TOPICS IN MODERN ALGEBRAIC GEOMETRY

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*To my brother **Haidar***

whose curiosity led me to the world of mathematics.

Abstract

This thesis consists of three separate parts namely, chapter one, chapter two and chapters three-four which correspond to different research activities. Chapter one is independent of the others.

In chapter one we consider an algebraic variety X over an algebraically closed field k and then study the nonstandard enlarged variety ${}^*X$. We study the shadow of internal subvarieties of ${}^*X$ (theorem 1.3.6). We prove the Nullstellensatz for infinite dimensional varieties (theorem 1.5.3). Then we study the enlargement of commutative rings.

In chapter two we give a survey of the fundamental paper of Shokurov [Sh4] on the existence of log flips in dimension 3 (no result of mine and no rigorous proof in this section).

In chapter three we outline Shokurov's program (see 3.1.16.1) to attack the log termination conjecture (3.1.16), the ACC conjecture on mlds (3.1.15) and the Alexeev-Borisovs conjecture on the boundedness of δ-lc weak log Fano varieties (3.1.11) in higher dimensions. The core of this program is the boundedness of ϵ-lc complements conjecture due to Shokurov (conjecture 3.1.7). We prove the latter conjecture in dimension two . In other words, we prove that for any $\delta > 0$ there exist a finite set $\mathcal{N}$ of positive integers and $\epsilon > 0$ such that any 2-dimensional δ-lc weak log Fano pair $(X/P \in Z, B)$, where $B \in \{\frac{m-1}{m}\}_{m\in\mathbb{N}}$ if $\dim Z \geq 1$ and $B = 0$ if $Z = \mathrm{p}t.$, is (ϵ, n)-complementary$/P \in Z$ for some $n \in \mathcal{N}$ (theorem 3.7.1 and theorem 3.10.1). As a corollary, We give a completely new proof of the Alexeev-Borisovs conjecture in dimension two, that is, we prove the boundedness of δ-lc log del Pezzo surfaces (corollary 3.7.9). We also prove that the bound-

edness of lc complements due to Shokurov (theorem 3.1.24) can be proved only using the theory of complements. However, our most important result is the method used to prove the boundedness of ϵ-lc complements conjecture (3.7.1 and 3.10.1).

In chapter four we outline separate plans proposed by myself and Shokurov toward the boundedness of ϵ-lc complements conjecture in dimension three.

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Contents

1 Nonstandard algebraic geometry

1.1 Introduction

Methods of nonstandard mathematics have been successfully applied to many parts of mathematics such as real analysis, functional analysis, topology, probability theory, mathematical physics etc. But just a little bit has been done in foundations of nonstandard algebraic geometry so far. Robinson indicated some ideas in [4] and [5] to prove Nullstellensatz (Rückert's Theorem) and Oka's Theorem, using nonstandard methods, in the case of analytic varieties. In this chapter we try to formulate first elements of nonstandard algebraic geometry.

Consider an enlargement ${}^*X$ of an affine variety X over an algebraically closed field k. We often take $k = \mathbb{C}$ to be able to define the shadow of limited points of ${}^*X$.

As one of the first results in section 1.3 (Theorem 1.3.6) we prove the following:

- *Let X be an algebraic closed subset of $\mathbb{C}^n$ and the polynomial $f \in ({}^*\mathbb{C})[z_1, \ldots, z_n]$ with limited coefficients. Then there is a polynomial $g \in ({}^*\mathbb{C})[z_1, \ldots, z_n]$ with limited coefficients such that:*

  $$V(g) = V(f), \quad {}^\circ V(f) = V({}^\circ g).$$

  where zeros of these polynomials are taken in ${}^*X$ and X correspondingly.

In the same section (Theorem 1.3.2) we show that the shadow of any internal open subset of ${}^*X$ equals X, which in turn implies that every point on X has an internal nonsingular point in its halo.

In section 1.4 we discuss an error in Robinson's paper [4, th.5.3] and indicate a way to fix it.

In section 1.5 we introduce the notion of a countable infinite dimensional affine variety and prove Nullstellensatz in the case of an uncountable underlying algebraically closed field, in particular for the field of complex numbers.

Finally in section 1.6 we investigate enlargements of a commutative ring R and R-modules M. We use flatness of ${}^*R$ over R to prove ${}^*M \simeq {}^*R \otimes_R M$ for R a Noetherian commutative ring R and a finitely generated R-module M.

1.2 Basic definitions

We consider the enlargement of a set which contains an algebraically closed field k and the real numbers. Then we can consider the enlargement of affine, projective and quasiprojective varieties over k. Let X be a variety over k and let ${}^*X$ be its enlargement. By ${}^*X_{{}^*k}$ we mean ${}^*X$ as a variety over the field ${}^*k$. Note that this is completely different from ${}^*X$ with the induced internal structure.

