ where U is an open subset of Z containing P.

(X, B) is δ-lc if $a(E, X, B) \geq \delta$ for any exceptional/X prime divisor E. Moreover, it is *totally δ-lc* if $a(E, X, B) \geq \delta$ holds for any exceptional/X and nonexceptional/X prime divisor E. Note that if (X, B) is totally δ-lc then $\delta \leq 1$ because if E is a divisor on X which is not a component of B, then $a(E, X, B) = 1$.

For a real number a, $\langle a \rangle$ denotes the fractional part of a.

3 The case of curves

In this section we prove Conjecture 1.3 in dimension one. Note that 1-dimensional global weak log Fano pairs are just $(\mathbb{P}^1, B)$ for a boundary $B = \sum b_i B_i$ where $\sum b_i - 2 < 0$. The local case for curves is trivial.

Theorem 3.1 *Conjecture 1.3 holds in dimension* 1*; more precisely, if* $\frac{m-1}{m} \leq 1 - \delta < \frac{m}{m+1}$ *for* $m \in \mathbb{N}$*, then we can take* $\varepsilon = \frac{1}{m+1}$ *and*

$$\mathcal{N}_{\delta,1,[0,1]} = \{1, \ldots, m+2\}$$

Proof Let $(\mathbb{P}^1, B = \sum b_i B_i)$ be a totally δ-lc weak log Fano pair and let $b_j = \max\{b_i\}$. If $\delta = 1$, then $B = 0$ and this case is trivial.

So we can assume that $\frac{k-1}{k} \leq b_j < \frac{k}{k+1}$ for a natural number $k \leq m$. If $k = 1$, then $b_i < \frac{1}{2}$ and there can be at most 5 of the b_i in $[\frac{1}{3}, \frac{1}{2})$. Thus,

$$\sum \llcorner 3b_i \lrcorner < 6$$

If $\sum \llcorner 3b_i \lrcorner \leq 4$, then we can take $n = 2$. Otherwise, $\sum \llcorner 3b_i \lrcorner = 5$ and it is easy to see that

$$\sum \llcorner 4b_i \lrcorner \leq 6$$

and so we can take $n = 3$.

Now assume that $k > 1$ and define $a_{i,t} = \llcorner (t+1)b_i \lrcorner$. Note that since $\sum b_i < 2$, then

$$\sum a_{i,k} = \sum \llcorner (k+1)b_i \lrcorner \leq \sum (k+1)b_i < 2k + 2$$

If we have

$$\sum \lfloor (k+1)b_i \rfloor \leq 2k$$

then we take $n = k$. If not, then $\sum a_{i,k} = 2k+1$. Since $\frac{k-1}{k} \leq b_j < \frac{k}{k+1}$ we have

$$\frac{(k+1)(k-1)}{k} = k+1-\frac{k+1}{k} = k-\frac{1}{k} \leq (k+1)b_j < \frac{(k+1)k}{k+1} = k$$

Then $a_{j,k} = \llcorner (k+1)b_j \lrcorner = k-1$ and $1-\frac{1}{k} \leq \langle (k+1)b_j \rangle < 1$. Now

$$a_{i,k+1} = \llcorner (k+2)b_i \lrcorner = \llcorner (k+1)b_i + b_i \lrcorner$$

So $a_{i,k+1}$ is either equal to $a_{i,k}$ or $a_{i,k}+1$. The latter happens if and only if $1 \leq b_i + \langle (k+1)b_i \rangle$. In particular,

$$b_j + \langle (k+1)b_j \rangle \geq \frac{k-1}{k} + 1 - \frac{1}{k} \geq 1$$

so $a_{j,k+1} = a_{j,k}+1$. On the other hand since

$$\sum a_{i,k} = \llcorner (k+1)b_i \lrcorner = 2k+1$$

and since

$$\sum (k+1)b_i = \sum a_{i,k} + \sum \langle (k+1)b_i \rangle = 2k+1+\sum \langle (k+1)b_i \rangle < 2k+2$$

