

ACC for log canonical thresholds and termination of log flips

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May 28, 2006

Abstract

We prove that the ascending chain condition (ACC) for log canonical (lc) thresholds in dimension d , existence of log flips in dimension d and the Log Minimal Model Program (LMMP) in dimension $d - 1$ imply termination of any sequence of log flips starting with a d -dimensional effective lc pair, and also imply termination of flops in dimension d . In particular, the latter terminations in dimension 4 follow from Alexeev-Borisov conjecture in dimension 3.

1 Introduction

Following the fundamental work of Shokurov [15] on the existence of log flips, one of the main open problems of the LMMP is the termination of log flips, in particular, in dimension 4.

Conjecture 1.1 (Termination of log flips) *Any sequence of log flips, with respect to a lc log divisor $K_X + B$, terminates.*

Conjecture 1.2 (Termination of D -flops) *Any sequence of D -flops, with respect to an effective divisor D , terminates.*

In this paper we prove the following

*This work was supported by the EPSRC. 2000 Mathematics Subject Classification. Primary 14E30; Secondary 14J35.

Main Theorem 1.3 *Assume ACC for lc thresholds in dimension d , existence of log flips in dimension d and the LMMP in dimension $d - 1$. Then, Conjecture 1.1 in dimension d holds for effective pairs (see Definition 2.1), that is, any sequence of log flips/ Z starting with a d -dimensional effective lc pair $(X_1/Z, B_1)$, terminates. Moreover, Conjecture 1.2 holds in dimension d , that is, any sequence of D -flops in dimension d terminates.*

Corollary 1.4 *Assume Alexeev-Borisov Conjecture in dimension $d - 1$, existence of log flips in dimension d and the LMMP in dimension $d - 1$. Then, Conjecture 1.1 in dimension d holds for effective pairs. Moreover, Conjecture 1.2 holds in dimension d .*

Corollary 1.5 *Assume Alexeev-Borisov Conjecture in dimension 3. Then, Conjecture 1.1 in dimension 4 holds for effective pairs. Moreover, Conjecture 1.2 holds in dimension 4.*

Remark 1.6 Recently, Hacon and McKernan [8] announced that they have proved that the LMMP in dimension $d - 1$ implies the existence of log flips in dimension d . In that case, in the Main Theorem and Corollary 1.4, we can drop the assumption of existence of log flips in dimension d . Thus, our results fit even better into an inductive approach to the establishment of the LMMP in all dimensions. Moreover, Shokurov [17] pointed out that existence of log flips in dimension 4 and termination of log flips for effective pairs in dimension 4 imply existence of log flips in dimension 5. Therefore, after our results, it is expected that Alexeev-Borisov Conjecture in dimension 3 implies also existence of log flips in dimension 5. For recent developments on the existence of log flips, see [4].

Here we recall the ACC for lc thresholds Conjecture and Special Termination Conjecture due to Shokurov, and the Alexeev-Borisov Conjecture due to Alexeev, A. Borisov and L. Borisov. For a set $S \subseteq \mathbb{R}$, and an \mathbb{R} -divisor B , $B \in S$ means that all nonzero coefficients of B are in S .

Conjecture 1.7 (ACC for lc thresholds) *Suppose that $\Gamma \subseteq [0, 1]$ satisfies the descending chain condition (DCC), $S \subseteq \mathbb{R}$ is a finite set of positive numbers and d a natural number. Then, the set*

$$\{\text{lct}(M, X, B) \mid (X, B) \text{ is lc of dimension } d, B \in \Gamma \text{ and } M \in S\}$$

satisfies the ACC where M is an \mathbb{R} -Cartier divisor on X and $\text{lct}(M, X, B)$ is the lc threshold of M with respect to (X, B) .

Conjecture 1.8 (Special Termination) *Let $X_i \dashrightarrow X_{i+1}/Z_i$ be a sequence of log flips/ Z starting from a lc pair $(X_1/Z, B_1)$ of dimension d . Then, there is an $I \in \mathbb{N}$ such that the flipping locus does not intersect the locus of lc singularities $\text{LCS}(X_i, B_i)$ for any $i \geq I$.*

Remark 1.9 Shokurov [16, Corollary 4, Addendum 4] proved that LMMP in dimension $d - 1$ and existence of log flips in dimension d imply Special Termination in dimension d .

