

**Part IID RIEMANN SURFACES (2011–2012)**

**Example Sheet 3**

**c.birkar@dpmms.cam.ac.uk**

- (1) Let  $f: X \rightarrow Y$  be a covering map, in the sense of complex analysis, of compact connected Riemann surfaces. Prove that it is a covering in the sense of topology.
- (2) Let  $f: X \rightarrow Y$  be a covering map, in the sense of complex analysis, where  $Y$  is a connected Riemann surface and  $X$  a connected topological surface. Show that there is a unique complex structure on  $X$  with respect to which  $\pi$  is a holomorphic map.
- (3) Consider the complete holomorphic function on  $\mathbb{C}$  determined by  $\sqrt{1 + \sqrt{z}}$ . Show that the corresponding Riemann surface  $\mathcal{F}$  contains exactly two germs  $[z, f]$  with  $z = 1$  and exactly four germs  $[z, f]$  for each  $z$  such that  $0 < |z - 1| < \frac{1}{2}$ . [Hint: consider the possible values  $f(z)$  for the function elements.]  
Let  $0 < \epsilon < 1/2$  and consider the associated holomorphic map  $\pi: \mathcal{F} \rightarrow \mathbb{C}$ . Verify that the path  $\gamma(t) = 1 - \epsilon/2 + \epsilon t$ ,  $0 \leq t \leq 1$ , does not have a lift to  $\mathcal{F}$  from  $[1 - \epsilon/2, g(1 - h(z))]$ , where  $g, h$  are holomorphic functions near  $1 - \epsilon/2$  and  $1 - h(1 - \epsilon/2)$ , respectively, satisfying  $g(z)^2 = z$ ,  $h(z)^2 = z$ ,  $h(1) = 1$ .
- (4) Let  $X$  be a connected Riemann surface and  $Y = \mathbb{C}$ . Show that  $\mathcal{G}_x$  the set of all germs at  $x \in X$  (into  $Y$ ) has a 'natural' ring structure. Identify a maximal ideal of this ring.
- (5) Prove that an open disc is simply connected. More generally, prove that any convex open subset of  $\mathbb{C}$  is simply connected. In particular, then  $\mathbb{C}$  is simply connected.
- (6) \* Try to show that the Riemann sphere is simply connected directly from the definition of simple-connectedness. [Hint: choose a closed path and prove that it is homotopic to a closed path which does not pass through every point of the Riemann sphere]
- (7) Let  $\Lambda$  be a lattice in  $\mathbb{C}$ . Prove that the quotient map  $\pi: \mathbb{C} \rightarrow \mathbb{C}/\Lambda$  is a universal covering map. Conclude that the torus is not simply connected without using the uniformization theorem; but you may use the monodromy theorem.

- (8) Let  $U \subset \mathbb{C}$  be a region and  $a \in U$ . Prove that  $U \setminus \{a\}$  is not simply connected. [You are allowed to use the uniformization theorem]
- (9) Let  $G$  be a discrete subgroup of  $\mathbb{C}$ . Show that  $G$  is one of the following:
- (i)  $\{0\}$ ,
  - (ii)  $\mathbb{Z}\lambda$ ,  $\lambda \in \mathbb{C}^*$ , or
  - (iii)  $\mathbb{Z}\lambda_1 + \mathbb{Z}\lambda_2$ ,  $\lambda_i \in \mathbb{C}$ , and  $\lambda_1, \lambda_2$  are linearly independent over  $\mathbb{R}$ .
- (10) Show, using the uniformization theorem, that any holomorphic map from  $\mathbb{C}$  to a compact connected Riemann surface of genus greater than 1 is constant.
- (11) Let  $X$  be the Riemann sphere. Show that there is no non-trivial subgroup of  $\text{Aut}(X)$  acting on  $X$  properly discontinuously.
- (12) (i) Let  $\mathcal{M}$  be the set of complex tori up to conformal equivalence. Show that  $\mathcal{M}$  is not countable.  
(ii) Let  $PSL(2, \mathbb{Z}) = \{A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid \det A = 1 \text{ and } a, b, c, d \in \mathbb{Z}\} / \pm I$  where  $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ . You can check that  $PSL(2, \mathbb{Z})$  acts on the upper half plane  $\mathbb{H} = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$  by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d}$$

Show that there is a bijection between  $\mathcal{M}$  and the quotient  $\mathbb{H}/PSL(2, \mathbb{Z})$ , the set of orbits of  $\mathbb{H}$  under the above action.

- (13) Show that any holomorphic map  $f$  of degree 2 from a complex torus  $\mathbb{C}/\Lambda$  to the Riemann sphere is given by a ‘Möbius transformation of a shifted  $\wp$ -function’:

$$f(z) = \frac{a\wp(z - z_0) + b}{c\wp(z - z_0) + d},$$

for some  $a, b, c, d, z_0 \in \mathbb{C}$ .

- (14) Show that a compact connected Riemann surface  $X$  is a complex torus iff there is a holomorphic map  $f: X \rightarrow Y$  of degree 2 with 4 branch points where  $Y$  is the Riemann sphere.
- (15) Let  $X = \mathbb{C}/\Lambda$  be the complex torus defined by a lattice  $\Lambda$  and write  $X_0 = X \setminus \{\Lambda\}$  for the complement of the coset of  $\Lambda$ . Show that

$$\Phi : z + \Lambda \in X_0 \rightarrow (\wp(z), \wp'(z)) \in \mathbb{C}^2$$

maps the punctured complex torus  $X_0$  biholomorphically onto a smooth algebraic curve in  $\mathbb{C}^2$ .

[Hint: the differential equation for  $\wp$ .]

- (16) (i) Let  $f$  and  $g$  be two elliptic functions (with the same lattice of periods) and  $N$  a positive integer. By considering the poles of  $f$  and  $g$ , estimate from above the dimension of the complex vector space spanned by  $f(z)^m g(z)^n$ , for  $0 \leq m, n \leq N$ . Deduce that when  $N$  is sufficiently large there must be a non-trivial linear dependence,

$$\sum_{m,n=0}^N a_{m,n} f(z)^m g(z)^n \equiv 0, \quad \text{for some } a_{m,n} \in \mathbb{C}.$$

Hence show that any two meromorphic functions  $f, g$  on a complex torus  $\mathbb{C}/\Lambda$  are ‘**algebraically related**’: there is a polynomial  $Q$  in two variables, so that  $Q(f(z), g(z)) = 0$  for all  $z$ .

- (ii)\* Show that in fact (i) holds for meromorphic functions on any compact connected Riemann surface.