Elements of Nonstandard Algebraic Geometry

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Abstract

I investigate the algebraic geometry of nonstandard varieties, using techniques of nonstandard mathematics.

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1 Introduction

Methods of nonstandard mathematics have been successfully applied to many parts of mathematics such as real analysis, functional analysis, topology, probability theory, mathematical physics etc. But just a little bit has been done in foundations of nonstandard algebraic geometry so far. Robinson indicated some ideas in [R1] and [R2] to prove Nullstellensatz (Rückert's theorem) and Oka's theorem, using nonstandard methods, in the case of analytic varieties.

In this paper we try to formulate first elements of nonstandard algebraic geometry.

Consider an enlargement ${}^*X$ of an affine variety X over an algebraically closed field k. We often take $k = \mathbb{C}$ to be able to define the shadow of limited points of ${}^*X$.

As one of the first results in section 4 (th 4.6) we shall show that the shadow of any 1-codimensional principal (given by an internal polynomial with a finite number of monomials) subvariety of ${}^*X_{{}^*\mathbb{C}}$ is closed in X where ${}^*X_{{}^*\mathbb{C}}$ is the ${}^*X$ as a variety over the field ${}^*\mathbb{C}$.

Also in section 4 (th 4.2) we show that the shadow of any internal open subset of ${}^*X$ equals X, which in turn implies that every point on X has an internal nonsingular point in its halo.

In section 5 we discuss an error in Robinson's paper [R1, th.5.3] and indicate a way to fix it.

In section 6 we introduce the notion of a countable infinite dimensional affine variety and prove Nullstellensatz in the case of an uncountable underlying algebraically closed field, in particular for the field of complex numbers.

Finally in section 7 we investigate enlargements of a commutative ring R and R-modules M. We use flatness of ${}^*R$ over R to prove ${}^*M \simeq {}^*R \otimes_R M$ for R a Noetherian commutative ring R and a finitely generated R-module M.

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2 List of Notation

${}^*\mathbb{C}[z_1, \ldots, z_n]$ .... internal polynomials over ${}^*\mathbb{C}$ in n variables
$({}^*\mathbb{C})[z_1, \ldots, z_n]$ .... polynomials over ${}^*\mathbb{C}$ in n variables
$\mathcal{P}_F(A)$ ....the set of finite subsets of A
$zh_X(a)$...Zariski halo of a in ${}^*X$
$h_X(a)$...halo of a in ${}^*X$
${}^*X_{{}^*k}$...${}^*X$ as a variety over the field ${}^*k$

3 Basic Definitions

We consider the enlargement of a set which contains an algebraically closed field k and real numbers. Then we can speak of the enlargement of affine and projective spaces and more ,the enlargement of any quasiprojective variety. Let X be a variety over k and let ${}^*X$ be its enlargement. ${}^*X_{{}^*k}$ denotes ${}^*X$ as a variety over the field ${}^*k$. Note that this is completely different from ${}^*X$.

Definition 3.1 Let $a \in X$ then the halo of a in Zariski topology is

$$zh_X(a) = \bigcap_{a \in U} {}^*U.$$

where U is Zariski open in X.

We distinguish it from $h_X(a)$ which stands for the halo of a when $k = \mathbb{C}$ and U is open in the sense of usual topology. In this case ${}^*X^{lim}$ shows the elements with limited coordinates.
The map ${}^* : X \longrightarrow {}^*X$ is the natural map which takes a to ${}^*a$ and usually we denote the image of a by the same a. And also we have another important map ${}^o : {}^*X^{lim} \longrightarrow X$ which takes each point to its shadow.
We would get two different "topologies" on ${}^*X$. One is the internal Zariski topology such that its opens are the internal open subsets of ${}^*X$. In order this is not always a topology. That is the intersection of a collection of closed subsets may not be a closed subset. For example let $X = \mathbb{A}^1_k$ and $B_M = \{x \in {}^*\mathbb{N} : 1 \leq x \leq M\}$. Now let $\mathfrak{B} = \{B_M\}_{M \leq N}$ where N, M are unlimited hypernatural numbers and k is an algebraically closed field with characteristic 0. All B_M in $\mathfrak{B}$ are hyperfinite and then by transfer internal closed subsets of ${}^*X$. Now consider $\bigcap_{B \in \mathfrak{B}} B = \mathbb{N}$ which is not an internal subset of ${}^*X$ and then not internal closed subset.
Another topology is the usual Zariski topology on ${}^*X_{{}^*k}$ as a variety over the field ${}^*k$.

4 Properties of the * and ^{o} maps

X shows an affine variety through this section. Consider on ${}^*X$ the internal topology in which a basis of open subsets consists of complements of all zeros of an internal polynomial (i.e. an element of ${}^*\mathbb{C}[z]$).

The first thing which draw our attention is the continuity of the * map. We shall show that this map is not continuous.

Example 4.1 Let $X = k = \mathbb{C}$, then there is an internal closed subset of ${}^*X$ with a non-closed preimage under the * map.

The following formula is true:

$$(\forall A \in \mathcal{P}_F(\mathbb{C}))(\exists p \in \mathbb{C}[z])(\forall a \in \mathbb{C})(a \in A \longleftrightarrow p(a) = 0).$$

By transfer we have:

$$(\forall A \in {}^*\mathcal{P}_F(\mathbb{C}))(\exists p \in {}^*\mathbb{C}[z])(\forall a \in {}^*\mathbb{C})(a \in A \longrightarrow p(a) = 0).$$

Now let $A = \{x \in {}^*\mathbb{N} : 1 \leq x \leq N\}$ for an unlimited hypernatural number N. A is a hyperfinite subset of ${}^*\mathbb{C}$.
Then there is an internal polynomial in ${}^*\mathbb{C}[z]$ which vanishes exactly on A. The preimage of A is $\mathbb{N}$ which is not a closed subset of $\mathbb{C}$.

