

**MINIMAL MODELS, FLIPS AND
FINITE GENERATION :
A TRIBUTE TO V.V. SHOKUROV AND Y.-T. SIU**

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ABSTRACT. In this paper, we discuss a proof of existence of log minimal models or Mori fibre spaces for klt pairs $(X/Z, B)$ with B big/ Z . This then implies existence of klt log flips, finite generation of klt log canonical rings, and most of the other results of Birkar-Cascini-Hacon-McKernan paper [3].

1. INTRODUCTION

We consider pairs $(X/Z, B)$ where B is an \mathbb{R} -boundary and $X \rightarrow Z$ is a projective morphism of normal quasi-projective varieties over an algebraically closed field k of characteristic zero. We call a pair $(X/Z, B)$ effective if there is an \mathbb{R} -divisor $M \geq 0$ such that $K_X + B \equiv M/Z$.

Theorem 1.1. *Let $(X/Z, B)$ be a klt pair where B is big/ Z . Then,*

- (1) *if $K_X + B$ is pseudo-effective/ Z , then $(X/Z, B)$ has a log minimal model,*
- (2) *if $K_X + B$ is not pseudo-effective/ Z , then $(X/Z, B)$ has a Mori fibre space.*

Corollary 1.2 (Log flips). *Log flips exist for klt (hence \mathbb{Q} -factorial dlt) pairs.*

Corollary 1.3 (Finite generation). *Let $(X/Z, B)$ be a klt pair where B is a \mathbb{Q} -divisor and $f: X \rightarrow Z$ the given morphism. Then, the log canonical sheaf*

$$\mathcal{R}(X/Z, B) := \bigoplus_{m \geq 0} f_* \mathcal{O}_X(m(K_X + B))$$

is a finitely generated \mathcal{O}_Z -algebra.

The proof of the above theorem is divided into two independent parts. First we have

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Theorem 1.4 (Log minimal models). *Assume (1) of Theorem 1.1 in dimension $d - 1$ and let $(X/Z, B)$ be a klt pair of dimension d where B is big/ Z . If $(X/Z, B)$ is effective, then it has a log minimal model.*

A proof of this theorem is given in section 2 based on ideas in [2][3]. Second we have

Theorem 1.5 (Nonvanishing). *Let $(X/Z, B)$ be a klt pair where B is big/ Z . If $K_X + B$ is pseudo-effective/ Z , then $(X/Z, B)$ is effective.*

A proof of this theorem is given in section 3; it is mainly based on the arguments in [27] (see [28] and [35] as well). We give a rough comparison of the proof of this theorem with the proof of the corresponding theorem in [3], that is [3, Theorem D].

One crucial feature of the proof of Theorem 1.5 is that it does not use the log minimal model program. The proof goes as follows:

(a) We first assume that Z is a point, and using Zariski type decompositions one can create lc centres and pass to plt pairs, more precisely, by going on a sufficiently high resolution we can replace $(X/Z, B)$ by a plt pair $(X/Z, B + S)$ where S is a smooth prime divisor, $K_X + B + S|_S$ is pseudo-effective, B is big and its components do not intersect.

(b) By induction, the \mathbb{R} -bundle $K_X + B + S|_S$ has an \mathbb{R} -section say T , which can be assumed to be singular enough for the extension purposes.

(c) Diophantine approximation of the couple (B, T) : we can find pairs (B_i, T_i) with rational coefficients and sufficiently close to (B, T) (in a very precise sense) such that

$$K_X + B + S = \sum r_i(K_X + B_i + S)$$

for certain real numbers $r_i \in [0, 1]$ and such that all the pairs $(X/Z, B_i + S)$ are plt and each $K_X + B_i + S|_S$ is numerically equivalent with T_i . Moreover, one can improve this to $K_X + B_i + S|_S \sim_{\mathbb{Q}} T_i \geq 0$.

(d) Using the *invariance of plurigenera techniques*, one can lift this to $K_X + B_i + S \sim_{\mathbb{Q}} M_i \geq 0$ and then a relation $K_X + B + S \equiv M \geq 0$.

(e) Finally, we get the theorem in the general case, i.e. when Z is not a point, using positivity properties of direct image sheaves and another application of extension theorems.

In contrast, the log minimal model program is an important ingredient of the proof of [3, Theorem D] which proceeds as follows: (a') This step is the same as (a) above. (b') By running the log minimal model program appropriately and using induction on finiteness of log minimal models and termination with scaling, one constructs a model Y birational to X such that $K_Y + B_Y + S_Y|_{S_Y}$ is nef where $K_Y + B_Y + S_Y$ is the push down of $K_X + B + S$. Moreover, by Diophantine approximation,

we can find boundaries B_i with rational coefficients and sufficiently close to B such that $B = \sum r_i B_i$ for certain real numbers $r_i \in [0, 1]$ and such that each $K_Y + B_{i,Y} + S_Y$ is plt and $K_Y + B_{i,Y} + S_Y|_{S_Y}$ is nef. (c') By applying induction and the base point free theorem one gets $K_Y + B_{i,Y} + S_Y|_{S_Y} \sim_{\mathbb{Q}} N_i \geq 0$. (d') The Kawamata-Viehweg vanishing theorem now gives $K_Y + B_{i,Y} + S_Y \sim_{\mathbb{Q}} M_{i,Y} \geq 0$ from which we easily get a relation $K_X + B + S \sim_{\mathbb{R}} M \geq 0$. (e') Finally, we get the theorem in the general case, i.e. when Z is not a point, by restricting to the generic fibre and applying induction.

2. LOG MINIMAL MODELS

In this section we prove Theorem 1.4 (cf. [3, Theorems A, B, C, E]). The results in this section are also implicitly or explicitly proved in [3]. We hope that this section also helps the reader to read [3].

Preliminaries. Let k be an algebraically closed field of characteristic zero fixed throughout this section. When we write an \mathbb{R} -divisor D as $D = \sum d_i D_i$ (or similar notation) we mean that D_i are distinct prime divisors. The norm $\|D\|$ is defined as $\max\{|d_i|\}$. For a birational map $\phi: X \dashrightarrow Y$ and an \mathbb{R} -divisor D on X we often use D_Y to mean the birational transform of D , unless specified otherwise.

A *pair* $(X/Z, B)$ consists of normal quasi-projective varieties X, Z over k , an \mathbb{R} -divisor B on X with coefficients in $[0, 1]$ such that $K_X + B$ is \mathbb{R} -Cartier, and a projective morphism $X \rightarrow Z$. For a prime divisor E on some birational model of X with a nonempty centre on X , $a(E, X, B)$ denotes the log discrepancy.

An \mathbb{R} -divisor D on X is called *pseudo-effective*/ Z if up to numerical equivalence/ Z it is the limit of effective \mathbb{R} -divisors, i.e. for any ample/ Z \mathbb{R} -divisor A and real number $a > 0$, $D + aA$ is big/ Z . A pair $(X/Z, B)$ is called *effective* if there is an \mathbb{R} -divisor $M \geq 0$ such that $K_X + B \equiv M/Z$; in this case, we call $(X/Z, B, M)$ a *triple*. By a log resolution of a triple $(X/Z, B, M)$ we mean a log resolution of $(X, \text{Supp } B + M)$. A triple $(X/Z, B, M)$ is log smooth if $(X, \text{Supp } B + M)$ is log smooth. When we refer to a triple as being lc, dlt, etc, we mean that the underlying pair $(X/Z, B)$ has such properties.

For a triple $(X/Z, B, M)$, define

$$\theta(X/Z, B, M) := \#\{i \mid m_i \neq 0 \text{ and } b_i \neq 1\}$$

where $B = \sum b_i D_i$ and $M = \sum m_i D_i$.

Let $(X/Z, B)$ be a lc pair. By a *log flip*/ Z we mean the flip of a $K_X + B$ -negative extremal flipping contraction/ Z , and by a *pl flip*/ Z we mean a log flip/ Z when $(X/Z, B)$ is \mathbb{Q} -factorial dlt and the log flip

is also an S -flip for some component S of $\lfloor B \rfloor$, i.e. S is numerically negative on the flipping contraction.

A sequence of log flips/ Z starting with $(X/Z, B)$ is a sequence $X_i \dashrightarrow X_{i+1}/Z_i$ in which $X_i \rightarrow Z_i \leftarrow X_{i+1}$ is a $K_{X_i} + B_i$ -flip/ Z , B_i is the birational transform of B_1 on X_1 , and $(X_1/Z, B_1) = (X/Z, B)$.

Definition 2.1 (Log minimal models and Mori fibre spaces). *Let $(X/Z, B)$ be a dlt pair, $(Y/Z, B_Y)$ a \mathbb{Q} -factorial dlt pair, $\phi: X \dashrightarrow Y/Z$ a birational map such that ϕ^{-1} does not contract divisors, and $B_Y = \phi_*B$.*

(1) *We say that $(Y/Z, B_Y)$ is a nef model of $(X/Z, B)$ if $K_Y + B_Y$ is nef/ Z . We say that $(Y/Z, B_Y)$ is a log minimal model of $(X/Z, B)$ if in addition*

$$a(D, X, B) < a(D, Y, B_Y)$$

for any prime divisor D on X which is exceptional/ Y .

(2) *Let $(Y/Z, B_Y)$ be a log minimal model of $(X/Z, B)$ such that $K_Y + B_Y$ is semi-ample/ Z so that there is a contraction $f: Y \rightarrow S/Z$ and an ample/ Z \mathbb{R} -divisor H on S such that $K_Y + B_Y \sim_{\mathbb{R}} f^*H/Z$. We call S the log canonical model of $(X/Z, B)$ which is unique up to isomorphism/ Z .*

(3) *On the other hand, we say that $(Y/Z, B_Y)$ is a Mori fibre space of $(X/Z, B)$ if there is a $K_Y + B_Y$ -negative extremal contraction $Y \rightarrow T/Z$ such that $\dim T < \dim Y$, and if*

$$a(D, X, B) \leq a(D, Y, B_Y)$$

for any prime divisor D on birational models of X with strict inequality for any prime divisor D on X which is exceptional/ Y .

Note that in [2], it is not assumed that ϕ^{-1} does not contract divisors. However, since in this paper we are mainly concerned with constructing models for klt pairs, in that case our definition here is equivalent to that of [2].

Lemma 2.2. *Let $(X/Z, B+C)$ be a \mathbb{Q} -factorial lc pair where $B, C \geq 0$, $K_X + B + C$ is nef/ Z , and $(X/Z, B)$ is dlt. Then, either $K_X + B$ is also nef/ Z or there is an extremal ray R/Z such that $(K_X + B) \cdot R < 0$, $(K_X + B + \lambda C) \cdot R = 0$, and $K_X + B + \lambda C$ is nef/ Z where*

$$\lambda := \inf\{t \geq 0 \mid K_X + B + tC \text{ is nef}/Z\}$$

Proof. This is proved in [2, Lemma 2.7] assuming that $(X/Z, B+C)$ is dlt. We extend it to the lc case.

Suppose that $K_X + B$ is not nef/ Z and let $\{R_i\}_{i \in I}$ be the set of $(K_X + B)$ -negative extremal rays/ Z and Γ_i an extremal curve of R_i

[32, Definition 1]. Let $\mu := \sup\{\mu_i\}$ where

$$\mu_i := \frac{-(K_X + B) \cdot \Gamma_i}{C \cdot \Gamma_i}$$

Obviously, $\lambda = \mu$ and $\mu \in (0, 1]$. It is enough to prove that $\mu = \mu_l$ for some l . By [32, Proposition 1], there are positive real numbers r_1, \dots, r_s and a positive integer m (all independent of i) such that

$$(K_X + B) \cdot \Gamma_i = \sum_{j=1}^s \frac{r_j n_{i,j}}{m}$$

where $-2(\dim X)m \leq n_{i,j} \in \mathbb{Z}$. On the other hand, by [30, First Main Theorem 6.2, Remark 6.4] we can write

$$K_X + B + C = \sum_{k=1}^t r'_k (K_X + \Delta_k)$$

where r'_1, \dots, r'_t are positive real numbers such that for any k we have: $(X/Z, \Delta_k)$ is lc with Δ_k being rational, and $(K_X + \Delta_k) \cdot \Gamma_i \geq 0$ for any i . Therefore, there is a positive integer m' (independent of i) such that

$$(K_X + B + C) \cdot \Gamma_i = \sum_{k=1}^t \frac{r'_k n'_{i,k}}{m'}$$

where $0 \leq n'_{i,k} \in \mathbb{Z}$.