then $\sum \langle (k+1)b_i \rangle < 1$. Then, if $i \neq j$, then $\langle (k+1)b_i \rangle < \frac{1}{k}$ because

$$1-\frac{1}{k} \leq \langle (k+1)b_j \rangle$$

So if $i \neq j$ and if $1 \leq \langle (k+1)b_i \rangle + b_i$, then $1-\frac{1}{k} < b_i$. Now if

$$\sum a_{i,k+1} = \llcorner (k+2)b_i \lrcorner = 2k+2$$

then we take $n = k+1$. Otherwise,

$$\sum a_{i,k+1} = \llcorner (k+2)b_i \lrcorner = 2k+3$$

and so $1 \leq \langle (k+1)b_p \rangle + b_p$ must hold at least for one $p \neq j$ which implies that $1-\frac{1}{k} < b_p \leq b_j$. This in turn implies that $1-\frac{1}{k} \leq \langle (k+1)b_p \rangle$ and we get a contradiction.

Therefore we can take $\mathcal{N}_{\delta,1,[0,1]} = \{1,2,\ldots,m+2\}$ and $\varepsilon = \frac{1}{m+2}$. $\square$

4 The case of surfaces

We need some preparations before we prove the main theorem.

Definition 4.1 Let (X, B) be a lc pair. A variety Y/X is a *crepant model* of (X, B) if $K_Y + B_Y$, the pullback of $K_X + B$, is lc.

Main Lemma 4.2 *Let $\varepsilon > 0$ be a real number. Suppose that $\mathcal{U} = \{(U, \operatorname{Supp} D)\}$ is a bounded family of pairs of dimension 2 where $K_U + D$ is antinef and totally ε-lc and U is projective and $\mathbb{Q}$-factorial. Then, the set of crepant models of all $(U, D) \in \mathcal{U}$ is bounded.*

Note that here we do not assume the set of all (U, D) to be bounded, that is, the coefficients of D may not necessarily be in a finite set. A similar lemma is proved by M^{c}Kernan and Prokhorov [11] where the coefficients are assumed to be in a finite set.

Proof Using Noetherian induction we can assume that $(U, \operatorname{Supp} D)$ is fixed. We can consider any divisor supported in D as a point in a real finite dimensional space $\mathbb{R}^q$. Let $D = \sum_{i=1}^{q} d_i D_i$ and define

$$\mathcal{H} := \{H = (h_1, \dots, h_q) \in \mathbb{R}^q \mid K_U + D_H \text{ is antinef and totally } \varepsilon\text{-lc}\}$$

where $D_H = \sum_{i=1}^{q} h_i D_i$.

So $\mathcal{H}$ is a subset of the cube $[0, 1]^q$ and since being ε-lc and antinef are closed conditions, $\mathcal{H}$ is a closed and hence compact subset of $[0, 1]^q$. For each $H \in [0, 1]^q$ let R_H be the set of exceptional/U prime divisors E with $a(E, U, D_H) \leq 1$. It is enough to prove that the union of all R_H is a finite set when H runs through $\mathcal{H}$. Suppose otherwise, and let $\{H_j\}_{j \in \mathbb{N}} \subseteq \mathcal{H}$ be a sequence such that the union of all R_{H_j} is not finite. Since $\mathcal{H}$ is compact, there is at least an accumulation point for the sequence in $\mathcal{H}$, say $\overline{H}$ and we can assume that this is the only accumulation point. By construction, $(U, D_{\overline{H}})$ is totally ε-lc.