Conjecture 1.10 (Alexeev-Borisov) *Let $\delta > 0$ be a real number and d a natural number. Then, projective varieties X for which (X, B) is a d -dimensional δ -lc weak log Fano (WLF) pair for a boundary B , with coefficients $\leq 1 - \delta$, form a bounded family.*

This Conjecture is proved up to dimension 2 [1].

The Termination Conjecture (1.1) has been proved in the following cases: 3-dimensional terminal case by Shokurov [13]; 3-dimensional Klt and 4-dimensional terminal cases by Kawamata ([9], [10]) for \mathbb{Q} -boundaries; 4-dimensional canonical and semi-stable cases by Fujino ([5], [6], [7]). 3-dimensional lc and 4-dimensional canonical cases by Shokurov ([14, Theorem 5.2], [16, Example 6]) for \mathbb{R} -boundaries. Alexeev also has proved some special cases [2]; but the main development toward a general solution is the paper by Shokurov [16] which proves that ACC and lower-semicontinuity for minimal log discrepancies (mld's) in dimension d imply the Termination Conjecture in dimension d . In particular, ACC for mld's in dimension 4 is enough to prove termination in dimension 4 [16]. Alternatively, in this paper, we use ACC for lc thresholds to prove termination for effective pairs. Note that effective pairs are exactly those which the LMMP predicts to have a *log minimal model*.

Termination of 4-fold flops in the terminal case was proved by Matsuki [11].

2 Notations and conventions

We assume the base field k to be an algebraically closed field of characteristic zero. A pair (X, B) consists of a normal variety X over k and B an \mathbb{R} -divisor

on X (boundary) with coefficients in $[0, 1]$ such that $K_X + B$ is an \mathbb{R} -Cartier divisor. More generally, $(X/Z, B)$ is a pair (X, B) equipped with a projective contraction $X \rightarrow Z$ onto a normal variety Z over k . We use the usual definitions of singularities of pairs such as Klt, dlt and lc, and $\text{LCS}(X, B)$ denotes the locus of lc singularities of (X, B) , that is, the locus where (X, B) is not Klt. By LMMP we mean the version outlined in [14, 5.1] which works with lc pairs and \mathbb{R} -boundaries. A pair $(X/\text{pt.}, B)$ is called *weak log Fano (WLF)* if it is lc and $-(K_X + B)$ is nef and big. We say (X, B) is \mathbb{Q} -factorial if X is.

Definition 2.1 A pair $(X/Z, B)$ is called *effective* if there is an effective \mathbb{R} -Cartier divisor M such that $K_X + B \equiv M/Z$.

Note that if $K_X + B$ is a \mathbb{Q} -divisor and the Kodaira dimension of (X, B) is nonnegative (i.e. $|m(K_X + B)| \neq \emptyset$ for some $m \in \mathbb{N}$), then $(X/Z, B)$ is effective.

Definition 2.2 Let M be an effective \mathbb{R} -Cartier divisor on X . The *lc threshold* of M with respect to a lc pair (X, B) is the real number

$$\text{lct}(M, X, B) := \sup\{t \mid (X, B + tM) \text{ is lc}\}$$

Definition 2.3 Let $(X/Z, B)$ be a pair and D an \mathbb{R} -Cartier divisor on X . A D -flip/ Z is a diagram

$$\begin{array}{ccc} X & \dashrightarrow & X^+ \\ & \searrow f & \swarrow f^+ \\ & & Z' \end{array}$$

such that

- X^+ and Z' are normal varieties $/Z$.
- f and f^+ are small projective birational contractions, where small means that they contract subvarieties of codimension ≥ 2 .
- f is extremal, that is, the relative Picard number $\rho(X/Z') = 1$.
- $-D$ is ample/ Z' and D^+ is \mathbb{R} -Cartier and ample/ Z' where D^+ is the birational transform of D .

Note that if X is \mathbb{Q} -factorial, then X^+ is also \mathbb{Q} -factorial. A *log flip*/ Z is a D -flip/ Z where $D = K_X + B$ and (X, B) is lc. We also call such a log flip, a $K_X + B$ -flip. For an effective divisor D , a flop directed with respect to D or a *D -flop*/ Z is a D -flip/ Z such that (X, B) is Klt and $K_X + B$ is numerically zero/ Z' for a boundary B .