The set $\{n_{i,j}\}_{i,j}$ is finite. Moreover,

$$\frac{1}{\mu_i} = \frac{C \cdot \Gamma_i}{-(K_X + B) \cdot \Gamma_i} = \frac{(K_X + B + C) \cdot \Gamma_i}{-(K_X + B) \cdot \Gamma_i} + 1 = -\frac{m \sum_k r'_k n'_{i,k}}{m' \sum_j r_j n_{i,j}} + 1$$

Thus, $\inf\{\frac{1}{\mu_i}\} = \frac{1}{\mu_l}$ for some l and so $\mu = \mu_l$. \square

Definition 2.3 (LMMP with scaling). *Let $(X/Z, B + C)$ be a lc pair such that $K_X + B + C$ is nef/ Z , $B \geq 0$, and $C \geq 0$ is \mathbb{R} -Cartier. Suppose that either $K_X + B$ is nef/ Z or there is an extremal ray R/Z such that $(K_X + B) \cdot R < 0$, $(K_X + B + \lambda_1 C) \cdot R = 0$, and $K_X + B + \lambda_1 C$ is nef/ Z where*

$$\lambda_1 := \inf\{t \geq 0 \mid K_X + B + tC \text{ is nef}/Z\}$$

When $(X/Z, B)$ is \mathbb{Q} -factorial dlt, the last sentence follows from Lemma 2.2. If R defines a Mori fibre structure, we stop. Otherwise assume that R gives a divisorial contraction or a log flip $X \dashrightarrow X'$. We can now consider $(X'/Z, B' + \lambda_1 C')$ where $B' + \lambda_1 C'$ is the birational transform of $B + \lambda_1 C$ and continue the argument. That is, suppose that either $K_{X'} + B'$ is nef/ Z or there is an extremal ray R'/Z such that

$(K_{X'} + B') \cdot R' < 0$, $(K_{X'} + B' + \lambda_2 C') \cdot R' = 0$, and $K_{X'} + B' + \lambda_2 C'$ is nef/ Z where

$$\lambda_2 := \inf\{t \geq 0 \mid K_{X'} + B' + tC' \text{ is nef}/Z\}$$

By continuing this process, we obtain a special kind of LMMP/ Z which is called the LMMP/ Z on $K_X + B$ with scaling of C ; note that it is not unique. This kind of LMMP was first used by Shokurov [29]. When we refer to termination with scaling we mean termination of such an LMMP.

Special termination with scaling means termination near $[B]$ of any sequence of log flips/ Z with scaling of C , i.e. after finitely many steps, the locus of the extremal rays in the process do not intersect $\text{Supp } [B]$.

When we have a lc pair $(X/Z, B)$, we can always find an ample/ Z \mathbb{R} -Cartier divisor $C \geq 0$ such that $K_X + B + C$ is lc and nef/ Z , so we can run the LMMP/ Z with scaling assuming that all the necessary ingredients exist, eg extremal rays, log flips.

Finiteness of models.

(P) Let $X \rightarrow Z$ be a projective morphism of normal quasi-projective varieties, $A \geq 0$ a \mathbb{Q} -divisor on X , and V a rational (i.e. with a basis consisting of rational divisors) finite dimensional affine subspace of the space of \mathbb{R} -Weil divisors on X . Define

$$\mathcal{L}_A(V) = \{B \mid 0 \leq (B - A) \in V, \text{ and } (X/Z, B) \text{ is lc}\}$$

By [29, 1.3.2], $\mathcal{L}_A(V)$ is a rational polytope (i.e. a polytope with rational vertices) inside the rational affine space $A + V$.

Remark 2.4. *With the setting as in (P) above assume that A is big/ Z . Let $B \in \mathcal{L}_A(V)$ such that $(X/Z, B)$ is klt. Let $A' \geq 0$ be an ample/ Z \mathbb{Q} -divisor. Then, there is a rational number $\epsilon > 0$ and an \mathbb{R} -divisor $G \geq 0$ such that $A \sim_{\mathbb{R}} \epsilon A' + G/Z$ and $(X/Z, B - A + \epsilon A' + G)$ is klt. Moreover, there is a neighborhood of B in $\mathcal{L}_A(V)$ such that for any B' in that neighborhood $(X/Z, B' - A + \epsilon A' + G)$ is klt. The point is that we can change A and get an ample part $\epsilon A'$ in the boundary. So, when we are concerned with a problem locally around B we feel free to assume that A is actually ample by replacing it with $\epsilon A'$.*

Lemma 2.5. *With the setting as in (P) above assume that A is big/ Z , $(X/Z, B)$ is klt of dimension d , and $K_X + B$ is nef/ Z where $B \in \mathcal{L}_A(V)$. Then, there is $\epsilon > 0$ (depending on X, Z, V, A, B) such that if R is a $K_X + B'$ -negative extremal ray/ Z for some $B' \in \mathcal{L}_A(V)$ with $\|B - B'\| < \epsilon$ then $(K_X + B) \cdot R = 0$.*

Proof. This is proved in [32, Corollary 9] in a more general situation. Since B is big/ Z and $K_X + B$ is nef/ Z , the base point free theorem implies that $K_X + B$ is semi-ample/ Z hence there is a contraction $f: X \rightarrow S/Z$ and an ample/ Z \mathbb{R} -divisor H on S such that $K_X + B \sim_{\mathbb{R}} f^*H/Z$. We can write $H \sim_{\mathbb{R}} \sum a_i H_i/Z$ where $a_i > 0$ and the H_i are ample/ Z Cartier divisors on S . Therefore, there is $\delta > 0$ such that for any curve C/Z in X either $(K_X + B) \cdot C = 0$ or $(K_X + B) \cdot C > \delta$.

Now let $\mathcal{C} \subset \mathcal{L}_A(V)$ be a rational polytope of maximal dimension which contains an open neighborhood of B in $\mathcal{L}_A(V)$ and such that $(X/Z, B')$ is klt for any $B' \in \mathcal{C}$. Pick $B' \in \mathcal{C}$ and let B'' be the point on the boundary of \mathcal{C} such that B' belongs to the line segment determined by B, B'' . Let R be a $K_X + B'$ -negative extremal ray R/Z . If $(K_X + B) \cdot R > 0$, then $(K_X + B'') \cdot R < 0$ and there is a rational curve Γ in R such that $(K_X + B'') \cdot \Gamma \geq -2d$. Since $(K_X + B') \cdot \Gamma < 0$, $(B' - B) \cdot \Gamma < -\delta$. If $\|B - B'\|$ is too small we cannot have

$$(K_X + B'') \cdot \Gamma = (K_X + B') \cdot \Gamma + (B'' - B') \cdot \Gamma \geq -2d$$

because $(B'' - B') \cdot \Gamma$ would be too negative. \square

Theorem 2.6. *Assume (1) of Theorem 1.1 in dimension d . With the setting as in (P) above assume that A is big/ Z . Let $\mathcal{C} \subseteq \mathcal{L}_A(V)$ be a rational polytope such that $(X/Z, B)$ is klt for any $B \in \mathcal{C}$ where $\dim X = d$. Then, there are finitely many birational maps $\phi_i: X \dashrightarrow Y_i/Z$ such that for any $B \in \mathcal{C}$ with $K_X + B$ pseudo-effective/ Z , there is i such that $(Y_i/Z, B_{Y_i})$ is a log minimal model of $(X/Z, B)$.*

Proof. Remember that as usual B_{Y_i} is the birational transform of B . We may proceed locally, so fix $B \in \mathcal{C}$. If $K_X + B$ is not pseudo-effective/ Z then the same holds in a neighborhood of B inside \mathcal{C} , so we may assume that $K_X + B$ is pseudo-effective/ Z . By assumptions, $(X/Z, B)$ has a log minimal model $(Y/Z, B_Y)$. Moreover, the polytope \mathcal{C} determines a rational polytope \mathcal{C}_Y of \mathbb{R} -divisors on Y by taking birational transforms of elements of \mathcal{C} . If we shrink \mathcal{C} around B we can assume that the inequality in (1) of Definition 2.1 is satisfied for every $B' \in \mathcal{C}$, that is,

$$a(D, X, B') < a(D, Y, B'_Y)$$

for any prime divisor $D \subset X$ contracted/ Y . Moreover, a log minimal model of $(Y/Z, B'_Y)$ would also be a log minimal model of $(X/Z, B')$, for any $B' \in \mathcal{C}$. Therefore, we can replace $(X/Z, B)$ by $(Y/Z, B_Y)$ and assume from now on that $(X/Z, B)$ is a log minimal model of itself, in particular, $K_X + B$ is nef/ Z .

Since B is big/ Z , by the base point free theorem, $K_X + B$ is semi-ample/ Z so there is a contraction $f: X \rightarrow S/Z$ such that $K_X + B \sim_{\mathbb{R}}$

f^*H/Z for some ample/ Z \mathbb{R} -divisor H on S . Now by induction on the dimension of \mathcal{C} , we may assume that the theorem already holds over S for all the points on the proper faces of \mathcal{C} , that is, there are finitely many birational maps $\psi_j: X \dashrightarrow Y_j/S$ such that for any B'' on the boundary of \mathcal{C} with $K_X + B''$ pseudo-effective/ S , there is j such that $(Y_j/S, B''_{Y_j})$ is a log minimal model of $(X/S, B'')$.

By Lemma 2.5, if we further shrink \mathcal{C} around B , then for any $B' \in \mathcal{C}$, any j , and any $K_{Y_j} + B'_{Y_j}$ -negative extremal ray R/Z we have the equality $(K_{Y_j} + B_{Y_j}) \cdot R = 0$. Note that all the pairs $(Y_j/Z, B_{Y_j})$ are klt and $K_{Y_j} + B_{Y_j} \equiv 0/S$ and nef/ Z because $K_X + B \equiv 0/S$.

Assume that $B \neq B' \in \mathcal{C}$ such that $K_X + B'$ is pseudo-effective/ Z , and let B'' be the unique point on the boundary of \mathcal{C} such that B' belongs to the line segment given by B and B'' . Since $K_X + B \equiv 0/S$, $K_X + B''$ is pseudo-effective/ S , and $(Y_j/S, B''_{Y_j})$ is a log minimal model of $(X/S, B'')$ for some j . So, $(Y_j/S, B'_{Y_j})$ is a log minimal model of $(X/S, B')$. Furthermore, $(Y_j/Z, B'_{Y_j})$ is a log minimal model of $(X/Z, B')$ because any $K_{Y_j} + B'_{Y_j}$ -negative extremal ray R/Z would be over S by the last paragraph. \square

Termination with scaling.

Theorem 2.7. *Assume (1) of Theorem 1.1 in dimension d and let $(X/Z, B + C)$ be a klt pair of dimension d where $B \geq 0$ is big/ Z , $C \geq 0$ is \mathbb{R} -Cartier, and $K_X + B + C$ is nef/ Z . Then, we can run the LMMP/ Z on $K_X + B$ with scaling of C and it terminates.*

Proof. Note that existence of klt log flips in dimension d follows from the assumptions (see the proof of Corollary 1.2). Run the LMMP/ Z on $K_X + B$ with scaling of C and assume that we get an infinite sequence $X_i \dashrightarrow X_{i+1}/Z_i$ of log flips/ Z . We may assume that $X = X_1$. Let λ_i be as in Definition 2.3 and put $\lambda = \lim \lambda_i$. So, by definition, $K_{X_i} + B_i + \lambda_i C_i$ is nef/ Z and numerically zero over Z_i where B_i and C_i are the birational transforms of B and C respectively. By taking a \mathbb{Q} -factorialisation of X , which exists by induction on d and Lemma 2.10, we can assume that all the X_i are \mathbb{Q} -factorial.

Let H_1, \dots, H_m be general ample/ Z Cartier divisors on X which generate the space $N^1(X/Z)$. Since B is big/ Z , we may assume that $B - \epsilon(H_1 + \dots + H_m) \geq 0$ for some rational number $\epsilon > 0$ (see Remark 2.4). Put $A = \frac{\epsilon}{2}(H_1 + \dots + H_m)$. Let V be the space generated by the components of $B + C$, and let $\mathcal{C} \subset \mathcal{L}_A(V)$ be a rational polytope of maximal dimension containing neighborhoods of B and $B + C$ such that $(X/Z, B')$ is klt for any $B' \in \mathcal{C}$. Moreover, we can choose \mathcal{C} such that for each i there is an ample/ Z \mathbb{Q} -divisor $G_i = \sum g_{i,j} H_{i,j}$ on X_i

with sufficiently small coefficients, where $H_{i,j}$ on X_i is the birational transform of H_j , such that $(X_i, B_i + G_i + \lambda_i C_i)$ is klt and the birational transform of $B_i + G_i + \lambda_i C_i$ on X belongs to \mathcal{C} .