We can prove a stronger assertion that for any subset B of $\mathbb{C}$, there is an internal closed subset of ${}^*\mathbb{C}$ which has B as its preimage. To prove it we can consider a hyperfinite approximation of B in ${}^*\mathbb{C}$, say H. $B \subseteq H \subseteq {}^*B$. The preimage of ${}^*B$ is B and then the preimage of H is also B.

Now we look at images of subsets of ${}^*X$ under the ^{o} map in the case of $k = \mathbb{C}$. Note that we defined the ^{o} from X^{lim} to X, but we can consider the image of subsets of ${}^*X$ by taking the images of its limited points. Unexpectedly the image of any nonempty internal open set is the whole X.

Theorem 4.2 *Let A be a nonempty internal open set in ${}^*X$ then ${}^oA$ is X.*

Proof It is sufficient to prove the theorem for principal internal open subsets. Then let $A = {}^*X_f$ be a nonempty internal principal open subset where f is an internal polynomial. If the shadow of A is not X there should be some point $a \in X$ for which $f(h_X(a)) = 0$. Hence the following formula is true:

$$(\exists g \in {}^*\mathbb{C}[z_1, \ldots, z_n])(\exists \varepsilon \in {}^*\mathbb{R}^+)(\forall z \in {}^*X)((\exists w \in {}^*X)(g(w) \neq 0) \wedge (\mid z - a \mid \leq \varepsilon \to g(z) = 0)).$$

And by transfer:

$$(\exists g \in \mathbb{C}[z_1, \ldots, z_n])(\exists \varepsilon \in \mathbb{R}^+)(\forall z \in X)((\exists w \in X)(g(w) \neq 0) \wedge (\mid z - a \mid \leq \varepsilon \to g(z) = 0)).$$

It is easy to see that the latter is not true.

Corollary 4.3 *There is a nonsingular point ξ in $h_X(a)$ for every $a \in X$.*

Theorem 4.4 *Let $f : X \longrightarrow Y$ be a regular map of varieties over $\mathbb{C}$. Then we have:*

1. *(i) ${}^o({}^*Z) = Z$ for every closed subset of X;*
2. *(ii) $({}^*f)^{-1}({}^*Z) = {}^*(f^{-1}(Z))$ for every subset Z of Y.*

Proof (i) Obviously $Z \subseteq {}^o({}^*Z)$. Let $Z = V(g_1, \ldots, g_l)$ and let $x \in {}^o({}^*Z)$, $x = {}^o\xi$ with $\xi \in {}^*Z$. Then $g_i(\xi) = 0$ for $1 \leq i \leq l$. Clearly $g_i({}^o\xi) = 0$ for $1 \leq i \leq l$ and this proves that $x \in Z$.
(ii) consider the formula :

$$(\forall x \in X)(x \in (f)^{-1}(Z) \longleftrightarrow f(x) \in Z).$$

and by transfer:

$$(\forall x \in {}^*X)(x \in {}^*((f)^{-1}(Z)) \longleftrightarrow {}^*f(x) \in {}^*Z).$$

and on the other hand we have:

$$(\forall x \in {}^*X)(x \in ({}^*f)^{-1}({}^*Z) \longleftrightarrow {}^*f(x) \in {}^*Z).$$

which proves the equality.

It is well known that the shadow of any subset of ${}^*\mathbb{R}$ is a closed subset in $\mathbb{R}$, the field of real numbers, in the sense of real topology. But that is not such easy in the case of algebraic sets. Now we show that the shadow of an internal closed subset of ${}^*X$ is not always closed in X. For example consider B_M in ${}^*\mathbb{A}^1_{\mathbb{C}}$, which was introduced in section 1, with M an unlimited hypernatural number. Obviously ${}^oB_M = \mathbb{N}$ which is not closed in $\mathbb{A}^1_{\mathbb{C}}$. A better deal is to consider closed subsets of ${}^*X_{\mathbb{C}}$.

Theorem 4.5 *Let $f \in ({}^*\mathbb{C})[z_1, \ldots, z_n]$ be a polynomial with limited coefficients and let ${}^of$ be nonzero. Then we have:*

$$^\circ(V(f)) = V(^\circ f).$$

Proof The shadow of f, $^\circ f$, may be a constant i.e the coefficients of nonzero degree monomials in f are infinitesimal. This implies that no limited point could be in $V(f)$. On the other hand $V(^\circ f) = \emptyset$. Then the equality is proved in this case. Otherwise let $\xi \in {}^*\mathbb{C}^n$ be a limited point and $f(\xi) = 0$. Then $^o f(^o\xi) = 0$ hence $^o\xi \in V(^o f)$.
Now let $a \in V(^o f)$ then $f(a) \simeq 0$. Using hypotheses, $f(h_{\mathbb{C}^n}(a)) \subseteq h_{\mathbb{C}^n}(0)$. It is sufficient to find a point in the halo of a such that f vanishes at that point. Now if $f(a) \neq 0$ we can change variables linearly such that a is transferred to origin. Note that the new polynomial, say g has also limited coefficients and this translation takes $h_{\mathbb{C}^n}(a)$ to $h_{\mathbb{C}^n}(0)$. We have $g = g^{inf} + g^{ap}$ such that g^{inf} has infinitesimal coefficients and g^{ap} has appreciable coefficients. Then $^o g = {}^o g^{ap}$.
Now we use induction on the number of variables. If $n = 1$ Robinson–Callot theorem[DD, ch. 2,th. 2.1.1] shows that $g(h_{\mathbb{C}}(0)) = h_{\mathbb{C}}(0)$ because g is S-continuous as it has limited coefficients. If $1 < n$ we consider the form with highest degree appeared in g^{ap}, say h. h is a sum of monomials of the same degree.
If $h = \alpha z_1 \ldots z_n$ where α is a hypercomplex number, then we change variables such that $z_1 = w_1$ and $z_i = w_i + w_1$. This change, obviously maps the halo of origin on itself and we get a new polynomial e with limited coefficients from g. Now consider $e(w_1, \ldots, w_{n-1}, 0)$, clearly the shadow of this polynomial in a smaller than n number of variables, is not constant, and we use induction. In the remaining cases we can again replace one of the variables by zero and reduce the number of variables, if necessary, and use induction.
This proves the existence of a zero for f and completes proof of the theorem.

