# Minimal model program and moduli spaces

Caucher Birkar

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# Minimal model program

In this talk, all the varieties are over the complex numbers.

**Problem (Classification)**

*Classify varieties up to isomorphism.*

**Problem (Birational classification)**

*Classify smooth projective varieties up to birational isomorphism.*

**Problem**

*Find good models in each birational class and classify those good models.*

# Minimal model program: dimension one

Every curve is birational to a unique smooth projective curve.

$$g(X) = \begin{cases} 0 & \text{iff } X \simeq \mathbb{P}^1 \text{ iff } \deg K_X < 0 \\ 1 & \text{iff } X \text{ is elliptic iff } \deg K_X = 0 \\ \geq 2 & \text{iff } X \text{ is of general type iff } \deg K_X > 0 \end{cases}$$

**Theorem**

*Construct $M_g$, the moduli space of smooth projective curves of genus $g$.*

**Problem**

*Study $M_g$.*

# Minimal model program: dimension two

Every surface is birational to many smooth projective surfaces.

Let $X$ be a smooth projective surface. If $X$ contains a $-1$-curve $E$ (i.e. $E \simeq \mathbb{P}^1$ and $E^2 = -1$), then there is a smooth projective surface $X_1$ and a birational morphism $f \colon X \to X_1$ such that it contracts $E$ and $\rho(X_1) = \rho(X) - 1$.

By continuing the process we get a birational morphism $g \colon X \to Y$ where $Y$ is a smooth projective surface which does not contain any $-1$-curve.

# Minimal model program: dimension two

**Theorem (Minimal model)**

*Every surface $X$ is birational to a smooth projective surface $Y$ such that either*

- *$Y$ is* **minimal:** *$K_Y$ is nef, i.e., $K_Y \cdot C \geq 0$ for every curve $C$ on $Y$, or*
- *$Y$ is a* **Mori fibre space:** *there is a projective morphism $f\colon Y \to Z$ with connected fibres such that $\dim Z < \dim X$ and $K_Y \cdot C < 0$ for any curve $C$ contracted by $f$.*

**Theorem (Abundance)**

*Let $Y$ be a smooth projective minimal surface. Then, there is a projective morphism $f\colon Y \to T$ with connected fibres such that $K_Y \equiv f^*H$ for an ample divisor $H$ on $T$.*

# Minimal model program: higher dimesion

Let $X$ be a variety. Up to birational equivalence, we can assume that $X$ is smooth and projective. If $X$ is not already minimal or a Mori fibre space, then there is a projective birational morphism $f\colon X \to X_1$ to a normal variety such that $K_X \cdot C < 0$ for any curve $C$ contracted by $f$.

**Singularities.** $X_1$ may be singular. If $f$ contracts a divisor, the singularities of $X_1$ are "manageable".

**Flips.** If $f$ does not contract a divisor, singularities of $X_1$ are not manageable. We hope to get another projective birational morphism $f_1\colon X_2 \to X_1$ from a normal variety $X_2$ with manageable singularities such that $K_{X_2} \cdot C > 0$ for any curve $C$ contracted by $f_1$. We call this operation a flip.

# Minimal model program: higher dimesion

**Theorem (Flip)**

*Flips exist.*

Now replace $X$ with $X_2$ if we have a flip, or with $X_1$ otherwise. Repeat the process.

**Conjecture (Termination)**

*Prove that there is no infinite sequence of flips.*

If this conjecture holds, then we end up with a projective variety $Y$ with manageable singularities such that it is minimal or a Mori fibre space.

# Minimal model program: higher dimesion

**Conjecture (Minimal model)**

*Every variety $X$ is birational to a projective variety $Y$ with manageable singularities such that either*

- *$Y$ is* **minimal:** *$K_Y$ is nef, i.e., $K_Y \cdot C \geq 0$ for every curve $C$ on $Y$, or*
- *$Y$ is a* **Mori fibre space:** *there is a projective morphism $f\colon Y \to Z$ with connected fibres such that $\dim Z < \dim X$ and $K_Y \cdot C < 0$ for any curve $C$ contracted by $f$.*

**Conjecture (Abundance)**

*Let $Y$ be a minimal projective variety. Then, there is a projective morphism $f\colon Y \to T$ with connected fibres such that $K_Y \equiv f^*H$ for an ample divisor $H$ on $T$.*

# Minimal model program: higher dimesion

Let $X$ be a normal variety. A divisor $B$ with coefficients in $[0, 1]$ is called a boundary.