Let $H' = \sum_{i=1}^{q} D_i$. Then, there is $\alpha > 0$ such that $K_U + D_{\overline{H}+\alpha H'}$ is totally $\frac{\varepsilon}{2}$-lc. On the other hand, except for finitely many j, $H_j \leq \overline{H} + \alpha H'$. This contradicts the way we chose the sequence because $R_{\overline{H}+\alpha H'}$ is finite. □

Proposition 4.3 (???) *Let $0 < \delta < 1$. Then, Fano type projective surfaces X for which (X, B) is a δ-lc pair and $K_X + B \equiv 0$ for some $B \in [\delta, 1-\delta]$, are bounded.*

Proof Suppose that X satisfies the assumptions of the proposition. Let $\phi\colon W \to X$ be a minimal resolution of X and $\psi\colon W \to S$ be the map obtained by running the LMMP on K_W. Since X is Fano type, so is W. Let $K_W + B_W$ be the crepant pullback of $K_X + B$. Possibly after decreasing δ (but independent of X, W), we can assume that $B_W \in [\delta, 1-\delta]$. Let $B_S = \sum b_{i,S} B_{i,S}$ be the pushdown of B_W on S. We know that S is isomorphic to $\mathbb{P}^2$ or a smooth ruled surface with no -1-curves.

In any case, it is a simple excercise to get the boundedness of $(S, \operatorname{Supp} B_S)$ from well known works or from [**??**]. Now Lemma 4.2 implies the boundedness of W and so of X. $\square$

Remark 4.4 Let $\operatorname{Accum}(2, \Phi_{\mathrm{sm}})$ be the set of accumulation points of the following set

$$\{\operatorname{mld}(P, T, B) \mid (T, B) \text{ is of dimension 2 and lc at } P \text{ and } B \in \Phi_{\mathrm{sm}}\}$$

By [17][2, Corollary 3.4] we have

$$\operatorname{Accum}(2, \Phi_{\mathrm{sm}}) = \{\frac{1}{k}\}_{k \in \mathbb{N}} \cup \{0\}$$

Let $m \in \mathbb{N}$, $\tau > 0$ and let (T, B) be a 2-dimensional pair which is totally $\frac{1}{m}$-lc at $P \in T$. Suppose that $a(E, T, B) \notin (\frac{1}{k}, \frac{1}{k} + \tau)$ for any natural number $k > 1$ and any exceptional$/X$ prime divisor E whose centre on X is P. Then, there are only finitely many possibilities for the index of $K_T + B$ at P depending on m and τ but independent of (T, B).[4.4.1]

Definition 4.5 Let $\Gamma \subset \mathbb{R}$ and $\tau > 0$ a real number. We define

$$\Gamma^\tau = \bigcup_{a \in \Gamma} (a - \tau, a)$$

Now for a divisor $D = \sum d_i D_i$ define

$$D^\tau := \sum_{d_i \notin {\Phi_{\mathrm{sm}}}^\tau} d_i D_i + \sum_{d_i \in {\Phi_{\mathrm{sm}}}^\tau} \frac{k-1}{k} D_i$$

[4.4.1] ref??

where in the second sum k is the smallest natural number satisfying $d_i \in (\frac{k-1}{k} - \tau, \frac{k-1}{k})$.

Lemma 4.6 *For any natural number m, there is a real number $\tau > 0$ (depending only on m) such that if (T, B) is a surface pair which is $\frac{1}{m}$-lc at $P \in T$ and $B^\tau \in \Phi_{\mathrm{sm}}$, then $K_T + B^\tau$ is also $\frac{1}{m}$-lc at P.*

Proof See [4, Main Proposition 2.1]. □

Construction 4.7 (Cf. [15]) Let $m \in \mathbb{N}$ and $\delta \geq \frac{1}{m}$ be a real number. Let $h = \frac{1}{(r+2)!}$ where $r = \max\{m, 6\}$. Now choose a τ for m as in Lemma 4.6 such that $\tau < h$.