We can generalize this result by replacing $\mathbb{C}^n$ with its affine subvariety, X. The theorem is again true. Although the previous theorem is a particular case of the next theorem, but their proofs are of different nature and we prefer to keep the previous proof.

Theorem 4.6 *Let X be an algebraic closed subset of $\mathbb{C}^n$ and $f \in ({}^*\mathbb{C})[z_1, \ldots, z_n]$ be a polynomial with limited coefficients. Then there is a g in $({}^*\mathbb{C})[z_1, \ldots, z_n]$ with limited coefficients such that:*

$$V(g) = V(f), {}^\circ V(f) = V(^\circ g).$$

where zeros of these polynomials are taken in ${}^*X$ and X correspondingly.

Remark 4.7 It is not always true if we take g to be f itself. For example let $X = V(z_1)$ in $\mathbb{C}^2$ and $f = z_1 + \varepsilon z_2$ in which ε is an infinitesimal hypernatural number. Then ${}^o f = z_1$ which is identically zero on X. But the shadow of $V(f)$ is just a single point.

Proof If $V(f) = {}^*X$ then the theorem is trivial.
In other cases if ${}^\circ f$ is not identically zero on X we take $g = f$. Otherwise let $\ddot{f}$ be f divided by one of its coefficients with maximum absolute value. If ${}^o\ddot{f}(X) \neq 0$ then put $g = \ddot{f}$. Otherwise assume that ${}^o\ddot{f}({}^*X) = 0$ then $V(\ddot{f} - {}^o\ddot{f}) = V(\ddot{f}) = V(f)$. $\ddot{f} - {}^o\ddot{f}$ has smaller number of monomials than f. By continuing this process eventually we get a polynomial g such that its shadow is not identically zero on X and $V(g) = V(f)$.
Now let $x \in {}^\circ V(g)$, then $x = {}^o\xi$ for some $\xi \in V(g)$. From $g(\xi) = 0$ we get ${}^o g({}^o\xi) = 0$ and then $x \in V({}^o g)$. Conversely let $x \in V({}^\circ g)$. We want to prove that $h_X(x) \cap V(g) \neq \emptyset$. Let $Y \subseteq X$ be an irreducible curve containing x. It is sufficient to prove that $h_Y(x) \cap V(g) \neq \emptyset$. It is proved if $g({}^*Y) = 0$, otherwise change variables such that x be transferred to the origin and then consider ${}^*Y_{{}^*\mathbb{C}}$. $V(g) \cap {}^*Y_{{}^*\mathbb{C}}$ is a finite set i.e a zero dimensional subvariety, say $\mathcal{A} = \{\xi_1, \ldots, \xi_l\}$. Since ${}^*X \subseteq {}^*\mathbb{C}^n$, then every point of ${}^*X$ is as $(b_1, \ldots, b_n)$, with n coordinates, $b_1, \ldots, b_n$. Now if no point in $\mathcal{A}$ is infinitesimal, with infinitesimal coordinates, then every ξ_i has at least a non-infinitesimal coordinate, say a_{i_j}. The index j means that a_{i_j} has appeared in the j-th coordinate of ξ_i. Now put $h_i = (z_j - a_{i_j})/a_{i_j}$. And let $h = h_1 \times \cdots \times h_l$. Now obviously $\mathcal{A} \subseteq V(h)$. Then we have $h^t = eg$ on ${}^*Y$, for some polynomial e and natural number t. By construction h and g have limited coefficients. e should also have limited coefficients, otherwise $h^t/s = (e/s)g$ on ${}^*Y$ where s is a coefficient appeared in e with maximum absolute value. Then ${}^o(h^t/s) = 0 = {}^o(e/s){}^o g$ on Y. But Y is irreducible, hence ${}^o(e/s) = 0$ on Y. Now we can use the method by which we constructed g and reduce the number of monomials appeared in e. Then we get a new e with limited coefficients which satisfies ${}^o e \neq 0$, $h^t = eg$ and ${}^o h^t = {}^o e {}^o g$. This is a contradiction, because ${}^o h$ is not zero at origin.

5 Generic Points for Prime Ideals

Let Γ be the ring of analytic functions at origin (origin of $\mathbb{C}^n$). An important theorem in complex analysis says that every prime ideal of Γ has a generic point in the halo of origin. We prove a similar theorem in the algebraic context.