Let $\phi_{i,j}: X_i \dashrightarrow X_j$ be the birational map induced by the above sequence of log flips. Since $K_{X_i} + B_i + G_i + \lambda_i C_i$ is ample/ Z and since the log canonical model is unique, by Theorem 2.6, there exist an infinite set $J \subseteq \mathbb{N}$ and a birational map $\phi: X = X_1 \dashrightarrow Y/Z$ such that $\psi_j := \phi_{1,j} \phi^{-1}$ is an isomorphism for any $j \in J$. This in turn implies that $\phi_{i,j}$ is an isomorphism for any $i, j \in J$. This is not possible as any log flip increases some log discrepancies. \square

Theorem 2.8. *Assume (1) of Theorem 1.1 in dimension $d - 1$ and let $(X/Z, B + C)$ be a \mathbb{Q} -factorial dlt pair of dimension d where $B - A \geq 0$ for some ample/ Z \mathbb{R} -divisor $A \geq 0$, and $C \geq 0$. Assume that $(Y/Z, B_Y + C_Y)$ is a log minimal model of $(X/Z, B + C)$. Then, the special termination holds for the LMMP/ Z on $K_Y + B_Y$ with scaling of C_Y .*

Proof. Note that we are assuming that we are able to run a specific LMMP/ Z on $K_Y + B_Y$ with scaling of C_Y otherwise there is nothing to prove, i.e. here we do not prove that such an LMMP/ Z exists but assume its existence. Suppose that we get a sequence $Y_i \dashrightarrow Y_{i+1}/Z_i$ of log flips/ Z for such an LMMP/ Z . Let S be a component of $\lfloor B \rfloor$ and let S_Y and S_{Y_i} be its birational transform on Y and Y_i respectively.

First suppose that we always have $\lambda_j = 1$ in every step where λ_j is as in Definition 2.3. Since $B - A \geq 0$ and since A is ample/ Z , we can write $B + C \sim_{\mathbb{R}} A' + B' + C'/Z$ such that $A' \geq 0$ is ample/ Z , $B', C' \geq 0$, $\lfloor B' \rfloor = \lfloor A' + B' + C' \rfloor = S$, $(X/Z, A' + B' + C')$ and $(Y/Z, A'_Y + B'_Y + C'_Y)$ are plt, and the LMMP/ Z on $K_Y + B_Y$ with scaling of C_Y induces an LMMP/ Z on $K_Y + A'_Y + B'_Y$ with scaling of C'_Y . If $S_{Y_1} \neq 0$, by restricting to S_Y and using a standard argument (cf. proof of [2, Lemma 2.11]) together with Theorem 2.7, we deduce that the log flips in the sequence $Y_i \dashrightarrow Y_{i+1}/Z_i$ do not intersect S_{Y_i} for $i \gg 0$.

Now assume that we have $\lambda_j < 1$ for some j . Then, $\lfloor B + \lambda_j C \rfloor = \lfloor B \rfloor$ for any $j \gg 0$. So, we may assume that $\lfloor B + C \rfloor = \lfloor B \rfloor$. Since $B - A \geq 0$ and since A is ample/ Z , we can write $B \sim_{\mathbb{R}} A' + B'/Z$ such that $A' \geq 0$ is ample/ Z , $B' \geq 0$, $\lfloor B' \rfloor = \lfloor A' + B' + C \rfloor = S$, and $(X/Z, A' + B' + C)$ and $(Y/Z, A'_Y + B'_Y + C_Y)$ are plt. The rest goes as before by restricting to S_Y . \square

Pl flips.

We need an important result of Hacon-McKernan [12] which in turn is based on important works of Shokurov [31][29], Siu [33] and Kawamata [16].

Theorem 2.9. *Assume (1) of Theorem 1.1 in dimension $d - 1$. Then, pl flips exist in dimension d .*

Proof. By Theorem 2.7 and Corollary 1.2 in dimension $d - 1$, [12, Assumption 5.2.3] is satisfied in dimension $d - 1$. Note that Corollary 1.2 in dimension $d - 1$ easily follows from (1) of Theorem 1.1 in dimension $d - 1$ (see the proof of Corollary 1.2). Now [12, Theorem 5.4.25, proof of Lemma 5.4.26] implies the result. \square

Log minimal models.

Lemma 2.10. *Assume (1) of Theorem 1.1 in dimension $d - 1$. Let $(X/Z, B)$ be a klt pair of dimension d and let $\{D_i\}_{i \in I}$ be a finite set of exceptional/ X prime divisors (on birational models of X) such that the log discrepancy $a(D_i, X, B) \leq 1$. Then, there is a \mathbb{Q} -factorial klt pair $(Y/X, B_Y)$ such that*

- (1) $Y \rightarrow X$ is birational and $K_Y + B_Y$ is the crepant pullback of $K_X + B$,
- (2) the set of exceptional/ X prime divisors of Y is exactly $\{D_i\}_{i \in I}$.

Proof. Let $f: W \rightarrow X$ be a log resolution of $(X/Z, B)$ and let $\{E_j\}_{j \in J}$ be the set of prime exceptional divisors of f . We can assume that for some $J' \subseteq J$, $\{E_j\}_{j \in J'} = \{D_i\}_{i \in I}$. Since f is birational, there is an ample/ X \mathbb{Q} -divisor $H \geq 0$ on W whose support is irreducible smooth and distinct from the birational transform of the components of B , and an \mathbb{R} -divisor $G \geq 0$ such that $H + G \sim_{\mathbb{R}} f^*(K_X + B)/X$. Moreover, there is $\epsilon > 0$ such that $(X/Z, B + \epsilon f_* H + \epsilon f_* G)$ is klt. Now define

$$K_W + \overline{B}_W := f^*(K_X + B + \epsilon f_* H + \epsilon f_* G) + \sum_{j \notin J'} a(E_j, X, B + \epsilon f_* H + \epsilon f_* G) E_j$$

for which obviously there is an exceptional/ X \mathbb{R} -divisor $\overline{M}_W \geq 0$ such that $K_W + \overline{B}_W \sim_{\mathbb{R}} \overline{M}_W/X$ and $\theta(W/X, \overline{B}_W, \overline{M}_W) = 0$. By running the LMMP/ X on $K_W + \overline{B}_W$ with scaling of some ample/ X \mathbb{R} -divisor, and using the special termination of Theorem 2.8 we get a log minimal model of $(W/X, \overline{B}_W)$ which we may denote by $(Y/X, \overline{B}_Y)$. Note that

here we only need pl flips to run the LMMP/ X because any extremal ray in the process intersects some component of $[\overline{B}_W]$ negatively.

The exceptional divisor E_j is contracted/ Y exactly when $j \notin J'$. By taking $K_Y + B_Y$ to be the crepant pullback of $K_X + B$ we get the result. \square

Proof. (of Theorem 1.4) We closely follow the proof of [2, Theorem 1.3]. Remember that the assumptions imply that pl flips exist in dimension d by Theorem 2.9 and that the special termination holds as in Theorem 2.8.

Step 1. Since B is big/ Z , we can assume that it has a general ample/ Z component which is not a component of M (see Remark 2.4). By taking a log resolution we can further assume that the triple $(X/Z, B, M)$ is log smooth. To construct log minimal models in this situation we need to pass to a more general setting.

Let \mathfrak{W} be the set of triples $(X/Z, B, M)$ which satisfy

- (1) $(X/Z, B)$ is dlt of dimension d and $(X/Z, B, M)$ is log smooth,
- (2) $(X/Z, B)$ does not have a log minimal model,
- (3) B has a component which is ample/ Z but it is not a component of $[B]$ nor a component of M .

Obviously, it is enough to prove that \mathfrak{W} is empty. Assume otherwise and choose $(X/Z, B, M) \in \mathfrak{W}$ with minimal $\theta(X/Z, B, M)$.

If $\theta(X/Z, B, M) = 0$, then either $M = 0$ in which case we already have a log minimal model, or by running the LMMP/ Z on $K_X + B$ with scaling of a suitable ample/ Z \mathbb{R} -divisor we get a log minimal model because by the special termination of Theorem 2.8, flips and divisorial contractions will not intersect $\text{Supp } [B] \supseteq \text{Supp } M$ after finitely many steps. This is a contradiction. Note that we need only pl flips here which exist by Theorem 2.9. We may then assume that $\theta(X/Z, B, M) > 0$.

Step 2. Notation: for an \mathbb{R} -divisor $D = \sum d_i D_i$ we define $D^{\leq 1} := \sum d'_i D_i$ in which $d'_i = \min\{d_i, 1\}$. Now put

$$\alpha := \min\{t > 0 \mid [(B + tM)^{\leq 1}] \neq [B] \}$$

In particular, $(B + \alpha M)^{\leq 1} = B + C$ for some $C \geq 0$ supported in $\text{Supp } M$, and $\alpha M = C + M'$ where M' is supported in $\text{Supp } [B]$. Thus, outside $\text{Supp } [B]$ we have $C = \alpha M$. The pair $(X/Z, B + C)$ is \mathbb{Q} -factorial dlt and $(X/Z, B + C, M + C)$ is a triple which satisfies (1)

and (3) above. By construction

$$\theta(X/Z, B + C, M + C) < \theta(X/Z, B, M)$$

so $(X/Z, B + C, M + C) \notin \mathfrak{M}$. Therefore, $(X/Z, B + C)$ has a log minimal model, say $(Y/Z, B_Y + C_Y)$. By definition, $K_Y + B_Y + C_Y$ is nef/ Z .

Step 3. Now run the LMMP/ Z on $K_Y + B_Y$ with scaling of C_Y . Note that we only need pl flips here because every extremal ray contracted in the process would have negative intersection with some component of $[B]$ by the properties of C mentioned in Step 2. By the special termination of Theorem 2.8, after finitely many steps, $\text{Supp } [B]$ does not intersect the extremal rays contracted by the LMMP hence we end up with a model Y' on which $K_{Y'} + B_{Y'}$ is nef/ Z . Clearly, $(Y'/Z, B_{Y'})$ is a nef model of $(X/Z, B)$ but may not be a log minimal model because the inequality in (1) of Definition 2.1 may not be satisfied.

Step 4. Let

$$\mathcal{T} = \{t \in [0, 1] \mid (X/Z, B + tC) \text{ has a log minimal model}\}$$

Since $1 \in \mathcal{T}$, $\mathcal{T} \neq \emptyset$. Let $t \in \mathcal{T} \cap (0, 1]$ and let $(Y_t/Z, B_{Y_t} + tC_{Y_t})$ be any log minimal model of $(X/Z, B + tC)$. Running the LMMP/ Z on $K_{Y_t} + B_{Y_t}$ with scaling of tC_{Y_t} shows that there is $t' \in (0, t)$ such that $[t', t] \subset \mathcal{T}$ because the inequality required in (1) of Definition 2.1 is an open condition. The LMMP terminates for the same reasons as in Step 3 and we note again that the log flips required are all pl flips.

Step 5. Let $\tau = \inf \mathcal{T}$. If $\tau \in \mathcal{T}$, then by Step 4, $\tau = 0$ and so we are done by deriving a contradiction. Thus, we may assume that $\tau \notin \mathcal{T}$. In this case, there is a sequence $t_1 > t_2 > \dots$ in $\mathcal{T} \cap (\tau, 1]$ such that $\lim_{k \rightarrow +\infty} t_k = \tau$. For each t_k let $(Y_{t_k}/Z, B_{Y_{t_k}} + t_k C_{Y_{t_k}})$ be any log minimal model of $(X/Z, B + t_k C)$ which exists by the definition of \mathcal{T} and from which we get a nef model $(Y'_{t_k}/Z, B_{Y'_{t_k}} + \tau C_{Y'_{t_k}})$ for $(X/Z, B + \tau C)$ by running the LMMP/ Z on $K_{Y_{t_k}} + B_{Y_{t_k}}$ with scaling of $t_k C_{Y_{t_k}}$. Let $D \subset X$ be a prime divisor contracted/ Y'_{t_k} . If D is contracted/ Y_{t_k} , then

$$\begin{aligned} a(D, X, B + t_k C) &< a(D, Y_{t_k}, B_{Y_{t_k}} + t_k C_{Y_{t_k}}) \\ &\leq a(D, Y_{t_k}, B_{Y_{t_k}} + \tau C_{Y_{t_k}}) \leq a(D, Y'_{t_k}, B_{Y'_{t_k}} + \tau C_{Y'_{t_k}}) \end{aligned}$$

but if D is not contracted/ Y_{t_k} we have

$$\begin{aligned} a(D, X, B + t_k C) &= a(D, Y_{t_k}, B_{Y_{t_k}} + t_k C_{Y_{t_k}}) \\ &\leq a(D, Y_{t_k}, B_{Y_{t_k}} + \tau C_{Y_{t_k}}) < a(D, Y'_{t_k}, B_{Y'_{t_k}} + \tau C_{Y'_{t_k}}) \end{aligned}$$

because $(Y_{t_k}/Z, B_{Y_{t_k}} + t_k C_{Y_{t_k}})$ is a log minimal model of $(X/Z, B + t_k C)$ and $(Y'_{t_k}/Z, B_{Y'_{t_k}} + \tau C_{Y'_{t_k}})$ is a log minimal model of $(Y_{t_k}/Z, B_{Y_{t_k}} + \tau C_{Y_{t_k}})$. Thus, in any case we have

$$a(D, X, B + t_k C) < a(D, Y'_{t_k}, B_{Y'_{t_k}} + \tau C_{Y'_{t_k}})$$

Replacing the sequence $\{t_k\}_{k \in \mathbb{N}}$ with a subsequence, we can assume that all the induced rational maps $X \dashrightarrow Y'_{t_k}$ contract the same components of $B + \tau C$. Now an easy application of the negativity lemma implies (cf. [2, Claim 2.10]) that the log discrepancy $a(D, Y'_{t_k}, B_{Y'_{t_k}} + \tau C_{Y'_{t_k}})$ is independent of k . Therefore, each $(Y'_{t_k}, B_{Y'_{t_k}} + \tau C_{Y'_{t_k}})$ is a nef model of $(X/Z, B + \tau C)$ such that

$$a(D, X, B + \tau C) = \lim_{l \rightarrow +\infty} a(D, X, B + t_l C) \leq a(D, Y'_{t_k}, B_{Y'_{t_k}} + \tau C_{Y'_{t_k}})$$

for any prime divisor $D \subset X$ contracted/ Y'_{t_k} .