Let $(X/Z, B)$ be a totally δ-lc Fano type surface pair where $B \in \Phi_{\mathrm{sm}}$ and $-(K_X + B)$ is nef/Z. Now suppose that there is an exceptional/X prime divisor E such that the log discrepancy satisfies $\frac{1}{k} < a(E, X, B) < \frac{1}{k} + \tau$ for some natural number $k > 1$. Let $f\colon Y \to X$ be the extraction of E. The crepant log divisor $K_Y + B_Y$ is totally $\frac{1}{m}$-lc and so $K_Y + B_Y^\tau$ is also totally $\frac{1}{m}$-lc by Lemma 4.6. By the Fano type assumption, we can run the LMMP/Z on $-(K_Y + B_Y^\tau)$; that is, unless $\rho(Y) = 1$, we contract an extremal ray R/Z via $Y \to Y_1$ such that $(K_Y + B_Y^\tau) \cdot R > 0$. Moreover, we can assume that there is a boundary B' such that $B_Y \leq B' \leq B_Y^\tau$, $-(K_Y + B')$ is nef/Z and $-(K_Y + B') \cdot R = 0$. We continue the process for Y_1 and so on.

Note that if in some step R is a fibre type extremal ray, then it defines a contraction $Y \to S/Z$ and by restricting to a general fibre $F \simeq \mathbb{P}^1$, we get a pair (F, B'_F) such that $K_F + B'_F \equiv 0$ and $B'_F \in {\Phi_{\mathrm{sm}}}^\tau \cup \Phi_{\mathrm{sm}}$. Now we can repeat the whole argument with $\frac{\tau}{2}, \frac{\tau}{3}, \ldots$ Either for some l, we get never get fibre type extremal rays in our construction using $\frac{\tau}{l}$ or we get a sequence of 1-dimensional lc pairs (F_i, B'_{F_i}) such that $K_{F_i} + B'_{F_i} \equiv 0$ and $B'_{F_i} \in {\Phi_{\mathrm{sm}}}^{\frac{\tau}{i}} \cup \Phi_{\mathrm{sm}}$. This contradicts [4, Proposition 4.1].

Now continue the process on Y_1 (as we did on Y) and so on. Thus, after finitely many steps, we get a model X_1 and the corresponding morphism $g\colon Y \to X_1$ such that either $K_{X_1} + B_1$ is antinef/Z or $\rho(X_1) = 1$ where B_1 is the pushdown of B_Y^τ. In any case, by Lemma 4.6, $K_{X_1} + B_1$ is totally $\frac{1}{m}$-lc and Y is a crepant model of (X_1, B'_1) for some boundary $g_*B \leq B'_1 \leq B_1$ such that $K_{X_1} + B'_1$ is antinef/Z. We call $(X_1/Z, B_1)$ a *τ-minimization* for $(X/Z, B)$. Note that by construction, $B_1 \in \Phi_{\mathrm{sm}}$ and $(X_1/Z, B_1)$ is FT.

Lemma 4.8 (Termination of τ-minimizations) *Let $m \in \mathbb{N}$ and τ as in Lemma 4.6. Then, there is no infinite sequence of τ-minimizations.*

Proof Suppose not and let $(X_i/Z, B_i)$ be a sequence of totally $\frac{1}{m}$-lc surface pairs index in i where $B_i \in \Phi_{\rm sm}$ and $(X_i/Z, B_i)$ is a τ-minimization of $(X_{i-1}/Z, B_{i-1})$. Note that, for each i, we blow up one exceptional/X_i divisor E_i via $f_i\colon Y_i \to X_i$ such that the log discrepancy satisfies $\frac{1}{k} < a(E_i, X_i, B_i) < \frac{1}{k} + \tau$ for some natural number $k > 1$. This in particular means that $f_i\colon Y_i \to X_i$ is not the blow up of a smooth point because $\tau < h$. Thus $a(E_i, X_i, 0) \le 1$. Moreover, a minimal resolution $h_i\colon T_i \to X_i$ factors through f_i. Therefore, there is a natural morphism $T_i \to T_{i+1}$. So, this natural morphism is an isomorphism for $i \ge l$ for some l. Thus, if F is contracted by g_i for $i \ge l$, then $a(F, X_{i+1}, 0) \le 1$. Now since $-(K_{X_i} + B_i)$ is semiample/Z and since $F \neq E_i$, the birational transform of F needs to be a component of B_i.