Theorem 5.1 *Let X be an irreducible affine variety and $x \in X$. Then every prime ideal in the ring of regular functions at x has a generic point in the Zariski halo of x.*

Proof Let $\mathfrak{p}$ be a prime ideal in $\mathcal{O}_{X,x}$, the ring of regular functions at x. Define:

$A_{f,g,U} = \{y \in U : U \text{ is open in } X \wedge g(y) \neq 0 \wedge f(y) = 0 \wedge f, g \text{ are regular on } U\}.$

Using Nullstellensatz $A_{f,g,U} \neq \emptyset$ where $x \in U$, $f \in \mathfrak{p}$ and $g \notin \mathfrak{p}$. Similarly the collection $\{A_{f,g,U}\}_{x \in U, f \in \mathfrak{p}, g \notin \mathfrak{p}}$ has finite intersection property. Then there would be a ξ in the following set:

$$\bigcap_{x \in U, f \in \mathfrak{p}, g \notin \mathfrak{p}} {}^*A_{f,g,U}.$$

So ξ is a generic point for $\mathfrak{p}$ and $\xi \in zh_X(x)$.

Thus, we deduce that the map $\pi : zh_X(x) \longrightarrow Spec(\mathcal{O}_{X,x})$ is surjective where $\pi(\xi) = m_\xi$, elements of $\mathcal{O}_{X,x}$ vanishing at ξ. This map demonstrates how close $zh_X(x)$ and $Spec(\mathcal{O}_{X,x})$ are.

Theorem 5.2 *With the hypotheses of the previous theorem we get:*

$$\pi^{-1}(V_S(I)) = V_{zh}(I).$$

in which I is an ideal of $\mathcal{O}_{X,x}$, $V_S(I)$ is the closed subset of $Spec(\mathcal{O}_{X,x})$ defined by I and $V_{zh}(I)$ is the zeros of I in $zh_X(x)$.

Proof Let $\xi \in zh_X(x)$ and $\pi(\xi) \in V_S(I)$. Then obviously $I \subseteq \pi(\xi)$, in other words every member of I vanishes at ξ. This shows that ξ is in the right side of the above equality.
Conversely let ξ be in the right side of the equality then every member of I vanishes at ξ. This implies that I is contained in $\pi(\xi)$ i.e ξ is in the left side of the equality.

In the analytic case the existence of the generic point is used to prove the Nullstellensatz theorem. That is if $f, g_1, \ldots, g_l \in \Gamma$ and $V(g_1, \ldots, g_l) \subseteq V(f)$ then some power of f should be in the ideal generated by g_i's [R1, sect. 4]. In [R1, th.5.1] the existence of a generic point was proved for infinite dimensional spaces $\mathbb{C}^\Lambda$, in which Λ is an arbitrary infinite set. Robinson used the previous result to deduce Nullstellensatz in this case [R1, th.5.3]. Unfortunately, his prove is erroneous. Now we indicate the gap.

Analysis of Robinson's Proof. Let Γ be the set of cylindrical analytic functions in the origin of $\mathbb{C}^\Lambda$ each one depending only on a finite number of variables. Let $\mathcal{A} \subseteq \Gamma$ be such that $V(\mathcal{A}) \subseteq V(f)$ in a neighborhood of origin. If no power of f is in $< \mathcal{A} >$ then there is a prime ideal, say P containing $\mathcal{A}$ and not f. P has a generic point in the halo of origin, say ξ. Robinson concludes that f is zero at ξ because $V(\mathcal{A}) \subseteq V(f)$ in a neighborhood of origin like U. But this is not true. Consider:

$$(\forall x \in U)((\forall h \in \mathcal{A})h(x) = 0 \longrightarrow f(x) = 0).$$

and by transfer:

$$(\forall x \in {}^*U)((\forall h \in {}^*\mathcal{A})h(x) = 0 \longrightarrow {}^*f(x) = 0).$$

This formula is true but it is different from:

$$(\forall x \in {}^*U)((\forall h \in {}^{im}\mathcal{A})h(x) = 0 \longrightarrow {}^*f(x) = 0).$$

which is a wrong formula Robinson applied to ξ.

Counter-Example 5.3 Let $\Lambda = \mathbb{C}$, $h_a = z_a(z_0 - a) - 1$, $\mathcal{A} = \{h_a : a \in \mathbb{C} \textit{ and } a \neq 0\}$ and $f = z_0$ in which z_a is a variable indexed by a. Then $V(\mathcal{A}) \subseteq V(f)$ and no power of f is in $< \mathcal{A} >$.

Let $\xi \in V(\mathcal{A})$, then $z_0(\xi) = 0$ because for every nonzero $a \in \mathbb{C}$, $h_a(\xi) = z_a(\xi)(z_0(\xi) - a) - 1 = 0$ and then $z_0(\xi) - a$ is nonzero. Hence $z_a(\xi) = 1/(z_0(\xi) - a) = 1/(-a)$. This means that $V(\mathcal{A}) = \{\xi\}$. Clearly $\xi \in V(z_0)$. But if a power of z_0, say z_0^l, be in $< \mathcal{A} >$ then $z_0^l = \sum_{i=1}^t e_i h_{a_i}$ where $h_{a_i} \in \mathcal{A}$. Now we can find a point at which all h_{a_i}'s are zero and z_0 is not. But this is a contradiction. Then no power of z_0 is in $< \mathcal{A} >$.

6 Varieties of Infinite Dimensions

The previous section demonstrates some peculiar features of varieties of infinite dimensions. In this section at first we show that Nullstellensatz does not hold in infinite dimensional algebraic geometry as well as in infinite dimensional complex analysis.

Counter-Example 6.1 There is a set Λ and a proper ideal $\mathfrak{J}$ in S, the ring of polynomials over $\mathbb{C}$ in variables indexed by Λ, such that $V(\mathfrak{J}) = \emptyset$.