A *sequence of flips*/ Z with respect to D_1 is a sequence $X_i \dashrightarrow X_{i+1}/Z_i$ of D_i -flips/ Z where $X_{i+1} = X_i^+$, $D_{i+1} = D_i^+$ and $Z_i = Z'$ as in Definition 2.3. A sequence of flops/ Z with respect to D_1 , is a sequence of flips/ Z with respect to D_1 such that B_i is the birational transform of B_1 as in the definition of a flop, that is, $K_{X_i} + B_i \equiv 0/Z_i$. A sequence of log flips/ Z with respect to $K_{X_1} + B_1$ is a sequence of flips/ Z with respect to $D_1 := K_{X_1} + B_1$.

3 Proof of the main theorem

Construction 3.1 Let $(X/Z, B)$ be a d -dimensional lc pair. Assuming existence of log flips in dimension d and the LMMP in dimension $d - 1$, it is well known that we can construct a dlt blow up $f: Y \rightarrow X$ of (X, B) as follows. Let $h: W \rightarrow X$ be a log resolution of (X, B) such that for each lc centre $S \subset \text{LCS}(X, B)$ we have a prime divisor E on W with log discrepancy $a(E, X, B) = 0$ and $h(E) = S$. Let

$$K_W + B_W := h^*(K_X + B) \quad \text{and} \quad D_W := B_W + \sum (1 - e_k)E_k$$

where E_k are prime exceptional/ X divisors and e_k are their coefficients in B_W . So, any exceptional/ X prime divisor has coefficient equal to one in D_W . Since h is a log resolution, (W, D_W) is \mathbb{Q} -factorial dlt.

Now we can run the LMMP/ X on $K_W + D_W$. Note that $K_W + B_W \equiv 0/X$ and so $K_W + D_W \equiv \sum (1 - e_k)E_k/X$ which is an exceptional/ X effective divisor. Thus, in each step of this LMMP, if an extremal ray R is contracted, then the reduced part of D_W intersects R . So, by Special Termination we can run such LMMP/ X . Note also that since $\sum (1 - e_k)E_k$ is exceptional, it is contracted after running the LMMP. In particular, if $e_k < 1$, then E_k is contracted and if $e_k = 1$, then E_k is not contracted by this LMMP.

At the end of the LMMP, we get a contraction $f: Y \rightarrow X$ with the following properties.

- The birational transform of $K_W + D_W$ on Y which is equal to $K_Y + B_Y := f^*(K_X + B)$, is \mathbb{Q} -factorial dlt,

- For each lc centre $S \subset \text{LCS}(X, B)$ we have a prime divisor E on Y with log discrepancy $a(E, X, B) = 0$ and $f(E) = S$, and
- f is small outside $\text{LCS}(X, B)$.

Let $X \dashrightarrow X^+/Z'$ be a $K_X + B$ -flip. Run the LMMP/ Z' on $K_Y + B_Y$. If the LMMP terminates, at the end, we get a model Y^+ where $K_{Y^+} + B_{Y^+}$, the birational transform of $K_Y + B_Y$ is nef/ Z' . By the uniqueness of the lc model, there is a morphism $f^+: Y^+ \rightarrow X^+$ such that $K_{Y^+} + B_{Y^+} = f^{+*}(K_{X^+} + B^+)$. So, Y^+ is a dlt blow up of (X^+, B^+) (see step 1 in the proof of 5.1.3 in [14]). In particular, if the flipping locus of X/Z' does not intersect $\text{LCS}(X, B)$, then $Y \dashrightarrow Y^+$ is a sequence of log flips.

Lemma 3.2 *Assume ACC for lc thresholds in dimension d , existence of log flips in dimension d and the LMMP in dimension $d - 1$. Let $X_i \dashrightarrow X_{i+1}/Z_i$ be a sequence of log flips with respect to a lc log divisor $K_{X_1} + B_1$ such that there is an effective \mathbb{R} -Cartier divisor M_1 on X_1 such that $-M_i$ is ample/ Z_i for all i , where $M_i \subset X_i$ is the birational transform of M_1 . Then, the sequence of log flips terminates.*

Proof Suppose that the sequence does not terminate; we derive a contradiction.

Step 1. Let t_1 be the lc threshold of M_1 with respect to (X_1, B_1) . By Special Termination (see Remark 1.9), there is an $I \in \mathbb{N}$ such that $\text{LCS}(X_i, B_i + t_1 M_i)$ does not intersect the flipping locus, that is the subvariety of X_i contracted/ Z_i , for any $i \geq I$.

