

Part IID RIEMANN SURFACES (2008–2009)

Example Sheet 3

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- (1) Let $f: X \rightarrow Y$ be a covering map, in the sense of complex analysis, of compact connected Riemann surfaces. Prove that it is a covering in the sense of topology.
- (2) Let $f: X \rightarrow Y$ be a covering map, in the sense of complex analysis, where Y is a connected Riemann surface and X a connected topological surface. Show that there is a unique complex structure on X with respect to which π is a holomorphic map.
- (3) Let X, Y be connected Riemann surfaces. As we have seen in the lectures, a function element (U, f) on X (into Y) determines a complete holomorphic function which corresponds to a connected component \mathcal{F} of the set of all germs \mathcal{G} . We also have the map $\pi: \mathcal{F} \rightarrow X$ given by $\pi([x, g]) = x$ with image $V = \pi(\mathcal{F})$ an open subset of X .
Show that \mathcal{F} satisfies the second countability condition of topology. In particular, $\pi^{-1}\{x\}$ is countable for any $x \in V$. [This completes the proof of \mathcal{F} being a Riemann surface]
- (4) Consider the complete holomorphic function on \mathbb{C} determined by $\sqrt{1 + \sqrt{z}}$. Show that the corresponding Riemann surface \mathcal{F} contains exactly two germs $[z, f]$ with $z = 1$ and exactly four germs $[z, f]$ for each z such that $0 < |z - 1| < \frac{1}{2}$. [Hint: consider the possible values $f(z)$ for the function elements.]
Let $0 < \epsilon < 1/2$ and consider the associated holomorphic map $\pi: \mathcal{F} \rightarrow \mathbb{C}$. Verify that the path $\gamma(t) = 1 - \epsilon/2 + \epsilon t$, $0 \leq t \leq 1$, does not have a lift to \mathcal{F} from $[1 - \epsilon/2, g(1 - h(z))]$, where g, h are holomorphic functions near $1 - \epsilon/2$ and $1 - h(1 - \epsilon/2)$, respectively, satisfying $g(z)^2 = z$, $h(z)^2 = z$, $h(1) = 1$.
- (5) Let X be a connected Riemann surface and $Y = \mathbb{C}$. Show that \mathcal{G}_x the set of all germs at $x \in X$ (into Y) has a 'natural' ring structure. Identify a maximal ideal of this ring.
- (6) Prove that an open disc is simply connected. More generally, prove that any convex open subset of \mathbb{C} is simply connected. In particular, then \mathbb{C} is simply connected.
- (7) * Try to show that the Riemann sphere is simply connected directly from the definition of simple-connectedness. [Hint: choose a closed path and prove that it is homotopic to a closed path which does not

pass through every point of the Riemann sphere]

- (8) Let Λ be a lattice in \mathbb{C} . Prove that the quotient map $\pi: \mathbb{C} \rightarrow \mathbb{C}/\Lambda$ is a universal covering map. Conclude that the torus is not simply connected without using the uniformization theorem; but you may use the monodromy theorem.
- (9) Let $U \subset \mathbb{C}$ be a region and $a \in U$. Prove that $U \setminus \{a\}$ is not simply connected. [You are allowed to use the uniformization theorem]
- (10) Let G be a discrete subgroup of \mathbb{C} . Show that G is one of the following:
- (i) $\{0\}$,
 - (ii) $\mathbb{Z}\lambda$, $\lambda \in \mathbb{C}^*$, or
 - (iii) $\mathbb{Z}\lambda_1 + \mathbb{Z}\lambda_2$, $\lambda_i \in \mathbb{C}$, and λ_1, λ_2 are linearly independent over \mathbb{R} .
- (11) Show, using the uniformization theorem, that any holomorphic map from \mathbb{C} to a compact connected Riemann surface of genus greater than 1 is constant.
- (12) Let X be the Riemann sphere. Show that there is no non-trivial subgroup of $\text{Aut}(X)$ acting on X properly discontinuously.
- (13) (i) Let \mathcal{M} be the set of complex tori up to conformal equivalence. Show that \mathcal{M} is not countable.
(ii) Let $PSL(2, \mathbb{Z}) = \{A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid \det A = 1 \text{ and } a, b, c, d \in \mathbb{Z}\} / \pm I$ where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. You can check that $PSL(2, \mathbb{Z})$ acts on the upper half plane $\mathbb{H} = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d}$$

Show that there is a bijection between \mathcal{M} and the quotient $\mathbb{H}/PSL(2, \mathbb{Z})$, the set of orbits of \mathbb{H} under the above action.

- (14) Show that any holomorphic map f of degree 2 from a complex torus \mathbb{C}/Λ to the Riemann sphere is given by a ‘Möbius transformation of a shifted \wp -function’:

$$f(z) = \frac{a\wp(z - z_0) + b}{c\wp(z - z_0) + d},$$

for some $a, b, c, d, z_0 \in \mathbb{C}$.

- (15) Show that a compact connected Riemann surface X is a complex torus iff there is a holomorphic map $f: X \rightarrow Y$ of degree 2 with 4

branch points where Y is the Riemann sphere.

- (16) Let $X = \mathbb{C}/\Lambda$ be the complex torus defined by a lattice Λ and write $X_0 = X \setminus \{\Lambda\}$ for the complement of the coset of Λ . Show that

$$\Phi : z + \Lambda \in X_0 \rightarrow (\wp(z), \wp'(z)) \in \mathbb{C}^2$$

maps the punctured complex torus X_0 biholomorphically onto a smooth algebraic curve in \mathbb{C}^2 .

[Hint: the differential equation for \wp .]

- (17) (i) Let f and g be two elliptic functions (with the same lattice of periods) and N a positive integer. By considering the poles of f and g , estimate from above the dimension of the complex vector space spanned by $f(z)^m g(z)^n$, for $0 \leq m, n \leq N$. Deduce that when N is sufficiently large there must be a non-trivial linear dependence,

$$\sum_{m,n=0}^N a_{m,n} f(z)^m g(z)^n \equiv 0, \quad \text{for some } a_{m,n} \in \mathbb{C}.$$

Hence show that any two meromorphic functions f, g on a complex torus \mathbb{C}/Λ are ‘**algebraically related**’: there is a polynomial Q in two variables, so that $Q(f(z), g(z)) = 0$ for all z .

- (ii)* Show that in fact (i) holds for meromorphic functions on any compact connected Riemann surface.