Let $\Lambda = \mathbb{C} \cup \{\mathbb{C}\}$, $h_a = z_a(z_0 - a) - 1$ for $a \neq 0$ in $\mathbb{C}$ and $h_{\mathbb{C}} = z_{\mathbb{C}} z_0 - 1$. Let $\mathfrak{J}$ be the ideal generated by all these functions in S. Then $V(\mathfrak{J}) = \emptyset$. If $\mathfrak{J} = S$ then there are $a_1, \ldots, a_l$ (a_l can be $\mathbb{C}$) and $f_1, \ldots, f_l$ such that

$$\sum_{i=1}^{l} f_i h_{a_i} = 1.$$

Now consider all variables which occur in this formula and let R be the ring of polynomials in these variables over $\mathbb{C}$ and $\mathbb{C}^m$ the corresponding affine space. Then the ideal generated by $h_{a_1}, \ldots, h_{a_l}$ in R, is R itself. That is $V(\mathfrak{J}) = \emptyset$ in $\mathbb{C}^m$. This is not possible, because we can find a point in $\mathbb{C}^m$ at which all h_{a_i}'s are zero. But right side of the above equation would not be zero at that point.

Fortunately this is not the end of the story. We prove a complete version of Nullstellensatz similar to the finite dimensions, in the particular case of $\Lambda = \mathbb{N}$. Let S be the ring $\mathbb{C}[z_1, z_2, \ldots]$.

Definition 6.2 Let $X \subseteq \mathbb{C}^{\mathbb{N}}$. We say X is an affine variety in $\mathbb{C}^{\mathbb{N}}$ if $X = V(\mathfrak{J})$ for some ideal $\mathfrak{J}$ of S and we call $\mathbb{C}[X] = S/I(X)$ the ring of regular functions on X. Similarly the field of fractions of $\mathbb{C}[X]$ denoted by $\mathbb{C}(X)$ is called the field of rational functions on X.

Theorem 6.3 *Let $\mathfrak{M}$ be a maximal ideal of S. Then $V(\mathfrak{M}) \neq \emptyset$.*

Proof If for every $n \in \mathbb{N}$ there be a $a_n \in \mathbb{C}$ such that $z_n - a_n \in \mathfrak{M}$ then $\mathfrak{M} =< z_n - a_n >_{n\in\mathbb{N}}$, because $< z_n - a_n >_{n\in\mathbb{N}}$ is a maximal ideal of S. Hence $V(\mathfrak{M}) = \{(a_n)_{n\in\mathbb{N}}\}$. Now suppose there is a $n \in \mathbb{N}$ such that $z_n - a \notin \mathfrak{M}$ for any $a \in \mathbb{C}$. For simplicity we can take $n = 1$. Now let $S_i = \mathbb{C}[z_1, \ldots, z_i]$ and $\mathfrak{M}_i$ the contraction of $\mathfrak{M}$ in S_i. $\mathfrak{M}_i$ is a prime ideal in S_i but our goal is to

prove that it is also a maximal ideal.
Let $Y_i = V(\mathfrak{M}_i)$ in $\mathbb{C}^i$. Then by our hypothesis $Y_1 = \mathbb{C}$, i.e $\mathfrak{M}_1 = 0$. For every i we have a projection:

$$\pi_i : Y_i \longrightarrow \mathbb{C}.$$

Where $\pi_i(y_1, \dots, y_i) = y_1$. Every member of S is a polynomial with a finite number of variables occurred in it. Then $\bigcup \mathfrak{M}_i = \mathfrak{M}$. By a theorem in algebraic geometry [SH, ch. I,§5,th.6] $\pi_i(Y_i)$ is open in $\mathbb{C}$ or a point in it. If $\pi_i(Y_i)$ is just a point for some i, say b, then $z_1 - b \in \mathfrak{M}_i$ which is a contradiction. If all $\pi_i(Y_i)$ are open, let $x \in \mathbb{C}$. Then there is a $h \in S$ such that $1-h(z_1-x) \in \mathfrak{M}$ and then $1-h(z_1-x) \in \mathfrak{M}_j$ for some j. x can't be in $\pi_j(Y_j)$ because $1-h(z_1-x)$ doesn't vanish at any point where its coordinate corresponding to 1 is x. This proves the following equality:

$$\mathbb{C} = \bigcup_{i=1}^{\infty} \mathbb{C} \setminus \pi_i(Y_i).$$

which is impossible.

This theorem shows that every proper ideal of S at least has a zero in $\mathbb{C}^{\mathbb{N}}$.

Corollary 6.4 *An ideal $\mathfrak{M}$ in S is maximal iff it is as $< z_i - a_i >_{i\in\mathbb{N}}$ for some $a_i \in \mathbb{C}$.*

In the proof of the previous theorem we haven't used any specific property of $\mathbb{C}$, we have just used that it is algebraically closed and uncountable. So

Corollary 6.5 *The theorem holds if we replace $\mathbb{C}$ by any uncountable algebraically closed field k.*

Now we look at other parts of Nullstellensatz.

Theorem 6.6 *Let $\mathfrak{J}$ be an ideal in S, then $I(V(\mathfrak{J})) = \sqrt{\mathfrak{J}}$.*

Proof One inclusion is obvious. Put $T = \mathbb{C}^{\mathbb{N}}$ and let $V(\mathfrak{J}) \subseteq V(g)$ where $g \in S$. Now we consider a new space of the same shape, say $W = \mathbb{C} \times T$. We will have a new variable like z_0 and a new coordinate corresponding to it (note that $0 \notin \mathbb{N}$ in this work). Consider the ideal $\mathfrak{J}^+ = \mathfrak{J}+ < 1 - z_0 g >$

in the ring $S[z_0]$. $\mathfrak{J}^+$ has no zero in W, so $\mathfrak{J}^+ = S[z_0]$. Hence there are $h_0, h_1, \dots, h_l$ in $S[z_0]$ and $f_1, \dots, f_l$ in $\mathfrak{J}$ for which we have:

$$\sum_{i=1}^{l} h_i f_i + h_0(1 - z_0 g) = 1.$$

Now we can put $z_0 = 1/g$ and conclude that either $\mathfrak{J} = S$ or some power of g is in $\mathfrak{J}$.