Definition 1.2.1 Let $a \in X$, then the *halo* of a in the Zariski topology is defined as

$$zh_X(a) = \bigcap_{a \in U} {}^*U.$$

where U is Zariski open in X.

We distinguish it from $h_X(a)$ which stands for the halo of a when $k = \mathbb{C}$ and U is open in the sense of usual topology.

${}^*X^{\lim}$ denotes the elements with limited coordinates. The map ${}^*\colon X \longrightarrow {}^*X$ is the natural map which takes a to ${}^*a$ and usually we denote the image of a by the same a. We also have another important map ^{o} : ${}^*X^{\lim} \longrightarrow X$ which takes each point to its shadow.

We get two different "topologies" on ${}^*X$. One is the internal Zariski topology such that its open subsets are the internal open subsets of ${}^*X$. In fact, this is not always a topology. That is, the intersection of a collection of closed subsets may not be a closed subset. For example, let $X = \mathbb{A}^1_k$ and $B_M = \{x \in {}^*\mathbb{N}\colon 1 \leq x \leq M\}$. Moreover, let $\mathfrak{B} = \{B_M\}_{M \leq N}$ where N, M are unlimited hypernatural numbers and k is an algebraically closed field with characteristic 0. All B_M in $\mathfrak{B}$ are hyperfinite, hence by transfer they are internal closed subsets of ${}^*X$. Now consider $\bigcap_{B \in \mathfrak{B}} B = \mathbb{N}$ which is not an internal subset of ${}^*X$ and then not internal closed subset.

The other topology is the usual Zariski topology on ${}^*X_{{}^*k}$ as a variety over the field ${}^*k$.

1.3 Properties of the * and ^{o} map

X denotes an affine variety through this section. Consider the internal topology on ${}^*X$, in which a basis of open subsets consists of complements of zeros of an internal polynomial (i.e. an element of ${}^*\mathbb{C}[z]$).

The first question which draws our attention is the continuity of the * map. We shall show that this map is not continuous.

Example 1.3.1 Let $X = k = \mathbb{C}$, then there is an internal closed subset of ${}^*X$ with a nonclosed preimage under the * map. Consider the following formula:

$$(\forall A \in \mathcal{P}_F(\mathbb{C}))(\exists p \in \mathbb{C}[z])(\forall a \in \mathbb{C})(a \in A \longleftrightarrow p(a) = 0).$$

By transfer we have:

$$(\forall A \in {}^*\mathcal{P}_F(\mathbb{C}))(\exists p \in {}^*\mathbb{C}[z])(\forall a \in {}^*\mathbb{C})(a \in A \longleftrightarrow p(a) = 0).$$

Now let $A = \{x \in {}^*\mathbb{N}\colon 1 \leq x \leq N\}$ for an unlimited hypernatural number N. A is a hyperfinite subset of ${}^*\mathbb{C}$. Then, there is an internal polynomial in ${}^*\mathbb{C}[z]$ which vanishes exactly on A. The preimage of A is $\mathbb{N}$, which is not a closed subset of $\mathbb{C}$.

We can prove a stronger assertion, that for any subset B of $\mathbb{C}$, there is an internal closed subset of ${}^*\mathbb{C}$ whose preimage is B. To prove this, let H be a hyperfinite approximation of B in ${}^*\mathbb{C}$. Hence $B \subseteq H \subseteq {}^*B$. The preimage of ${}^*B$ is B, thus the preimage of H is also B.

Now we look at images of subsets of ${}^*X$ under the ^{o} map in the case of $k = \mathbb{C}$. Note that we defined the ^{o} from $X^{\lim}$ to X, but we can consider the image of subsets of ${}^*X$ by taking the image of their limited points. Unexpectedly, the image of any nonempty internal open set is the whole X.

Theorem 1.3.2 *Let A be a nonempty internal open subset of ${}^*X$, then ${}^oA = X$.*

Proof It is sufficient to prove the Theorem for principal internal open subsets. Let $A = {}^*X_f$ be a nonempty internal principal open subset where f

is an internal polynomial. If the shadow of A is not X, there is some point $a \in X$ for which $f(h_X(a)) = 0$. Hence, we have the following:

$$(\exists g \in {}^*\mathbb{C}[z_1, \ldots, z_n])(\exists \varepsilon \in {}^*\mathbb{R}^+)(\forall z \in {}^*X)((\exists w \in {}^*X)(g(w) \neq 0) \wedge (\mid z - a \mid \leq \varepsilon \rightarrow g(z) = 0)).$$