We can define different classes of manageable singularities for $X, B$. When $X, B$ has such singularities we form $(X, B)$ and call it a pair.

In particular, if $X$ is smooth and the support of $B$ is with simple normal crossings, then $(X, B)$ is a pair.

We can also run a minimal model program for pairs $(X, B)$ by replacing $K_X$ with $K_X + B$. So, we expect the same standard conjectures as before.

# Minimal model program: higher dimesion

**Conjecture (Minimal model)**

*Let $(X, B)$ be a pair. Then, there is a projective pair $(Y, B_Y)$ where $Y$ is birational to $X$ and such that either*

- $(Y, B_Y)$ *is* **minimal:** $K_Y + B_Y$ *is nef, i.e.,* $(K_Y + B_Y) \cdot C \geq 0$ *for every curve $C$ on $Y$, or*
- $(Y, B_Y)$ *is a* **Mori fibre space:** *there is a projective morphism* $f\colon Y \to Z$ *with connected fibres such that* $\dim Z < \dim X$ *and* $(K_Y + B_Y) \cdot C < 0$ *for any curve $C$ contracted by $f$.*

# Minimal model program: higher dimesion

As before the minimal model conjecture follows from

**Conjecture (Termination)**

*Prove that there is no infinite sequence of flips.*

**Conjecture (Abundance)**

*Let $(Y, B_Y)$ be a projective minimal pair. Then, there is a projective morphism $f\colon Y \to T$ with connected fibres such that $(K_Y + B_Y) \equiv f^*H$ for an ample divisor $H$ on $T$.*

# Moduli: varieties of general type

By the minimal model program, each smooth projective variety of general type is birational to a projective variety $X$ with canonical singularities and ample canonical divisor $K_X$.

Fix $d, n \in \mathbb{N}$ and $H$ a polynomial in $\mathbb{Q}[t]$. Let $\mathcal{G}$ be the class of projective varieties $X$ such that
$\dim X = d$,
$X$ has canonical singularities,
$nK_X$ is Cartier and ample,
$H(l) = h^0(X, lK_X)$ for large and divisible $l \in \mathbb{N}$.

By Matsusaka theorem, $\mathcal{G}$ is bounded. We would like to find a moduli space for $\mathcal{G}$ and find a geometrically meaningful compactification for it.

# Moduli: varieties of general type

An *n*-stable variety with Hilbert polynomial $H$ of dimension $d$ is a projective variety $X$ of dimension $d$ such that
$X$ is connected but possibly reducible,
$X$ has only semi-log canonical singularities,
$nK_X$ is Cartier and ample,

An *n*-stable variety $X$ is called smoothable if there is a flat projective $\mathbb{Q}$-Gorenstein morphism $f\colon Y \to C$ to a smooth curve such that $X$ is the special fibre and the general fibre is in $\mathcal{G}$.

# Moduli: varieties of general type

For a scheme $S$, we define an $n$-stable variety with Hilbert polynomial $H$ of dimension $d$ over $S$ to be a flat projective $\mathbb{Q}$-Gorenstein morphism $f\colon Y \to S$ such that
every geometric fibre is an $n$-stable variety with Hilbert polynomial $H$ of dimension $d$,
the restriction of $\mathcal{O}_Y(lnK_{Y/S})$ to a geometric fibre $X$ is $\mathcal{O}_X(lnK_X)$ for all large $l \in \mathbb{N}$.

Similarly, one can define a smoothable $n$-stable variety with Hilbert polynomial $H$ of dimension $d$ over $S$.