Step 6. To get a log minimal model of $(X/Z, B + \tau C)$ we just need to extract those prime divisors D on X contracted/ Y'_{t_k} for which

$$a(D, X, B + \tau C) = a(D, Y'_{t_k}, B_{Y'_{t_k}} + \tau C_{Y'_{t_k}})$$

Since B has a component which is ample/ Z , we can find Δ on X such that $\Delta \sim_{\mathbb{R}} B + \tau C/Z$ and such that $(X/Z, \Delta)$ and $(Y'_{t_k}/Z, \Delta_{Y'_{t_k}})$ are klt (see Remark 2.4). Now we can apply Lemma 2.10 to construct a crepant model of $(Y'_{t_k}/Z, \Delta_{Y'_{t_k}})$ which would be a log minimal model of $(X/Z, \Delta)$. This in turn induces a log minimal model of $(X/Z, B + \tau C)$. Thus, $\tau \in \mathcal{T}$ and this gives a contradiction. Therefore, $\mathfrak{W} = \emptyset$. \square

3. NONVANISHING

In this section we are going to prove the theorem 1.5, which is a numerical version of the corresponding result obtained in [3] (see equally [17], [8] for interesting presentations of [3]).

Preliminaries.

During the following subsections, we will give a complete proof of the next particular case of the theorem 1.5 (the *absolute case* $Z = \{z\}$).

Theorem 3.1. *Let X be a smooth projective manifold, and let B be an \mathbb{R} -divisor such that :*

- (1) *The pair (X, B) is klt, and B is big ;*
- (2) *The adjoint bundle $K_X + B$ is pseudo-effective.*

Then there exist an effective \mathbb{R} -divisor $\sum_{j=1}^N \nu^j [Y_j]$ numerically equivalent with $K_X + B$. \square

We recall that by definition a big divisor B contains in its cohomology class a current

$$(1) \quad \Theta_B := \omega_B + [E]$$

where ω_B is a Kähler metric, and $[E]$ is the current of integration associated to an effective \mathbb{R} -divisor E . This is just a reformulation of the usual Kodaira lemma, except that in algebraic geometry one usually denotes the decomposition (1) by $B = H + E$, where H is ample ; the ω_B above is a smooth, positive representative of $c_1(H)$.

Moreover, the pair (X, B) is klt and X is assumed to be non-singular, thus we have

$$(2) \quad B = \sum b^j Z_j$$

where b^j are positive reals, (Z_j) is a finite set of hypersurfaces of X such that

$$(3) \quad \prod_j |f_j|^{-2b^j} \in L^1(\Omega)$$

for each coordinate set $\Omega \subset X$, where $Z_j \cap \Omega = (f_j = 0)$.

Therefore, by considering a convex combination of the objects in (1) and (2), we can assume from the beginning that the \mathbb{R} -divisor E satisfy the integrability condition (3) : this can be seen as a metric counterpart of the hypothesis (1) in the statement 3.1.

Let L be a pseudo-effective \mathbb{R} -divisor on X ; we denote its *numerical dimension* by $\text{num}(L)$. The formal definition will not be reproduced here (the interested reader can profitably consult the references [24], [4]), however, in order to gain some intuition about it, let us mention that if L has a Zariski decomposition, then $\text{num}(L)$ is the familiar numerical dimension of the nef part.

The statements which will follow assert the existence of geometric objects in the Chern class of L and its approximations, according to the size of its numerical dimension. The first one is due to N. Nakayama in [24] (see also the transcendental generalization by S. Boucksom, [4]).

Theorem 3.2. ([4], [24]) *Let L be a pseudo-effective \mathbb{R} -divisor such that $\text{num}(L) = 0$. Then there exist an effective \mathbb{R} -divisor*

$$\Theta := \sum_{j=1}^{\rho} \nu^j [Y_j] \in \{\alpha\}.$$

For a more complete discussion about the properties of the divisor Θ above we refer to the article [24]. \square

Concerning the pseudoeffective classes in $\text{NS}_{\mathbb{R}}(X)$ whose numerical dimension is strictly greater than 0, we have the following well-known statement.

Theorem 3.3. ([19]) *Let X be a projective manifold, let L be a pseudoeffective \mathbb{R} -divisor, such that $\text{num}(L) \geq 1$. Let B be a big \mathbb{R} -divisor. Then for any $x \in X$ and $m \in \mathbb{Z}_+$ there exist an integer k_m and a representative*

$$T_{m,x} := [D_m] + \omega_m \equiv mL + B$$

where D_m is an effective \mathbb{Q} -divisor and ω_m is a Kähler metric such that $\nu(D_m, x) \geq k_m$ and $k_m \rightarrow \infty$ as $m \rightarrow \infty$. \square

Dichotomy.

We start now the actual proof of 3.1 and denote by ν the numerical dimension of the divisor $K_X + B$. We proceed as in [28], [15], [3], [35].

- If $\nu = 0$, then the theorem 3.5 is an immediate consequence of 3.6, so this first case is completely settled. \square
- The second case $\nu \geq 1$ is much more involved ; we are going to use induction on the dimension of the manifold. Up to a certain point, our arguments are very similar to the classical approach of Shokurov (see [28]) ; perhaps the main difference is the use of the *invariance of plurigenera* extension techniques as a substitute for the Kawamata-Viehweg vanishing theorem in the classical case.

Let G be an ample bundle on X , endowed with a smooth metric whose curvature form is denoted by ω_G ; by hypothesis, the \mathbb{R} -divisor $K_X + B$ is pseudo-effective, thus for each positive ε , there exists an effective \mathbb{R} -divisor

$$\Theta_{K_X+B,\varepsilon} \equiv K_X + B + \varepsilon G.$$

We denote by W_ε the support of the divisor $\Theta_{K_X+B,\varepsilon}$ and we consider a point $x_0 \in X \setminus \cup_\varepsilon W_\varepsilon$. Then the statement 3.3 provides us with a current

$$T = [D_m] + \omega_m \equiv m(K_X + B) + B$$

such that $\nu(D_m, x_0) \geq 1 + \dim(X)$. The integer m will be fixed during the rest of the proof.

The next step in the classical proof of Shokurov would be to consider the log-canonical threshold of T , in order to use an inductive argument.

However, under the assumptions of 3.1 we cannot use exactly the same approach, since unlike in the nef context, the restriction of a pseudo-effective class to an arbitrary hypersurface may not be pseudo-effective. In order to avoid such an unpleasant surprise, we introduce now our substitute for the log canonical threshold (see [27] for an interpretation of the quantity below, and also [3] for similar considerations).

Let $\mu_0 : \tilde{X} \rightarrow X$ be a common log resolution of the singular part of T and Θ_B . By this we mean that μ_0 is the composition of a sequence of blow-up maps with non-singular centers, such that we have

$$(4) \quad \mu_0^*(\Theta_B) = \sum_{j \in J} a_B^j [Y_j] + \tilde{\Lambda}_B$$

$$(5) \quad \mu_0^*(T) = \sum_{j \in J} a_T^j [Y_j] + \tilde{\Lambda}_T$$

$$(6) \quad K_{\tilde{X}/X} = \sum_{j \in J} a_{\tilde{X}/X}^j [Y_j]$$

where the divisors above are assumed to be non-singular and to have normal crossings, and $\tilde{\Lambda}_B, \tilde{\Lambda}_T$ are smooth (1,1)-forms.

Now the family of divisors $\Theta_{K_X+B,\varepsilon}$ enter into the picture. Let us consider its inverse image via the map μ_0 :

$$\mu_0^*(\Theta_{K_X+B,\varepsilon}) = \sum_{j \in J} a_{K_X+B,\varepsilon}^j [Y_j] + \tilde{\Lambda}_{K_X+B,\varepsilon}$$

where $\tilde{\Lambda}_{K_X+B,\varepsilon}$ in the relation above is an effective \mathbb{R} -divisor, whose support does not contain any of the hypersurfaces $(Y_j)_{j \in J}$.

The set J is finite and given independently of ε , so we can assume that the following limit exists

$$a_{K_X+B}^j := \lim_{\varepsilon \rightarrow 0} a_{K_X+B,\varepsilon}^j.$$

For each $j \in J$, let α_j be non-singular representative of the Chern class of the bundle associated to Y_j ; by the preceding equality we have

$$(7) \quad \mu_0^*(\Theta_{K_X+B,\varepsilon}) \equiv \sum_{j \in J} a_{K_X+B}^j [Y_j] + \tilde{\Lambda}_{K_X+B,\varepsilon} + \sum_{j \in J} \delta_\varepsilon^j \alpha_j$$

where $\delta_\varepsilon^j := a_{K_X+B,\varepsilon}^j - a_{K_X+B}^j$. We denote along the next lines by \tilde{D} the ε -free part of the current above. In conclusion, we have organized the previous terms such that μ_0 appears as a *partial log-resolution* for the family of divisors $(\Theta_{K_X+B,\varepsilon})_{\varepsilon > 0}$.

Given any real number t , consider the following quantity

$$\mu_0^*(K_X + t(T - \Theta_B) + \Theta_B) ;$$

it is numerically equivalent to the current

$$K_{\tilde{X}} + (1 + mt)\tilde{D} + (1 - t)\tilde{\Lambda}_B + t\tilde{\Lambda}_T + \sum_{j \in J} \gamma^j(t)[Y_j],$$

where we use the following notations

$$\gamma^j(t) := ta_T^j + (1 - t)a_B^j - (1 + mt)a_{K_X+B}^j - a_{\tilde{X}/X}^j.$$

We have $\mu^*(\Theta_{K_X+B}) \equiv \tilde{D} + \tilde{\Lambda}_{K_X+B,\varepsilon} + \sum_{j \in J} \delta_\varepsilon^j \alpha_j$, and on the other hand the cohomology class of the current

$$t(T - \Theta_B) + \Theta_B - (1 + mt)(\Theta_{K_X+B,\varepsilon} - \varepsilon\omega_G)$$

is equal to the first Chern class of X , so by the previous relations we infer that the currents

$$(8) \quad (1 + mt)(\tilde{\Lambda}_{K_X+B,\varepsilon} + \sum_{j \in J} \delta_\varepsilon^j \alpha_j - \varepsilon\omega_G)$$

and

$$(9) \quad \Theta_{\tilde{\omega}}(K_{\tilde{X}}) + \sum_{j \in J} \gamma^j(t)[Y_j] + (1 - t)\tilde{\Lambda}_B + t\tilde{\Lambda}_T$$

are numerically equivalent, for any $t \in \mathbb{R}$.

We use next the strict positivity of B , in order to modify slightly the inverse image of Θ_B *within the same cohomology class*, so that we have :

(i) The real numbers

$$\frac{1 + a_{K_X+B}^j + a_{\tilde{X}/X}^j - a_B^j}{a_T^j - a_B^j - ma_{K_X+B}^j}$$

are distinct ;

(ii) The klt hypothesis in 3.1 is preserved, i.e. $a_B^j - a_{\tilde{X}/X}^j < 1$;

(iii) The $(1,1)$ -class $\{\Lambda_B\}$ contains a Kähler metric.

The arguments we use in order to obtain the above properties are quite standard : it is the so-called *tie-break method*, therefore we will skip the details.