Corollary 6.7 *Let $\mathfrak{J}_1, \mathfrak{J}_2$ be ideals in S then we have the following:*

(i) $V(\mathfrak{J}_1\mathfrak{J}_2) = V(\mathfrak{J}_1 \cap \mathfrak{J}_2) = V(\mathfrak{J}_1) \cup V(\mathfrak{J}_2)$*;*

(ii) $V(\mathfrak{J}_1 + \mathfrak{J}_2) = V(\mathfrak{J}_1) \cap V(\mathfrak{J}_2)$*;*

(iii) $\sqrt{\mathfrak{J}_1}$ *is prime iff* $V(\mathfrak{J}_1)$ *is irreducible.*

Proof Standard.

Definition 6.8 Let $\phi : X \longrightarrow Y$ be a map in which X and Y are affine varieties. ϕ is a regular map if $\phi = (\phi_1, \phi_2, \dots)$ in which ϕ_i is a regular function on X. Similarly if all ϕ_i are rationals on X and $\phi(Dom(\phi)) \subseteq Y$, ϕ is called a rational map.

It is easy to check that, there is an equivalence between the category of affine varieties over $\mathbb{C}$ (as defined in 4.2) and the category of reduced countably generated $\mathbb{C}$-algebras.

It is not obvious that every rational map has a nonempty domain.

Theorem 6.9 *$Dom(\phi) \neq \emptyset$ for any rational map $\phi : X \longrightarrow Y$.*

Proof Let $\phi = (\phi_1, \phi_2, \dots)$, $\phi_i = g_i/f_i$ and $T = \mathbb{C}^{\mathbb{N}}$. It is sufficient to prove that there is a point at which none of f_i's vanishes. Suppose there is no such point i.e.

$$\bigcup_{i=1}^{\infty} V(f_i) = \mathbb{C}^{\mathbb{N}}.$$

Now let $W = \mathbb{C} \times \mathbb{C} \times \mathbb{C} \times \mathbb{C} \ldots$. We define a coordinate system on W such that the $(2i-1)$th component in it is same as the ith component of T i.e we associate the variable z_i to the component with number $2i-1$, and the variable w_i to the $2i$th component.

Now consider the set:

$$\mathcal{A} = \{1 - w_i f_i : i \in \mathbb{N}\}.$$

This set has no zero in W. Then by theorem 5.3, $< \mathcal{A} >= \mathbb{C}[z_1, w_1, z_2, w_2, \ldots]$ and hence there are $h_1, \ldots, h_l$ in $\mathbb{C}[z_1, w_1, z_2, w_2, \ldots]$ such that:

$$\sum_{j=1}^{l} h_j(1 - w_{i_j} f_{i_j}) = 1.$$

But this is a contradiction because we know that there is some $\xi \in T$ such that $f_{i_j}(\xi) \neq 0$ for $1 \leq j \leq l$. By putting $w_{i_j}(\xi) = 1/f_{i_j}(\xi)$ we get a point at which all $(1 - w_{i_j} f_{i_j})$ are zero.

Corollary 6.10 *Neither $\mathbb{C}^{\mathbb{N}}$ or $\mathbb{C}^n$ (n is finite) is the union of a countable set of proper subvarieties.*

We just proved it for $\mathbb{C}^{\mathbb{N}}$. Suppose that $\mathbb{C}^n = \bigcup_{i=1}^{\infty} V(f_i)$ in which f_i is in $\mathbb{C}[z_1, \ldots, z_n]$. Now extend it to $\mathbb{C}^{\mathbb{N}}$ and we get the result.

Let $S_{\mathbb{N}} = \mathbb{C}[z_1, z_2, \ldots]$ and $S_i = \mathbb{C}[z_1, \ldots, z_i]$. We have inclusions when $n < m$:

$$S_n \longrightarrow S_{\mathbb{N}}$$

$$S_n \longrightarrow S_m$$

and by transfer we have

$$^*S_n \longrightarrow {}^*S_{\mathbb{N}}$$

$$^*S_n \longrightarrow S_N$$

in which $S_N = \mathbb{C}[z_1, \ldots, z_N]$ is the set of internal polynomials over $^*\mathbb{C}$ in variables $z_1, \ldots, z_N$ with an unlimited hypernatural number N.
Now let $\mathfrak{J}$ be an ideal in $S_{\mathbb{N}}$, $\mathfrak{J}_n$ its contraction in S_n and $\mathfrak{J}_N$ the corresponding internal ideal in S_N. We have the following diagram:

$$\begin{array}{ccc} S_n & \xrightarrow{\alpha_{n,\mathbb{N}}} & S_{\mathbb{N}} \\ \downarrow & & \downarrow \\ {}^*S_n & \xrightarrow{\alpha_{n,N}} & S_N \\ & & \downarrow \\ & & {}^*S_{\mathbb{N}} \end{array}$$

By using transfer we can see that $\alpha_{n,N}^{-1}(\mathfrak{J}_N) = {}^*\mathfrak{J}_n$, for all $n \in \mathbb{N}$. And then $\alpha_{\mathbb{N},N}^{-1}(\mathfrak{J}_N) = \mathfrak{J}$.