Step 2. By construction, $(X_i, B_i + t_1 M_i)$ is Klt near the flipping locus for all $i \geq I$. Let $f: Y_I \rightarrow X_I$ be a dlt blow up as in Construction 3.1 for the pair $(X_I, B_I + t_1 M_I)$. Let B_I^Y be the divisor on Y_I such that there is no component F of B_I^Y with $f(F) \subset \text{LCS}(X_I, B_I + t_1 M_I)$ and such that $f_*(B_I^Y) = B_I$ outside $\text{LCS}(X_I, B_I + t_1 M_I)$. Similarly, define M_I^Y . Thus, $(Y_I, B_I^Y + t_1 M_I^Y)$ is Klt. As in Construction 3.1, either after a finite sequence of $K_{Y_I} + B_{Y_I} := f^*(K_{X_I} + B_I + t_1 M_I)$ -flips/ Z_I , we end up with Y_{I+1} which is a dlt blow up of $(X_{I+1}, B_{I+1} + t_1 M_{I+1})$, or we get an infinite sequence of $K_{Y_I} + B_{Y_I}$ -flips. In either case, the sequence (finite or infinite) is also a sequence of $K_{Y_I} + B_I^Y + t_1 M_I^Y$ -flips/ Z_I because $\text{LCS}(X_I, B_I + t_1 M_I)$ does not intersect the flipping locus. If the sequence is finite, then continue the

argument for $i > I$ and so on. Therefore, in any case, we get a sequence of $K_{Y_I} + B_I^Y + t_1 M_I^Y$ -flips/ Z_I which does not terminate. Moreover, the latter sequence is also a sequence of M_I^Y -flips.

Step 3. Let t_2 be the lc threshold of M_I^Y with respect to (Y_I, B_I^Y) . Obviously, $t_2 > t_1$. Similar to step 1, Special Termination implies that after finitely many $K_{Y_I} + B_I^Y + t_2 M_I^Y$ -flips the flipping locus does not intersect the locus of lc singularities. Now continue as in step 1 and 2.

Step 3. By repeating the above process, we get an increasing sequence $t_1 < t_2 < t_3 < \dots$ of lc thresholds. This contradicts the ACC for lc thresholds.

□

Proof (of Main Theorem) To get termination of log flips for effective pairs in dimension d , let M_1 be an effective \mathbb{R} -Cartier divisor such that $K_{X_1} + B_1 \equiv M_1/Z$ as in Definition 2.1. Now apply Lemma 3.2.

To get termination of a sequence of flops with respect to D_1 in dimension d such that $K_{X_i} + B_i \equiv 0/Z_i$ and Klt, let $\alpha > 0$ be a real number such that $(X_1, B_1 + \alpha D_1)$ is lc. Now apply Lemma 3.2 with $M_1 := D_1$ and the boundary $B_1 + \alpha D_1$. □

The following theorem is proved in [12] but with the extra assumption of termination of log flips in dimension d .

Theorem 3.3 *Alexeev-Borisov Conjecture in dimension $d - 1$, existence of log flips in dimension d and the LMMP in dimension $d - 1$ imply ACC for lc thresholds (1.7) in dimension d .*

To prove this theorem, McKernan and Prokhorov [12, Theorem 3.10] also assume termination of log flips in dimension d . In fact, this is not necessary. The only place where they use termination of log flips in dimension d is in [12, Lemma 6.1]. They need this termination to extract a divisor with log discrepancy zero. But Special Termination is enough to extract such a divisor, as this is verified in [14, Theorem 3.1, Remark 3.1.1] or one can simply use a dlt blow up as in Construction 3.1.

Proof (of Corollary 1.4) Immediate by Theorem 3.3. \square

Proof (of Corollary 1.5) LMMP in dimension 3 [14] and existence of log flips in dimension 4 [15] are known. Thus, by Corollary 1.4 we are done. \square

Remark 3.4 Our method gives a new proof of termination in dimension 3 for effective pairs which unlike other proofs ([9], [14]) does not use the classification of 3-fold terminal singularities, because Alexeev-Borisov Conjecture in dimension 2 [1][3], existence of log flips in dimension 3 and the LMMP in dimension 2 are all proved without using termination of log flips in dimension 3. It is quite important to avoid the classification of 3-fold terminal singularities, because such a classification is not known and not expected in higher dimension.

Remark 3.5 A plan to attack the Alexeev-Borisov Conjecture in dimension 3, using the theory of complements, is exposed in [3].

4 Acknowledgement

I thank Professor V.V. Shokurov for his valuable comments and discussions. I also thank the referees who helped to improve the paper considerably. This research was supported by the EPSRC.

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