Granted this, there exist a unique index say $0 \in J$ and a positive real τ such that $\gamma^0(\tau) = 1$ and $\gamma^j(\tau) < 1$ for $j \in J \setminus \{0\}$. Moreover,

we have $0 < \tau < 1$, by the klt hypothesis and the concentration of the singularity of T at the point x_0 .

We equally have the next numerical identity

$$(10) \quad (1 + m\tau)(\tilde{\Lambda}_{K_X+B,\varepsilon} + \sum_{j \in J} \delta_\varepsilon^j \alpha_j - \varepsilon \omega_G) + \tilde{H} \equiv K_{\tilde{X}} + \tilde{S} + \tilde{B}$$

where \tilde{B} is the \mathbb{R} -divisor

$$(11) \quad \tilde{B} := \sum_{j \in J_p} \gamma^j(\tau)[Y_j] + (1 - \tau)\tilde{\Lambda}_B + \tau\tilde{\Lambda}_T$$

and we also denote by

$$(12) \quad \tilde{H} := - \sum_{j \in J_n} \gamma^j(\tau)[Y_j].$$

The choice of the partition of $J = J_p \cup J_n \cup \{0\}$ is such that the coefficients of the divisor part in (11) are in $[0, 1[$, the \mathbb{R} divisor \tilde{H} is effective, and of course the coefficient $\gamma^0(\tau) = 1$ corresponds to \tilde{S} .

In order to apply induction, we collect here the main features of the objects constructed above.

- In the first place, the \mathbb{R} -divisor $\sum_{j \in J_p} \gamma^j(\tau)[Y_j]$ is klt, and the smooth $(1, 1)$ -form $(1 - \tau)\tilde{\Lambda}_B + \tau\tilde{\Lambda}_T$ is positive definite ; thus the \mathbb{R} -divisor in (?) is big and klt. Moreover, its restriction to \tilde{S} has the same properties.
- There exist an effective \mathbb{R} -divisor Δ such that

$$\tilde{B} + \tilde{S} + \Delta \equiv \mu_0^*(B + \tau m(K_X + B))$$

Indeed, the expression of Δ is easily obtained as follows

$$\Delta := \sum_{j \in J_p \cup \{0\}} ((1 + m\tau)a_{K_X+B}^j + a_{\tilde{X}/X}^j)[Y_j] + \sum_{j \in J_n} (\tau a_T^j + (1 - \tau)a_B^j)[Y_j].$$

Therefore, it is enough to produce an effective \mathbb{R} -divisor numerically equivalent to $K_{\tilde{X}} + \tilde{S} + \tilde{B}$ in order to complete the proof of the theorem 3.1.

- The adjoint bundle $K_{\tilde{X}} + \tilde{S} + \tilde{B}$ and its restriction to \tilde{S} are pseudo-effective by the relation (10).
- By using a sequence of blow-up maps, we can even assume that the components $(Y_j)_{j \in J_p}$ in the decomposition (11) have empty mutual intersections. Indeed, this is a simple –but nevertheless crucial!– classical result, which we recall next.

We denote by Ξ an effective \mathbb{R} -divisor, whose support do not contain \tilde{S} , such that $\text{Supp } \Xi \cup \tilde{S}$ has normal crossings and such that its coefficients are strictly smaller than 1.

Lemma 3.4. (see e.g. [12]) *There exist a birational map $\mu_1 : \widehat{X} \rightarrow \widetilde{X}$ such that*

$$\mu_1^*(K_{\widehat{X}} + \widetilde{S} + \Xi) + E_{\widehat{X}} = K_{\widehat{X}} + S + \Gamma$$

where $E_{\widehat{X}}$ and Γ are effective with no common components, $E_{\widehat{X}}$ is exceptional and S is the proper transform of \widetilde{S} ; moreover, the support of the divisor Γ has normal crossings, its coefficients are strictly smaller than 1 and the intersection of any two components is empty. \square

We apply this result in our setting with $\Xi := \sum_{j \in J_p} \gamma^j(\tau)[Y_j]$, and we summarize the discussion in this paragraph in the next statement (in which we equally adjust the notations).

Proposition 3.5. *There exist a birational map $\mu : \widehat{X} \rightarrow X$ and an \mathbb{R} -divisor*

$$\widehat{B} \equiv \sum_{j \in J} \nu^j Y_j + \widehat{\Lambda}_B$$

on \widehat{X} , where $0 < \nu^j < 1$, the hypersurfaces Y_j above are smooth, they have empty mutual intersection and moreover the following hold :

- (1) *There exist a family of closed $(1, 1)$ -currents*

$$\Theta_\varepsilon := \Delta_{K_X + B, \varepsilon} + \alpha_\varepsilon$$

numerically equivalent with $K_{\widehat{X}} + S + \widehat{B}$ where $S \subset \widehat{X}$ is a non-singular hypersurface which has transversal intersections with (Y_j) , and where $\Delta_{K_X + B, \varepsilon}$ is an effective \mathbb{R} -divisor whose support is disjoint from the set (S, Y_j) , and finally α_ε is a non-singular $(1, 1)$ -form, greater than $-\varepsilon\omega$;

- (2) *There exist a map $\mu_1 : \widehat{X} \rightarrow \widetilde{X}$ such that S is not μ_1 -exceptional, and such that $\widehat{\Lambda}_B$ is greater than the inverse image of a Kähler metric on \widetilde{X} via μ_1 . Therefore, the form $\widehat{\Lambda}_B$ is positive defined at the generic point of \widehat{X} , and so is its restriction to the generic point of S ;*
- (3) *There exist an effective \mathbb{R} -divisor Δ on \widehat{X} such that*

$$\widehat{B} + S + \Delta = \mu^*(B + \tau m(K_X + B)) + E$$

where E is μ -exceptional. \square

Restriction and induction.

We consider next the restriction to S of the currents Θ_ε above

$$(13) \quad \Theta_{\varepsilon|S} \equiv K_S + \widehat{B}|_S;$$

we have the following decomposition

$$(14) \quad \Theta_{\varepsilon|S} = \sum_{j \in J} \rho^{\varepsilon,j} Y_{j|S} + R_{\varepsilon}$$

where the coefficients $(\rho^{\varepsilon,j})$ are positive real numbers, and R_{ε} above is the closed current given by the restriction to S of the differential form

$$(15) \quad \alpha_{\varepsilon}$$

plus the part of the restriction to S of the \mathbb{R} -divisor

$$(16) \quad \Delta_{K_X+B,\varepsilon}$$

which is disjoint from the family $(Y_{j|S})$. Even if the differential form in (15) may not be positive, nevertheless we can assume that we have

$$R_{\varepsilon} \geq -\varepsilon \omega|_S$$

for any $\varepsilon > 0$. We remark that the coefficients $\rho^{\varepsilon,j}$ in (14) may be *positives*, despite of the fact the Y_j *does not belongs* to the support of Θ_{ε} , for any $j \in J$.

For each index $j \in J$ we will assume that the next limit

$$\rho^{\infty,j} := \lim_{\varepsilon \rightarrow 0} \rho^{\varepsilon,j}$$

exist, and we introduce the following notation

$$(17) \quad I := \{j \in J : \rho^{\infty,j} \geq \nu^j\}.$$

The numerical identity in 3.5, (1) restricted to S coupled with (14) show that we have

$$(18) \quad \sum_{j \in I} (\rho^{\infty,j} - \nu^j) [Y_{j|S}] + R_{\varepsilon} + \sum_{j \in J} (\rho^{\varepsilon,j} - \rho^{\infty,j}) [Y_{j|S}] \equiv K_S + B_S$$

where B_S is the current

$$\sum_{j \in J \setminus I} (\nu^j - \rho^{\infty,j}) [Y_{j|S}] + \widehat{\Lambda}_{B|S}.$$

We are now in good position to apply induction :

- The \mathbb{R} -divisor B_S is big and klt on S . Indeed, this follows by (2) and the properties of \widehat{B} in 3.5, and the definition of the set I , see (17).
- The adjoint divisor $K_S + B_S$ is pseudoeffective, by the relation (18).

Therefore, we can apply the induction hypothesis : there exist a non-zero, effective \mathbb{R} -divisor, which can be written as

$$T_S := \sum_{i \in K} \lambda^i [W_i]$$

(where $W_i \subset S$ are hypersurfaces) which is numerically equivalent to $K_S + B_S$. We consider now the current

$$(19) \quad \widehat{T}_S := \sum_{i \in K} \lambda^i [W_i] + \sum_{j \in J \setminus I} \rho^{\infty, j} [Y_{j|S}] + \sum_{j \in I} \nu^j [Y_{j|S}] ;$$

from the relation (18) we get

$$(20) \quad \widehat{T}_S \equiv K_{\widehat{X}} + S + \widehat{B}_{|S}.$$

It is precisely the \mathbb{R} -divisor \widehat{T}_S above who will “generate” the section we seek, in the following manner. We first use a diophantine argument, in order to obtain a *simultaneous approximation* of T_S and \widehat{B} with a \mathbb{Q} -divisor, respectively \mathbb{Q} -line bundle, such that the relation (20) above still holds. The next step is to use a trick by Shokurov (adapted to our setting) and finally the main ingredient is an extension result for pluricanonical forms. All this will be presented in full details in the next three subsections. \square

Approximation.

In this paragraph we recall the following diophantine approximation lemma (we refer to [27] for a complete proof).

Lemma 3.6. *For each $\eta > 0$, there exist a positive integer q_η , a \mathbb{Q} -line bundle \widehat{B}_η on \widehat{X} and a \mathbb{Q} -divisor*

$$(21) \quad \widehat{T}_{S, \eta} := \sum_{i \in K} \lambda_\eta^i [W_i] + \sum_{j \in J \setminus I} \rho_\eta^{\infty, j} [Y_{j|S}] + \sum_{j \in I} \nu_\eta^j [Y_{j|S}]$$

on S such that :

A.1 *The multiple $q_\eta \widehat{B}_\eta$ is a genuine line bundle, and the numbers*

$$(q_\eta \lambda_\eta^i)_{i \in K}, (q_\eta \nu_\eta^j)_{j \in J}, (q_\eta \rho_\eta^{\infty, j})_{j \in J}$$

are integers ;

A.2 *We have $\widehat{T}_{S, \eta} \equiv K_{\widehat{X}} + S + \widehat{B}_{\eta|S}$;*

A.3 *We have $\|q_\eta(\widehat{B} - \widehat{B}_\eta)\| < \eta$, $|q_\eta(\lambda_\eta^i - \lambda^i)| < \eta$ and the analog relation for the $(\rho^{\infty, j}, \nu^j)_{j \in J}$ (here $\|\cdot\|$ denotes any norm on the real Neron-Severi space of \widehat{X}) ;*

A.4 *For each $\eta_0 > 0$, there exist a finite family (η_j) such that the class $\{K_{\widehat{X}} + S + \widehat{B}\}$ belong to the convex hull of $\{K_{\widehat{X}} + S + \widehat{B}_{\eta_j}\}$ where $0 < \eta_j < \eta_0$. \square*

Remark. Even if we do not reproduce here the arguments of the proof (again, see [27]), we present an interpretation of it, due to S. Boucksom. Let $N := |J| + |K|$; we consider the map

$$l_1 : \mathbb{R}^N \rightarrow \mathrm{NS}_{\mathbb{R}}(S)$$

defined as follows. To each vector (x^1, \dots, x^N) , it corresponds the class of the \mathbb{R} -divisor $\sum_{i=1}^{|K|} x^i W_i + \sum_{j=1+|K|}^N x^j Y_{j|S}$. We define another linear map

$$l_2 : \mathrm{NS}_{\mathbb{R}}(X) \rightarrow \mathrm{NS}_{\mathbb{R}}(S)$$

which is given by the restriction to S . We are interested in the set

$$\mathcal{I} := (x^1, \dots, x^N; \tau) \in \mathbb{R}^N \times \mathrm{NS}_{\mathbb{R}}(X)$$

such that $l_1(x) = l_2(\tau)$; it is a vector space, which is moreover defined over \mathbb{Q} (since this is the case for both maps l_1 and l_2). Now our initial data $(T_S, \{K_X + S\} + \theta_{\widehat{L}})$ corresponds to a point of the above fibered product, and the claim of the lemma is that given a point in a vector subspace defined over \mathbb{Q} , we can approximate it with rational points satisfying the Dirichlet condition.

A trick by V. Shokurov.

Our concern in this paragraph will be to “convert” the effective \mathbb{Q} -divisor $\widehat{T}_{S,\eta}$ into a genuine section s_η of the bundle $q_\eta(K_{\widehat{X}} + S + \widehat{B}_\eta)$. To this end, we will apply a classical argument of Shokurov, in the version revisited by Siu in his recent work [35]. A crucial point is that by a careful choice of the metrics we use, the L^2 estimates will allow us to have a very precise information concerning the vanishing of s_η .