7 Enlargement of Commutative Rings

In this section we study the enlargement of commutative rings, especially Noetherian rings. In the theory of commutative rings localization and completion of rings and modules have some typical properties like preserving exactness of sequences and their closed relation with tensor product. That is, if R is Noetherian ring, $\mathfrak{p}$ a prime ideal and M is a finitely generated R-module then we have:

$$M_{\mathfrak{p}} \simeq R_{\mathfrak{p}} \otimes_R M.$$

$$\widehat{M} \simeq \widehat{R} \otimes_R M.$$

We prove similar properties of enlargement of modules. As usual we denote the enlargement of R and M as ${}^*R$ and ${}^*M$. For any ideal $\mathfrak{J}$ of R we have two notions of radical of ${}^*\mathfrak{J}$ in the ring ${}^*R$. One is the usual $\sqrt{{}^*\mathfrak{J}}$ when we consider ${}^*R$ as a ring and another is the internal notion of radical, say ${}^{int}\sqrt{{}^*\mathfrak{J}}$ which is exactly the enlargement of $\sqrt{\mathfrak{J}}$ i.e.

$${}^{int}\sqrt{{}^*\mathfrak{J}} = {}^*\sqrt{\mathfrak{J}}.$$

Similarly we have the same situation for many other notions. From now on we work with a Noetherian commutative ring R.

Theorem 7.1 *For any ideal $\mathfrak{J}$ in R we have:*

$${}^*min(\mathfrak{J}) = min_{int}({}^*\mathfrak{J}) = min({}^*\mathfrak{J}).$$

Proof R is Noetherian then $min(\mathfrak{I})$ is a finite set, say $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_l\}$. Then it is its own enlargement. Now let $\mathfrak{q}$ be a prime ideal in $^*R$ containing $^*\mathfrak{I}$. Hence its contraction $\mathfrak{q}^c$ in R is a prime ideal containing $\mathfrak{I}$. There is some j such that $\mathfrak{p}_\mathfrak{j} \subseteq \mathfrak{q}^\mathfrak{c}$. Then $^*\mathfrak{p}_\mathfrak{j} \subseteq \mathfrak{q}$. This implies the equalities.

It can also easily be proved that $^*J(R) = J(^*R)$ where $J(R)$ is the Jacobson radical of R and similarly $J(^*R)$ is the Jacobson radical of $^*R$.

Corollary 7.2 .
(i) $^{int}\sqrt{^*\mathfrak{I}} = \sqrt{^*\mathfrak{I}}$ *and* $nil_{int}(^*R) = {}^*nil(R) = nil(^*R)$*;*
(ii) $\mathfrak{q}$ *is* $\mathfrak{p}$*-primary iff* $^*\mathfrak{q}$ *is* $^*\mathfrak{p}$*-primary iff* $^*\mathfrak{q}$ *is internally* $^*\mathfrak{p}$*-primary.*

Let $\phi : M \longrightarrow N$ be a homeomorphism of R-modules. Then

Lemma 7.3 *(i)* $ker^*\phi = {}^*ker\phi$*;*
(ii) $im^*\phi = {}^*im\phi$*.*

Proof (i)

$$(\forall m \in M)(m \in \ker\phi \longleftrightarrow \phi(m) = 0).$$

and by transfer:

$$(\forall m \in {}^*M)(m \in {}^*\ker\phi \longleftrightarrow {}^*\phi(m) = 0).$$

(ii) Use a similar formula.

Corollary 7.4 *Let* M, N, L *and* K *be* R*-modules. Then*
(i) $0 \longrightarrow N \longrightarrow M \longrightarrow K \longrightarrow 0$ *is exact iff* $0 \longrightarrow {}^*N \longrightarrow {}^*M \longrightarrow {}^*K \longrightarrow 0$ *is exact;*
(ii) $^*M/^*K = {}^*(M/K)$*;*

Lemma 7.5 $^*R$ *is a faithfully flat* R*-algebra.*

Proof By [B,ch. I,§2,n^o11] $^*R$ is a faithfully flat R-algebra iff for any maximal ideal $\mathfrak{m}$ in $^*R$, $\mathfrak{m}^*R \neq {}^*R$ and any solution of an R-homogeneous linear equation $\sum_{i=1}^{l} a_iY_i = 0$ in $^*R^l$ is an $^*R$ linear combination of solutions in R^l.
Let $\mathfrak{m}$ be any maximal ideal of R. Since R is Noetherian, $\mathfrak{m}^*R = {}^*\mathfrak{m}$, then

$\mathfrak{m}^*R \neq {}^*R$.
Now let $f = \sum_{i=1}^{l} a_i Y_i = 0$ be an R-homogeneous linear equation. Let $\mathcal{A}$ be the module of solutions to f in R^l. $\mathcal{A}$ is an R-submodule of R^l. Since R is Noetherian then $\mathcal{A}$ is finitely generated, say $\mathcal{A} =< \beta_1, \dots, \beta_c >$. Then we have:

$$(\forall x_1, \dots, x_l \in R)[\sum_{i=1}^{l} a_i x_i = 0 \longleftrightarrow (\exists r_1, \dots, r_c \in R)(x_1, \dots, x_l) = \sum_{i=1}^{c} r_i \beta_i].$$

and using transfer:

$$(\forall x_1, \dots, x_l \in {}^*R)[\sum_{i=1}^{l} a_i x_i = 0 \longleftrightarrow (\exists r_1, \dots, r_c \in {}^*R)(x_1, \dots, x_l) = \sum_{i=1}^{c} r_i \beta_i].$$

This proves that ${}^*R$ is R-flat, and then faithfully flat R-algebra.