We define the moduli functor $\mathcal{M}^n_H\colon \{Schemes\} \to \{Sets\}$ by
$\mathcal{M}^n_H(S) = \{$isomorphism classes of smoothable
$n$-stable varieties with Hilbert polynomial $H$ of dim $d$ over $S\}$

# Moduli: varieties of general type

**Theorem**

*The moduli functor $\mathcal{M}^n_H$ is coarsely representable by a projective scheme.*

By Matsusaka's theorem, there is $r \in \mathbb{N}$ such that $rnK_X$ is very ample for any $X \in \mathcal{G}$.

So, we get a family $\pi\colon \mathcal{X} \to Z$ such that every $X \in \mathcal{G}$ appears as a geometric fibre of $\pi$.

After compactification of $Z, \mathcal{X}$ and applying the weakly semi-stable reduction, we can further assume that $\pi$ is weakly semi-stable: $Z$ is smooth and projective, all fibres are reduced, etc.

# Moduli: varieties of general type

By the minimal model program, we can construct a canonical model $\pi' \colon \mathcal{X}' \to Z$, i.e., $K_{\mathcal{X}'}$ is $\pi'$-ample and $\mathcal{X}'$ has canonical singularities.

Every geometric fibre of $\pi'$ is a smoothable $n$-stable variety with Hilbert polynomial $H$ of dimension $d$. Conversely every smoothable $n$-stable variety with Hilbert polynomial $H$ of dimension $d$ appears as a geometric fibre of $\pi'$.

In other words, smoothable $n$-stable varieties with Hilbert polynomial $H$ of dimension $d$ are bounded.

We can construct the moduli space $M$ from $Z$. Theorems of Kollar show that $M$ is a proper algebraic space and eventually a projective scheme. The moduli functor $\mathcal{M}^n_H$ is then coarsely represented by the projective scheme $M$.

# Fano varieties

A projective variety $X$ with log canonical singularities is a Fano variety if $-K_X$ is ample.

**Conjecture (BAB)**

*Let $d \in \mathbb{N}$ and $\epsilon \in \mathbb{R}^+$. Then, the class of Fano varieties $X$ of dimension $d$ with $\epsilon$-log canonical singularities, is bounded.*

Several important conjectures in minimal model program follow from this conjecture.

The BAB conjecture follows from conjectures in the theory of complement in the minimal model program.

# Fano varieties

If we assume $X$ smooth, then the BAB is proved in this case using the more geometric side of the minimal model program: by studying rational curves.

**Theorem**

*Let $d, n \in \mathbb{N}$. Then, the class of Fano varieties of dimension $d$ with klt singularities and $nK_X$ Cartier, is bounded.*

# Canonical bundle formula

Let $f\colon X \to Z$ be a minimal elliptic fibration of dimension 2. Kodaira's canonical bundle formula says that $K_X \equiv f^*(K_Z + B_Z)$ for some divisor $B_Z \geq 0$.

**Conjecture (Adjunction)**

*Let $f\colon X \to Z$ be a morphism of normal projective varieties with connected fibres. Assume that $(X, B)$ is klt and $K_X + B \equiv 0$ over $Z$. Then, there is $B_Z \geq 0$ such that $K_X + B \equiv f^*(K_Z + B_Z)$ and $(Z, B_Z)$ is klt.*

This conjecture is related to abundance, BAB, minimal model, and complements conjecture.

# Canonical bundle formula

In the conjecture, suppose that $\dim X = \dim Z + 1$ and $f(B) = Z$. Then, a general fibre $F$ of $f$ is $\mathbb{P}^1$.

Let $B_F$ be the support of $B|_F$. Then, $(F, B_F)$ is a stable pointed curve of genus zero. This induces a rational map $\phi\colon Z \dashrightarrow \overline{M}_{0,l}$ to the moduli space of $l$-pointed curves of genus zero.

In this case, using the explicit geometry of $\overline{M}_{0,l}$ and the universal family, one can prove the adjunction conjecture.

This plays a crucial role in the proof of the abundance theorem and theorems about complements in dimension 3, and the subadjunction formula for codimension 2 centres.