Proposition 3.7. *There exist a section*

$$s_\eta \in H^0\left(S, q_\eta(K_S + \widehat{B}_{\eta|S})\right)$$

whose zero set contains the divisor

$$q_\eta\left(\sum_{j \in J \setminus I} \rho_\eta^{\infty, j} [Y_{j|S}] + \sum_{j \in I} \nu_\eta^j [Y_{j|S}]\right)$$

for all $0 < \eta \ll 1$.

Proof of [3.8]. We first remark that we have

$$q_\eta(K_S + \widehat{B}_{\eta|S}) = K_S + (q_\eta - 1)(K_S + \widehat{B}_{\eta|S}) + \widehat{B}_{\eta|S};$$

in order to use the classical vanishing theorems, we have to endow the bundle

$$(q_\eta - 1)(K_S + \widehat{B}_{\eta|S}) + \widehat{B}_{\eta|S}$$

with an appropriate metric.

We first consider the \mathbb{Q} -bundle \widehat{B}_η ; we will construct a metric on it induced by the decomposition

$$\widehat{B}_\eta = \widehat{B} + (\widehat{B}_\eta - \widehat{B}).$$

The second term above admits a smooth representative whose local weights are bounded by $\frac{\eta}{q_\eta}$ in \mathcal{C}^∞ norm, by the approximation relation

A.3. As for the first one, we recall that we have

$$(26) \quad \widehat{B} = \sum_{j \in J} \nu^j Y_j + \widehat{\Lambda}_B;$$

where the (1,1)-form $\widehat{\Lambda}_B$ has the positivity properties in 3.5, 2.

Now, the first metric we consider on $\widehat{B}_{\eta|S}$ is defined such that its curvature current is equal to

$$(27) \quad \sum_{j \in I} \max(\nu^j, \nu_\eta^j) Y_{j|S} + \sum_{j \in J \setminus I} \nu^j Y_{j|S} + \widehat{\Lambda}_{B|S} + \Xi(\eta)|_S$$

where $\Xi(\eta)$ is a non-singular (1, 1)-form on \widehat{X} in the class of the current

$$\sum_{j \in I} \left(\nu^{(j)} - \max(\nu^{(j)}, \nu_\eta^{(j)}) \right) [Y_j]$$

plus $\widehat{B}_\eta - \widehat{B}$; we can assume that it is greater than $-C \frac{\eta}{q_\eta}$, where the constant C above is independent of η .

The smooth term $\widehat{\Lambda}_B$ is semi-positive on \widehat{X} and strictly positive at the generic point of S : thanks to this positivity properties we can find a representative of the class $\{\widehat{\Lambda}_B\}$ which dominates a Kähler metric. In general we cannot avoid that this representative acquire some singularities. However, in the present context we will show that there exist current in the above class which is “restrictable” to S .

Indeed, we consider the exceptional divisors (E_j) of the map μ_1 (see the proposition 3.5); the hypersurface S do not belong to this set, and then the class

$$\widehat{\Lambda}_B - \sum_j \varepsilon^j E_j$$

is ample on \widehat{X} , for some positive reals ε^j . Once a set of such parameters is chosen, we fix a Kähler form

$$\Omega \in \{\widehat{\Lambda}_B - \sum_j \varepsilon^j E_j\}$$

and for each $\delta \in [0, 1]$ we define

$$(28) \quad \widehat{\Lambda}_{B,\delta} := (1 - \delta)\widehat{\Lambda}_B + \delta(\Omega + \sum_j \varepsilon^j E_j).$$

For each $\eta > 0$, there exist $\delta > 0$ such that the differential form $\delta\Omega + \Xi(\eta)$ is positive defined. For example, we can take

$$(29) \quad \delta := C \frac{\eta}{q_\eta}$$

where the constant $C > 0$ does not depends on η .

With the choice of several parameters as indicated above, the current

$$\widehat{\Lambda}_{B,\delta} + \Xi(\eta)$$

dominates a Kähler metric, and since $\widehat{\Lambda}_{B,\delta}$ is in the same cohomology class as $\widehat{\Lambda}_B$, we have

$$(30) \quad \widehat{B} \equiv \sum_{j \in I} \max(\nu^j, \nu_\eta^j) Y_j + \sum_{j \in J \setminus I} \nu^j Y_j + \widehat{\Lambda}_{B,\delta} + \Xi(\eta).$$

We remark that the current in the expression above admits a well-defined restriction to S ; moreover, the additional singularities of the restriction (induced by $\widehat{\Lambda}_{B,\delta}$) are of order $C \frac{\eta}{q_\eta}$, thus it will clearly be klt as soon as $\eta \ll 1$. The current in the expression (30) induce a metric on $\widehat{B}_{\eta|S}$.

Next, we define a singular metric on the bundle $(q_\eta - 1)(K_S + \widehat{B}_{\eta|S})$ whose curvature form is equal to $(q_\eta - 1)\widehat{T}_{S,\eta}$ and we denote by h_η the resulting metric on the bundle

$$(q_\eta - 1)(K_S + \widehat{B}_{\eta|S}) + B_{\eta|S}.$$

The divisor $q_\eta \widehat{T}_{S,\eta}$ corresponds to the current of integration along the zero set of the section u_η of the bundle

$$q_\eta(K_S + \widehat{B}_{\eta|S}) + \rho$$

where ρ is a topologically trivial line bundle on S .

By the Kawamata-Viehweg-Nadel vanishing theorem (cf. [14], [40], [23]) we have

$$H^j(S, q_\eta(K_S + \widehat{B}_\eta) \otimes \mathcal{I}(h_\eta)) = 0$$

for all $j \geq 1$, and the same is true for the bundle $q_\eta(K_S + \widehat{B}_\eta) + \rho$, since ρ carries a metric with zero curvature. Moreover, the section u_η belong to the multiplier ideal of the metric h_η above, as soon as η is small enough, because the multiplier ideal of the metric on the bundle $\widehat{B}_\eta|_S$ will be trivial. Since the Euler characteristic of the two bundles is the same, we infer that

$$H^0\left(S, q_\eta(K_S + \widehat{B}_\eta) \otimes \mathcal{I}(h_\eta)\right) \neq 0$$

We denote by s_η any non-zero element in the group above ; we show now that its zero set satisfy the requirements in the lemma. Indeed, locally at any point of $x \in S$ we have

$$\int_{(S,x)} \frac{|f_s|^2}{\prod_{j \in J \setminus I} |f_j|^{2\rho_\eta^{\infty,j}(q_\eta-1)+2\tilde{\nu}_\eta^j} \prod_{j \in I} |f_j|^{2\nu_\eta^j(q_\eta-1)+2\tilde{\nu}_\eta^j}} d\lambda < \infty$$

where $\tilde{\nu}_\eta^j := \nu^j$ if $j \in J \setminus I$ and $\tilde{\nu}_\eta^j := \max\{\nu_\eta^j, \nu^j\}$ if $j \in I$; we denote by f_s the local expression of the section s_η , and we denote by f_j the local equation of $Y_j \cap S$.

But the we have

$$\int_{(S,x)} \frac{|f_s|^2}{\prod_{j \in J \setminus I} |f_j|^{2\rho_\eta^{\infty,j}q_\eta} \prod_{j \in I} |f_j|^{2\nu_\eta^j q_\eta}} d\lambda < \infty$$

for all $\eta \ll 1$ (by the definition of the set I and the construction of the metric on $\widehat{B}_\eta|_S$). Therefore, the lemma is proved. \square

Remark 3.8. Concerning the construction and the properties of $\widehat{\Lambda}_{B,\delta}$, we recall the very nice result in [9], stating that *if D is an \mathbb{R} -divisor which is nef and big, then its associated augmented base locus can be determined numerically.*

Remark 3.9. As one can easily see, the divisor we are interested in the previous proposition 3.7 is given by

$$E_\eta := \sum_{j \in J \setminus I} \rho_\eta^{\infty,j} [Y_j|_S] + \sum_{j \in I} \nu_\eta^j [Y_j|_S].$$

The crucial fact about it is that it is *smaller* than the singularities of the metric we construct for \widehat{B}_η ; this is the reason why we can infer

that the section s_η above vanishes on $q_\eta E_\eta$ –and not just on the round down of the divisor $(q_\eta - 1)E_\eta^-$, see [27], page 42 for some comments about this issue.

The method of Siu.

We have arrived at the last step in our proof : for all $0 < \eta \ll 1$, the section s_η admit an extension on \widehat{X} . Once this is done, we just use the point A.4 of the approximation lemma 3.8, in order to infer the existence of a \mathbb{R} -section of the bundle $K_{\widehat{X}} + S + \widehat{B}$, and then the relation (3) of 3.5 to conclude.

The extension of the section s_η will be obtained by using the *invariance of plurigenera* techniques, thus in the first paragraph of the current subsection, we will highlight some of the properties of the \mathbb{Q} -divisors \widehat{B}_η constructed above.

Uniformity properties of $(K_{\widehat{X}} + S + \widehat{B}_\eta)_{\eta>0}$.

We list below the pertinent facts which will ultimately enable us to perform the extension of (s_η) ; the constant C which appear in the next statement is independent of η .

(U₁) The section $s_\eta \in H^0(S, q_\eta(K_S + \widehat{B}_\eta))$ vanishes along the divisor

$$q_\eta \left(\sum_{j \in J \setminus I} \rho_\eta^{\infty, j} [Y_{j|S}] + \sum_{j \in I} \nu_\eta^j [Y_{j|S}] \right)$$

for all $0 < \eta \ll 1$ □

(U₂) There exist a closed (1,1)-current $\Theta_\eta \in \{K_{\widehat{X}} + S + \widehat{B}_\eta\}$ such that :

(2.1) It is greater than $-C \frac{\eta}{q_\eta} \omega$;

(2.2) Its restriction to S is well defined, and we have

$$\Theta_{\eta|S} = \sum_{j \in J} \theta_\eta^j [Y_{j|S}] + R_{\eta, S}.$$

Moreover, the support of the divisor part of $R_{\eta, S}$ is disjoint from the set $(Y_{j|S})$ and $\theta_\eta^j \leq \rho_\eta^{\infty, j} + C \frac{\eta}{q_\eta}$.

(**U₃**) The bundle \widehat{B}_η can be endowed with a metric whose curvature current is given by

$$\sum_{j \in J} \nu_\eta^j [Y_j] + \widehat{\Lambda}_{B,\eta} + \Xi(\eta)$$

where the hypersurfaces Y_j above verify $Y_j \cap Y_i = \emptyset$, if $i \neq j$ and moreover we have :

- (**3.1**) The current $\widehat{\Lambda}_{B,\eta} + \Xi(\eta)$ dominates a Kähler metric ;
- (**3.2**) The restriction $\widehat{\Lambda}_{B,\eta} + \Xi(\eta)|_S$ is well defined, and if we denote by ν_η the maximal multiplicity of the above restriction then we have

$$q_\eta \nu_\eta \leq C\eta.$$

The property (**U₁**) is a simple recapitulation of facts which were completely proved during the previous paragraphs.

The family of currents in (**U₂**) can be easily obtained thanks to the proposition 3.5, by the definition of the quantities $\rho^{\infty,j}$ and their approximations.

Finally, the construction of the metric on \widehat{B}_η as above is done precisely as in the previous paragraph, except that instead of taking the coefficients $\max(\nu^j, \nu_\eta^j)$, we simply consider ν_η^j . The negativity of the error term is the same (i.e. $C\eta/q_\eta$). \square

Let us introduce the next notations :

- $\Delta_1 := \sum_{j \in J \setminus I} \nu_\eta^j [Y_j]$. It is an effective and klt \mathbb{Q} -bundle ; notice that the multiple $q_\eta \nu_\eta^j$ is a positive integer strictly smaller than q_η , for each $j \in J \setminus I$;
- $\Delta_2 := \sum_{j \in I} \nu_\eta^j [Y_j] + \widehat{\Lambda}_{B,\eta} + \Xi(\eta)$. It is equally a effective and klt \mathbb{Q} -bundle such that $q_\eta \Delta_2$ is integral.

Precisely as in [7], [10], [1], there exist a decomposition

$$q_\eta \Delta_1 = L_1 + \dots + L_{q_\eta - 1}$$

such that for each $m = 1, \dots, q_\eta - 1$, we have

$$L_m := \sum_{j \in I_m \subset J \setminus I} Y_j.$$

We denote by $L_{q_\eta} := q_\eta \Delta_2$ and

$$(31) \quad L^{(p)} := p(K_X + S) + L_1 + \dots + L_p$$

where $p = 1, \dots, q_\eta$. By convention, $L^{(0)}$ is the trivial bundle.

We remark that it is possible to find an ample bundle (A, h_A) independent of η whose curvature form is positive enough such that the next relations hold.