Let M be a finitely generated R-module. Define a bilinear function

$$\omega : M \times {}^*R \longrightarrow {}^*M$$

such that $\omega(m, r) = rm$. This induces a unique R-homomorphism

$$\Omega_M : M \otimes_R {}^*R \longrightarrow {}^*M, \quad \Omega_M(\sum_{i=1}^{t} a_i(m_i \otimes r_i)) = \sum_{i=1}^{t} a_i r_i m_i.$$

where $a_i \in R$, $m_i \in M$ and $r_i \in {}^*R$. Clearly Ω is surjective.

Theorem 7.6 *Ω_M is an isomorphism.*

Proof We first assume that M is a free module, say $M = R^s$. Let $\{e_1, \dots, e_s\}$ be a basis for M over R. Then every element of $M \otimes_R {}^*R$ can be written as $\sum_{i=1}^{s} a_i(e_i \otimes r_i)$ and its image under Ω_M will be $\sum_{i=1}^{s} a_i r_i e_i$. Now assume $\sum_{i=1}^{s} a_i r_i e_i = 0$. By transfer all $a_i r_i$ should be zero. This proves the theorem when M is free.
Now in the general case, there is an l and a surjective homomorphism from R^l to M. Let K be the kernel of this homomorphism. Then we get an exact sequence of R-modules:

$$0 \longrightarrow K \longrightarrow R^l \longrightarrow M \longrightarrow 0.$$

and so

$$0 \longrightarrow {}^*K \longrightarrow {}^*R^l \longrightarrow {}^*M \longrightarrow 0.$$

And also by flatness of ${}^*R$ we have:

$$0 \longrightarrow K \otimes_R {}^*R \longrightarrow R^l \otimes_R {}^*R \longrightarrow M \otimes_R {}^*R \longrightarrow 0.$$

Now the maps Ω, namely Ω_K, Ω_{R^l} and Ω_M give us vertical homomorphisms between the two exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K \otimes_R {}^*R & \xrightarrow{\lambda} & R^l \otimes_R {}^*R & \xrightarrow{\gamma} & M \otimes_R {}^*R & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & {}^*K & \xrightarrow{\alpha} & {}^*R^l & \xrightarrow{\beta} & {}^*M & \longrightarrow & 0
\end{array}$$

Suppose $\Omega_M(a) = 0$. There is b such that $\gamma(b) = a$. And let $\Omega_{R^l}(b) = c$. By commutativity of the diagram $\beta(c) = 0$. Hence there is d such that $\alpha(d) = c$. Ω_K is surjective then there is e such that $\Omega_K(e) = d$. Then $\Omega_{R^l}\lambda(e) = c$. But Ω_{R^l} is an isomorphism, then $\lambda(e) = b$. And by exactness $\gamma(b) = \gamma\lambda(e) = 0$. This shows that Ω_M is an isomorphism of R-modules. This completes the proof.

We can consider $M \otimes_R {}^*R$ as a ${}^*R$-module. Ω_M is also ${}^*R$-homomorphism and then it is a ${}^*R$-isomorphism.

By [B, ch. IV,§2.6,th.2] we have

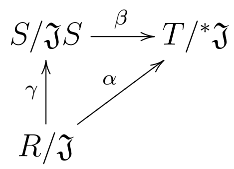
$$Ass_{{}^*R}{}^*M = Ass_{{}^*R}(M \otimes_R {}^*R) = \{{}^*\mathfrak{p} : \mathfrak{p} \in Ass_R M\}.$$

Corollary 7.7 *$Ass_{{}^*R}{}^*M = {}^*Ass_R M$.*

By [VS,th1.1] we can say that (as a particular case) $T = {}^*\mathbb{C}[z_1, \ldots, z_m]$ is a faithfully flat $S = ({}^*\mathbb{C})[z_1, \ldots, z_m]$-algebra. By lemma 6.6, T is also a faithfully flat $R = \mathbb{C}[z_1, \ldots, z_m]$-algebra:

$$\begin{array}{ccc}
S & \xrightarrow{\beta} & T \\
\gamma \uparrow & \nearrow_{\alpha} & \\
R & &
\end{array}$$

Now let $\mathfrak{J}$ be an ideal of R. $(\mathfrak{J}S)T = {}^*\mathfrak{J}$ and let $\mathfrak{J}_1 = \gamma^{-1}(\mathfrak{J}S)$. By flatness of β, $\beta^{-1}({}^*\mathfrak{J}) = \mathfrak{J}S$, hence $\gamma^{-1}[\beta^{-1}({}^*\mathfrak{J})] = \mathfrak{J}_1$. On the other hand by flatness of α, $\alpha^{-1}({}^*\mathfrak{J}) = \mathfrak{J}$, then we conclude that $\gamma^{-1}(\mathfrak{J}S) = \mathfrak{J}$. Then we get another diagram:



Corollary 7.8 $\mathfrak{J}$ *is prime iff* $\mathfrak{J}S$ *is prime.*

If $\mathfrak{J}$ be a radical ideal. Then $\mathfrak{J}S$ and ${}^*\mathfrak{J}$ are also radical.
These ideals correspondingly define closed subsets Y (in $\mathbb{C}^m$), ${}^*Y_{{}^*\mathbb{C}}$ (in ${}^*\mathbb{C}^m_{{}^*\mathbb{C}}$) and ${}^*Y$ (in ${}^*\mathbb{C}^m$). Moreover $R/\sqrt{\mathfrak{J}}, S/\sqrt{\mathfrak{J}S}$ and $T/\sqrt{{}^*\mathfrak{J}}$ are their coordinate rings. Now using the previous corollary we get

Corollary 7.9 ${}^*Y$ *is irreducible iff* ${}^*Y_{{}^*\mathbb{C}}$ *is irreducible iff* ${}^*Y$ *is internally irreducible.*

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