Now let B_i' on T_l, be the creapant pullback of B_i, for $i \ge l$. The observations above show that the birational transform of any divisor contracted by f_i or g_i is a component of B_i'. Moreover, the birational transform of E_{i+1} is also a component of B_i'. In other words, the support of B_i' is fixed for $i \ge l$. On the hand, each prime divisor E on T_l can be the birational transform of E_i only finitely many times, hence, a contradiction.

$$\begin{array}{ccccccc}
Y_1 & & Y_2 & & Y_3 & & \\
\downarrow{\scriptstyle f_1} & \overset{g_1}{\searrow} & \downarrow{\scriptstyle f_2} & \overset{g_2}{\searrow} & \downarrow & \searrow & \\
X_1 & & X_2 & & X_3 & & \cdots
\end{array}$$

□

Definition 4.9 Let $m \in \mathbb{N}$ and τ as in Lemma 4.6. Let $(X/Z, B)$ be a totally $\frac{1}{m}$-lc FT surface log pair where $B \in \Phi_{\rm sm}$ and $-(K_X + B)$ is nef/Z. We construct sucsessive τ-minimizations and when the process stops at $(X_s/Z, B_s)$ we call it a *final model*.

Proof (of Main Theorem) Let m be a natural number such that $1 - \delta \le 1 - \varepsilon := \frac{m}{m+1}$. Now fix a $\tau > 0$ as in Construction 4.7 and let (X_s, B_s) be a final model as in Definition 4.9.

First assume that $K_{X_s}+B_s$ is antinef$/Z$. Then, there is no exceptional$/X_s$ prime divisor E such that $\frac{1}{k} < a(E, X_s, B_s) < \frac{1}{k}+\tau$ for some $1 < k < m$. By Remark 4.4, the index of $K_{X_s} + B_s$ is bounded. Hence, by Kollár's effective base point freeness (ε, n)-complemements are bounded for such $(X_s/Z, B_s)$. Now we get bounded (ε, n)-complements on $(X/Z, B)$ by [12, Proposition 4.3.2].

Now assume that $K_{X_s} + B_s$ is not antinef$/Z$. Then, $Z = \text{pt.}$, $K_{X_s} + B_s$ is ample and $\rho(X_s) = 1$. Moreover, there is a boundary $B'_s \leq B_s$ with the same support as B_s such that $K_{X_s} + B'_s \equiv 0$ and $B'_s \in {\Phi_{\text{sm}}}^{\tau}$. Let $(T_1, \Delta_1) := (X_s, B'_s)$.

Now repeat the argument by taking $\frac{\tau}{2}$-minimizations. Either we always get $K_{X_s} + B_s$ antinef for any final model or we find a lc pair (T_2, Δ_2) such that $K_{T_2} + \Delta_2 \equiv 0$ and $\Delta_2 \in {\Phi_{\text{sm}}}^{\frac{\tau}{2}}$.

By repeating this process for $\frac{\tau}{3}, \frac{\tau}{4}, \ldots$ either for some $\frac{\tau}{l}$ we have the antinef case for all final models, or we an infinite sequence of pairs (T_i, Δ_i) such that $K_{T_i} + \Delta_i \equiv 0$ and $\Delta_i \in {\Phi_{\text{sm}}}^{\frac{\tau}{i}}$. This contradicts [4, Addendum 2.1 and Proposition 4.1]. Note that boundedness of canonical Fano surfaces in [4, Addendum 2.1] is classical and also follows immediately from [10].

□

Proof (of Corollary 1.7) First note that to prove Corollary 1.7, it is enough to consider the case $B = 0$. If $B \neq 0$, there is a contraction $X \to X'$ such that $-K_{X'}$ is nef and big, and boundedness of X' implies boundedness of X. Then, our Main Theorem and Lemma 4.3 imply the result. □

Proof (of Corollary 1.8) It follows from Corollary 1.7. □

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