- (†) For each $0 \leq p \leq q_\eta - 1$, the bundle $L^{(p)} + q_\eta A$ is generated by its global sections, which we denote by $(s_j^{(p)})$.
- (†²) Any section of the bundle $L^{(q_\eta)} + q_\eta A|_S$ admits an extension to \widehat{X} .
- (†³) We endow the bundle corresponding to $(Y_j)_{j \in J}$ with a non-singular metric, and we denote by $\tilde{\varphi}_m$ the induced metric on L_m . Then for each $m = 1, \dots, q_\eta$, the functions

$$\tilde{\varphi}_{L_m} + 1/3\varphi_A$$

are strictly psh.

- (†⁴) For any $\eta > 0$ we have

$$(32) \quad \Theta_\eta \geq -\frac{\eta}{q_\eta} \Theta_A.$$

□

Under the numerous assumptions/normalizations above, we formulate the next statement.

Claim 3.10. *There exist a constant $C > 0$ independent of η such that the section*

$$s_\eta^{\otimes k} \otimes s_j^{(p)} \in H^0(S, L^{(p)} + kL^{(q_\eta)} + q_\eta A|_S)$$

extend to \widehat{X} , for each $p = 0, \dots, q_\eta - 1$, $j = 1, \dots, N_p$ and $k \in \mathbb{Z}_+$ such that

$$k \frac{\eta}{q_\eta} \leq C.$$

□

The statement above can be seen as a natural generalization of the usual invariance of plurigenera setting (see [5], [7], [10], [16], [18], [26], [34], [37], [39]) ; in substance, we are about to say that the more general hypothesis we are forced to consider induce an *effective limitation* of the number of iterations we are allowed to perform.

Proof of the claim 3.10.

To start with, we recall the following very useful integrability criteria (see e.g. [12]).

Lemma 3.11. *Let D be an effective \mathbb{R} -divisor on a manifold S . We consider the non-singular hypersurfaces $Y_j \subset S$ for $j = 1, \dots, N$ such that $Y_j \cap Y_i = \emptyset$ if $i \neq j$, and such that the support of D is disjoint from the set (Y_j) . Then there exist a constant $\varepsilon_0 := \varepsilon_0(\{D\}, C)$ depending only on the cohomology class of the divisor D such that for all positive real numbers $\delta \in]0, 1]$ and $\varepsilon \leq \varepsilon_0$ we have*

$$\int_{(S,s)} \frac{d\lambda}{|f_D|^{2\varepsilon} \prod_j |f_j|^{2(1-\delta)}} < \infty$$

for all $s \in S$. □

In the statement above, we denote by f_j, f_D the local equations of Y_j , respectively D near $s \in S$ (with the usual abuse of notation).

We will equally need the following version of the Ohsawa-Takegoshi theorem (see [6], [22], [25], [34]) ; it will be our main technical tool in the proof of the claim.

Theorem 3.12 (22). *Let \widehat{X} be a projective n -dimensional manifold, and let $S \subset \widehat{X}$ be a non-singular hypersurface. Let F be a line bundle, equipped with a metric h_F . We assume that :*

- (a) *The curvature current $\frac{\sqrt{-1}}{2\pi} \Theta_F$ is greater than a Kähler metric on \widehat{X} ;*
- (b) *The restriction of the metric h_F on S is well defined.*

Then every section $u \in H^0(S, (K_{\widehat{X}} + S + F|_S) \otimes \mathcal{I}(h_{F|_S}))$ admits an extension U to \widehat{X} . □

We will use inductively the extension theorem 3.12, in order to derive a lower bound for the power k we can afford in the invariance of pluri-genera algorithm, under the conditions $(\mathbf{U}_j)_{1 \leq j \leq 3}$; the first steps are as follows.

Step 1. For each $j = 1, \dots, N_0$, the section $s_\eta \otimes s_j^{(0)} \in H^0(S, L^{(q_\eta)} + q_\eta A|_S)$ admits an extension $U_j^{(q_\eta)} \in H^0(X, L^{(q_\eta)} + q_\eta A)$, by the property ††.

Step 2. We use the sections $(U_j^{(q_\eta)})$ to construct a metric $\varphi^{(q_\eta)}$ on the bundle $L^{(q_\eta)} + q_\eta A$.

Step 3. Let us consider the section $s_\eta \otimes s_j^{(1)} \in H^0(S, L^{(1)} + L^{(q_\eta)} + q_\eta A|_S)$. We remark that the bundle

$$L^{(1)} + L^{(q_\eta)} + q_\eta A = K_{\widehat{X}} + S + L_1 + L^{(q_\eta)} + q_\eta A$$

can be written as $K_{\widehat{X}} + S + F$ where

$$F := L_1 + L^{(q_\eta)} + q_\eta A$$

thus we have to construct a metric on F which satisfy the curvature and integrability assumptions in the Ohsawa-Takegoshi-type theorem above.

Let δ, ε be positive real numbers ; we endow the bundle F with the metric given by

$$(33) \quad \varphi_{\delta, \varepsilon}^{(q_\eta)} := (1 - \delta)\varphi_{L_1} + \delta\widetilde{\varphi}_{L_1} + (1 - \varepsilon)\varphi^{(q_\eta)} + \varepsilon q_\eta(\varphi_A + \varphi_{\Theta_\eta})$$

where the metric $\widetilde{\varphi}_{L_1}$ is smooth (no curvature requirements) and φ_{L_1} is the weight of the singular metric induced by the divisors $(Y_j)_{j \in I_1}$. We denote by φ_{Θ_η} the local weight of the current Θ_η ; it induces a metric on the corresponding \mathbb{Q} -bundle $K_{\widehat{X}} + S + \widehat{B}_\eta$, which is used above.

We remark that the curvature conditions in the extension theorem will be fulfilled if

$$\delta < \varepsilon q_\eta$$

provided that $\eta \ll 1$: by the relations (\dagger^3) and (\dagger^4) the negativity of the curvature induced by the term $\delta\widetilde{\varphi}_{L_1}$ will be absorbed by A .

Next we claim that the sections $s_\eta \otimes s_j^{(1)}$ are integrable with respect to the metric defined in (33), provided that the parameters ε, δ are chosen in an appropriate manner. Indeed, we have to prove that

$$\int_S \frac{|s_\eta \otimes s_j^{(1)}|^2}{(\sum_r |s_\eta \otimes s_r^{(0)}|^2)^{1-\varepsilon}} \exp(- (1 - \delta)\varphi_{L_1} - \varepsilon q_\eta \varphi_{\Theta_\eta}) dV < \infty ;$$

since the sections $(s_r^{(0)})$ have no common zeroes, it is enough to show that

$$\int_S |s_\eta|^{2\varepsilon} \exp(- (1 - \delta)\varphi_{L_1} - \varepsilon q_\eta \varphi_{\Theta_\eta}) dV < \infty$$

(we have abusively removed the smooth weights in the above expressions, to simplify the writing).

Now the property (\mathbf{U}_1) concerning the zero set of s_η is used : the above integral is convergent, provided that we have

$$\int_S \exp(- (1 - \delta)\varphi_{L_1} - \varepsilon q_\eta(\varphi_{\Theta_\eta} - \sum_{j \in J \setminus I} \rho_\eta^{\infty, j} \varphi_{Y_j} - \sum_{j \in I} \nu_\eta^j \varphi_{Y_j})) dV < \infty.$$

In order to conclude the convergence of the above integral, we would like to apply the integrability lemma 3.11 ; therefore, we have to estimate the coefficients of the common part of the support of $L_{1|S}$ and

$$(34) \quad \Theta_\eta - \sum_{j \in J \setminus I} \rho_\eta^{\infty, j} [Y_j] - \sum_{j \in I} \nu_\eta^j [Y_j]$$

restricted to S . For any $j \in J \setminus I$, the coefficient associated to the divisor $Y_{j|S}$ in the expression above is equal to

$$(35) \quad \theta_\eta^j - \rho_\eta^{\infty, j}$$

and by the property \mathbf{U}_2 , the difference above is smaller than $C \frac{\eta}{q_\eta}$. The singular part corresponding to $j \in J \setminus I$ in the expression (34) will be incorporated into the $(1 - \delta)\varphi_{L_1}$, thus we have to impose the relation

$$1 - \delta + q_\eta \varepsilon C \frac{\eta}{q_\eta} < 1.$$

In conclusion, the positivity and integrability conditions will be satisfied provided that

$$(36) \quad C\eta\varepsilon < \delta < \varepsilon q_\eta \leq \varepsilon_0$$

We can clearly choose the parameters δ, ε such that (36) is verified.

Step 4. We apply the extension theorem and we get $U_j^{(q_\eta+1)}$, whose restriction on S is precisely $s_\eta \otimes s_j^{(1)}$. \square

The claim will be obtained by iterating the procedure (1)-(4) several times, and estimating carefully the influence of the negativity of Θ_η on this process. Indeed, assume that we already have the set of global sections

$$U_j^{(kq_\eta+p)} \in H^0(\widehat{X}, L^{(p)} + kL^{(q_\eta)} + q_\eta A)$$

which extend $s_\eta^{\otimes k} \otimes s_j^{(p)}$. They induce a metric on the above bundle, denoted by $\varphi^{(kq_\eta+p)}$.

If $p < q_\eta - 1$, then we define the family of sections

$$s_\eta^{\otimes k} \otimes s_j^{(p+1)} \in H^0(S, L^{(p+1)} + kL^{(q_\eta)} + q_\eta A|_S)$$

on S . As in the step (3) above we remark that we have

$$L^{(p+1)} = K_{\widehat{X}} + S + L_{p+1} + L^{(p)}$$

thus according to the extension result 3.12, we have to exhibit a metric on the bundle

$$F := L_{p+1} + L^{(p)} + kL^{(q_\eta)} + q_\eta A$$

for which the curvature conditions are satisfied, and such that the family of sections above are L^2 with respect to it. We define

$$(37) \quad \varphi_{\delta,\varepsilon}^{(kq_\eta+p+1)} := (1-\delta)\varphi_{L_{p+1}} + \delta\tilde{\varphi}_{L_{p+1}} + (1-\varepsilon)\varphi^{(kq_\eta+p)} + \varepsilon q_\eta \left(k\varphi_{\Theta_\eta} + \varphi_A + \frac{1}{q_\eta}\tilde{\varphi}_{L^{(p)}} \right)$$

and we check now the conditions that the parameters ε, δ have to satisfy.

We have to absorb the negativity in the smooth curvature terms in (37), and the one from Θ_η . The Hessian of the term

$$1/3\varphi_A + \frac{1}{q_\eta}\tilde{\varphi}_{L^{(p)}}$$

is assumed to be positive by \dagger^3 , but we also have a *huge* negative contribution

$$-Ck\frac{\eta}{q_\eta}\Theta_A$$

induced by the current Θ_η . However, we remark that we can assume that we have

$$(38) \quad Ck\frac{\eta}{q_\eta} < 1/3$$

since this is *precisely* the range of k for which we want to establish the claim. Then the curvature of the metric defined in (37) will be positive, provided that

$$\delta < \varepsilon q_\eta$$

again by (\dagger^3) .

Let us check next the L^2 condition ; we have to show that the integral below is convergent

$$\int_S \frac{|s^{\otimes k} \otimes s_j^{(p+1)}|^2}{(\sum_r |s^{\otimes k} \otimes s_r^{(p)}|^2)^{1-\varepsilon}} \exp(- (1-\delta)\varphi_{L_{p+1}} - kq_\eta\varepsilon\varphi_{\Theta_\eta}) dV.$$

This is equivalent with

$$\int_S |s|^{2\varepsilon k} \exp(- (1-\delta)\varphi_{L_{p+1}} - kq_\eta\varepsilon\varphi_{\Theta_\eta}) dV < \infty.$$

In order to show the above inequality, we use the same trick as before : the vanishing set of the section s_η as in (\mathbf{U}_1) will allow us to apply the integrability lemma—the computations are strictly identical with those discussed in the point 3) above, but we give here some details.

By the vanishing properties of the section s_η , the finiteness of the previous integral will be implied by the inequality

$$\int_S \exp\left(- (1-\delta)\varphi_{L_{p+1}} - k\varepsilon q_\eta(\varphi_{\Theta_\eta} - \sum_{j \in J \setminus I} \rho_\eta^{\infty, j} \varphi_{Y_j} - \sum_{j \in I} \nu_\eta^j \varphi_{Y_j})\right) dV < \infty.$$

In the first place, we have to keep the poles of $k\varepsilon q_\eta \Theta_\eta$ “small” in the expression of the metric (37), thus we impose

$$k\varepsilon q_\eta \leq \varepsilon_0.$$

The hypothesis in the integrability lemma will be satisfied provided that

$$1 - \delta + \varepsilon k q_\eta C \frac{\eta}{q_\eta} < 1$$

(this is the contribution of the common part of $\text{Supp } L_{p+1}$ and Θ_η). Combined with the previous relations, the conditions for the parameters become

$$(39) \quad C\varepsilon k \eta < \delta < \varepsilon q_\eta < \varepsilon_0/k.$$

Again we see that the inequalities above are *compatible* if k satisfy the inequality

$$Ck\eta < q_\eta$$

which is precisely what the claim (3.10) states.

In conclusion, we can choose the parameters ε, δ so that the integrability/positivity conditions in the extension theorem are verified ; for example, we can take

$$\bullet \varepsilon := \frac{\varepsilon_0}{2kq_\eta}$$

and

$$\bullet \delta := \left(1 + kC \frac{\eta}{q_\eta}\right) \frac{\varepsilon_0}{4k}. \quad \square$$

Finally, let us indicate how to perform the induction step if $p = q_\eta - 1$: we consider the family of sections

$$s_\eta^{k+1} \otimes s_j^{(0)} \in H^0(S, (k+1)L^{(q_\eta)} + q_\eta A|_S),$$

In the case under consideration, we have to exhibit a metric on the bundle

$$L_{q_\eta} + L^{(q_\eta-1)} + kL^{(q_\eta)} + q_\eta A ;$$

however, this is easier than before, since we can simply take

$$(40) \quad \varphi_\eta^{q_\eta(k+1)} := q_\eta \varphi_{\Delta_2} + \varphi^{(kq_\eta + q_\eta - 1)}$$

where the metric on Δ_2 is induced by its expression in the preceding subsection. With this choice, the curvature conditions are satisfied ; as for the L^2 ones, we remark that we have

$$\int_S |s_\eta^{k+1} \otimes s_j^{(0)}|^2 \exp(-\varphi^{q_\eta(k+1)}) dV < C \int_S |s_\eta \otimes s_j^{(0)}|^2 \exp(-q_\eta \varphi_{\Delta_2}) dV ;$$

moreover, by the vanishing of s_η along the divisor

$$q_\eta \left(\sum_{j \in I} \nu_\eta^j [Y_j|_S] \right),$$

the right hand side term of the inequality above is dominated by

$$C \int_S \exp(-q_\eta \varphi_{\widehat{\Lambda}_{B,\eta}}) dV$$

where the last integral is convergent because of the fact that $q_\eta \nu < 1$, see **(U₃)**. The proof of the extension claim is therefore finished.

□

End of the proof.

We show next that the sections s_η can be lifted to \widehat{X} as soon as η is small enough, by using the claim 3.10.

Indeed, we consider the extensions $U_j^{(kq_\eta)}$ of the sections $s_\eta^{\otimes k} \otimes s_j^{(0)}$; they can be used to define a metric on the bundle

$$kq_\eta(K_{\widehat{X}} + S + \widehat{B}_\eta) + q_\eta A$$

whose kq_η^{th} root it is defined to be $h_k^{(\eta)}$.

We write the bundle we are interested in i.e. $q_\eta(K_{\widehat{X}} + S + \widehat{B}_\eta)$ as an adjoint bundle, as follows

$$q_\eta(K_{\widehat{X}} + S + \widehat{B}_\eta) = K_{\widehat{X}} + S + (q_\eta - 1)(K_{\widehat{X}} + S + \widehat{B}_\eta) + \widehat{B}_\eta$$

and this last expression equals

$$K_{\widehat{X}} + S + (q_\eta - 1)(K_{\widehat{X}} + S + \widehat{B}_\eta + 1/kA) + \widehat{B}_\eta - \frac{q_\eta - 1}{k} A$$

Given the extension theorem 3.12, we need to construct a metric on the bundle

$$(q_\eta - 1)(K_{\widehat{X}} + S + \widehat{B}_\eta + 1/kA) + \widehat{B}_\eta - \frac{q_\eta - 1}{k} A.$$

On the first factor of the above expression we will use $(q_\eta - 1)\varphi_k^{(\eta)}$ (that is to say, the $(q_\eta - 1)^{\text{th}}$ power of the metric given by $h_k^{(\eta)}$).

We endow the bundle \widehat{B}_η with a metric whose curvature is given by the expression

$$\sum_{j \in J \setminus I} \nu_\eta^j [Y_j] + \sum_{j \in I} \nu_\eta^j [Y_j] + \widehat{\Lambda}_{B,\delta} + \Xi(\eta) ;$$

here we take δ independent of η , but small enough such that the restriction $\widehat{B}_{\eta|S}$ is still klt. Finally, we multiply with the $\frac{q_\eta - 1}{k}$ times h_A^{-1} .

By the claim 3.10, we are free to choose k e.g. such that $k = q_\eta [\eta^{-1/2}]$ (where $[x]$ denotes the integer part of the real x). Then the metric above is not identically ∞ when restricted to S , and its curvature will be strongly positive as soon as $\eta \ll 1$. Indeed, the curvature of \widehat{B}_η is greater than a Kähler metric on \widehat{X} which is independent of η because of the factor $\widehat{\Lambda}_{B,\delta}$.

Moreover, the L^2 conditions in the theorem 3.12 are satisfied, since the norm of the section s_η with respect to the metric $q_\eta \varphi_k^{(\eta)}$ is *pointwise bounded*, and by the choice of the metric on $\widehat{B}_{\eta|S}$. In conclusion, we obtain an extension of the section s_η , and the theorem 1.5 is completely proved. □

The relative case.

We will explain along the next lines the nonvanishing result 1.5 in its general form ; to the end, we first review the notion of *relative bigness* from metric point of view.

Let $p : X \rightarrow Z$ be a projective map and let B be a \mathbb{R} -divisor on X . The pair (X, B) is klt by hypothesis, so we can assume that X is non-singular and that

$$(41) \quad B = \sum_{j=1}^N a^j W_j$$

where $0 < a^j < 1$ and (W_j) have normal crossings. Moreover, it is enough to prove 1.5 for non-singular manifolds Z (since we can desingularize it if necessary, and modify further X).

The \mathbb{R} -divisor B is equally p -big, thus there exist an ample bundle A_X , an effective divisor E on X and an ample divisor A_Z on Z such that

$$(42) \quad B + p^* A_Z \equiv A_X + E.$$

By a suitable linear combination of the objects give by the relations (41) and (42) above, we see that there exist a klt current

$$\Theta_B \in \{B + p^*A_Z\}$$

which is greater than a Kähler metric. Thus modulo the inverse image of a suitable bundle, the cohomology class of B has precisely the same metric properties as in the absolute case.

The main technique we will use in order to settle 1.5 in full generality is the positivity properties of the twisted relative canonical bundles of projective surjections ; more precisely, the result we need is the following.

Theorem 3.13 (1). *Let $p : X \rightarrow Z$ be a projective surjection, and let $L \rightarrow X$ be a line bundle endowed with a metric h_L with the following properties.*

- (1) *The curvature current of (L, h_L) is positive ;*
- (2) *There exist a generic point $z \in Z$, an integer m and a non-zero section $u \in H^0(X_z, mK_{X_z} + L)$ such that*

$$\int_{X_z} |u|^{\frac{2}{m}} \exp\left(-\frac{\varphi_L}{m}\right) d\lambda < \infty.$$

Then the twisted relative bundle $mK_{X/Z} + L$ is pseudo-effective, and it admits a positively curved metric $h_{X/Z}$ whose restriction to the generic fiber of p is less singular than the metric induced by the holomorphic sections who verify the $L^{2/m}$ condition in (2) above. \square

Given this result, the end of the proof of 1.5 goes as follows. A point $z \in Z$ will be called *very generic* if the restriction of Θ_B to the fiber X_z dominates a Kähler metric and its singular part is klt, and moreover if the sections of all multiples of rational approximations of $K_X + B$ restricted to X_z do extend near z . We see that the set of very generic points of z is the complement of a countable union of Zariski closed algebraic sets ; in particular, it is non-empty.

Let $z \in Z$ be a generic point. The adjoint \mathbb{R} -bundle $K_{X_z} + B|_{X_z}$ is pseudo-effective, thus by the absolute case of 1.5 we obtain an effective \mathbb{R} -divisor

$$\Theta := \sum_{j=1}^N \nu^j W_j$$

within the cohomology class of $K_{X_z} + B|_{X_z}$. By diophantine approximation we obtain a family of \mathbb{Q} -bundles (B_η) and a family of non-zero

holomorphic sections

$$u_\eta \in H^0(X_z, q_\eta(K_{X_z} + B_{\eta|X_z}))$$

induced by the rational approximations of Θ (see 3.6, 3.8 above).

With these datum, the theorem 3.17 provide the bundle

$$q_\eta(K_{X/Z} + B_\eta)$$

with a positively curved metric $h_{X/Z}$, together with a crucial *quantitative* information : the section u_η is *bounded* with respect to it.

The last step is yet another application of the Ohsawa-Takegoshi type theorem 3.12. Indeed, we consider the bundle

$$q_\eta(K_X + B_\eta) + p^*A$$

where $A \rightarrow Z$ is a positive enough line bundle, such that $A - (q_\eta - 1)K_Z$ is ample. We have the decomposition

$$q_\eta(K_X + B_\eta) + p^*A = K_X + (q_\eta - 1)(K_{X/Z} + B_\eta) + B_\eta + p^*(A - (q_\eta - 1)K_Z)$$

and we have to construct a metric on the bundle

$$F := (q_\eta - 1)(K_{X/Z} + B_\eta) + B_\eta + p^*(A - (q_\eta - 1)K_Z)$$

with the curvature conditions as in 3.12. The first term in the sum above is endowed with the multiple $\frac{q_\eta - 1}{q_\eta}$ of the metric $h_{X/Z}$. The \mathbb{Q} -bundle B_η is endowed with the metric given by Θ_B plus a smooth term corresponding to the difference $B_\eta - B$. Finally, the last term has a non-singular metric with positive curvature, thanks to the choice of A ; one can see that with this choice, the curvature assumptions in 3.12 are satisfied.

The klt properties of B are inherited by B_η ; thus we have

$$\int_{X_z} |u_\eta|^2 \exp\left(-\frac{q_\eta - 1}{q_\eta} \varphi_{X/Z} - \varphi_{B_\eta}\right) d\lambda \leq C \int_{X_z} \exp(-\varphi_{B_\eta}) d\lambda < \infty.$$

In conclusion, we can extend u_η to the whole manifold X by 3.12. The convexity argument in the lemma 3.6 ends the proof of the nonvanishing. \square

Remark 3.14. In fact, V. Lazic informed us that given the non-vanishing statement 1.5 in *numerical setting*, he can infer the original non-vanishing statement in [3] (see [20], as well as [21]). As a consequence, one can infer the relative version of 1.5 in the same way as in [3].

4. PROOF OF MAIN RESULTS

Proof. (of Theorem 1.1) We proceed by induction on d . Suppose that the theorem holds in dimension $d - 1$ and let $(X/Z, B)$ be a klt pair of dimension d such that B is big/ Z . First assume that $K_X + B$ is pseudo-effective/ Z . Then, by Theorem 1.5 in dimension d , $K_X + B$ is effective/ Z . Theorem 1.4 then implies that $(X/Z, B)$ has a log minimal model.

Now assume that $K_X + B$ is not pseudo-effective/ Z and let A be a general ample/ Z \mathbb{Q} -divisor such that $K_X + B + A$ is klt and nef/ Z . Let t be the smallest number such that $K_X + B + tA$ is pseudo-effective/ Z . By part (1), there is a log minimal model $(Y/Z, B_Y + tA_Y)$ for $(X/Z, B + tA)$. Run the LMMP/ Z on $K_Y + B_Y$ with scaling of tA_Y . By Theorem 2.7, we end up with a Mori fibre space for $(X/Z, B)$. \square

Proof. (of Corollary 1.2) Let $(X/Z, B)$ be a klt pair of dimension d and $f: X \rightarrow Z'$ a $(K_X + B)$ -flipping contraction/ Z . By (1) of Theorem 1.1, there is a log minimal model $(Y/Z', B_Y)$ of $(X/Z', B)$. By the base point free theorem, $(Y/Z', B_Y)$ has a log canonical model which gives the flip of f . \square

Proof. (of Corollary 1.3) If $K_X + B$ is not effective/ Z , then the corollary trivially holds. So, assume otherwise. By [11] there exist a klt pair $(S/Z, B_S)$ of dimension $\leq \dim X$ with big/ Z \mathbb{Q} -divisor B_S , and $p \in \mathbb{N}$ such that locally over Z we have

$$H^0(mp(K_X + B)) \simeq H^0(mp(K_S + B_S))$$

for any $m \in \mathbb{N}$. By Theorem 1.1, we may assume that $K_S + B_S$ is nef/ Z . The result then follows as $K_S + B_S$ is semi-ample/ Z by the base point free theorem. \square

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