So, by transfer:

$$(\exists g \in \mathbb{C}[z_1, \ldots, z_n])(\exists \varepsilon \in \mathbb{R}^+)(\forall z \in X)((\exists w \in X)(g(w) \neq 0) \wedge (\mid z - a \mid \leq \varepsilon \rightarrow g(z) = 0)).$$

It is easy to see that the latter is not true. $\square$

Corollary 1.3.3 *There is a nonsingular point ξ in $h_X(a)$ for every $a \in X$.*

Theorem 1.3.4 *Let $f \colon X \longrightarrow Y$ be a regular map of varieties over $\mathbb{C}$. Then, we have:*

(i) ${}^o({}^*Z) = Z$ *for every closed subset* $Z \subseteq X$*;*

(ii) $({}^*f)^{-1}({}^*Z) = {}^*(f^{-1}(Z))$ *for every subset Z of Y.*

Proof (i) Obviously $Z \subseteq {}^o({}^*Z)$. Let $Z = V(g_1, \ldots, g_l)$ and let $x \in {}^o({}^*Z)$ and $x = {}^o\xi$ for some $\xi \in {}^*Z$. Then, $g_i(\xi) = 0$ for $1 \leq i \leq l$. Clearly $g_i({}^o\xi) = 0$ for $1 \leq i \leq l$ which in turn proves that $x \in Z$.

(ii) consider the formula :

$$(\forall x \in X)(x \in (f)^{-1}(Z) \longleftrightarrow f(x) \in Z).$$

so by transfer:

$$(\forall x \in {}^*X)(x \in {}^*((f)^{-1}(Z)) \longleftrightarrow {}^*f(x) \in {}^*Z).$$

On the other hand we have:

$$(\forall x \in {}^*X)(x \in ({}^*f)^{-1}({}^*Z) \longleftrightarrow {}^*f(x) \in {}^*Z).$$

which proves (ii). □

It is well known that the shadow of any subset of ${}^*\mathbb{R}$ is a closed subset in $\mathbb{R}$, the field of real numbers, in the sense of real topology. But a similar fact for algebraic sets, is much more complicated. We first show that the shadow of an internal closed subset of ${}^*X$ is not necessarily closed in X. For example, consider $B_M \subset {}^*\mathbb{A}^1_{\mathbb{C}}$ which was defined in section 1.2 for an unlimited hypernatural number M. Obviously, ${}^oB_M = \mathbb{N}$ is not closed in $\mathbb{A}^1_{\mathbb{C}}$. A better deal is to consider closed subsets of ${}^*X_{{}^*\mathbb{C}}$.

Theorem 1.3.5 *Let $f \in ({}^*\mathbb{C})[z_1, \ldots, z_n]$ be a polynomial with limited coefficients and let ${}^of$ be nonzero. Then, we have:*

$$^\circ(V(f)) = V(^\circ f).$$

Proof The shadow of f, ${}^\circ f$, may happen to be a constant, that is, the coefficients of nonzero degree monomials in f are infinitesimal. This implies that no limited point can be in $V(f)$. On the other hand, $V({}^\circ f) = \emptyset$. Then, the equality is proved in this case.

So we may assume that ${}^\circ f$ is not constant. Let $\xi \in {}^*\mathbb{C}^n$ be a limited point such that $f(\xi) = 0$. Then, ${}^of({}^o\xi) = 0$ hence ${}^o\xi \in V({}^of)$.

If $a \in V({}^o f)$ then $f(a) \simeq 0$. Moreover, $f(h_{\mathbb{C}^n}(a)) \subseteq h_{\mathbb{C}^n}(0)$. It is sufficient to find a point in the halo of a such that f vanishes at that point. Now if $f(a) \neq 0$ we can linearly change variables so that a is transferred to the origin. Note that the new polynomial, say g has also limited coefficients and this translation takes $h_{\mathbb{C}^n}(a)$ to $h_{\mathbb{C}^n}(0)$. We can write $g = g^{inf} + g^{ap}$ such that g^{inf} has infinitesimal coefficients and g^{ap} has noninfinitesimal limited coefficients. Then, ${}^o g = {}^o g^{ap}$.

Now we use induction on the number of variables. If $n = 1$ Robinson–Callot Theorem [3, ch.2, th.2.1.1] shows that $g(h_{\mathbb{C}}(0)) = h_{\mathbb{C}}(0)$ because g is S-continuous as it has limited coefficients. If $1 < n$ we consider the homogeneous form with highest degree appearing in g^{ap}, say h. h is a sum of monomials of the same degree.

If $h = \alpha z_1 \ldots z_n$, where α is a hypercomplex number, then we change variables such that $z_1 = w_1$ and $z_i = w_i + w_1$. This change, obviously maps the halo of origin on itself and from g we get a new polynomial e with limited coefficients. Now consider $e(w_1, \ldots, w_{n-1}, 0)$, clearly the shadow of this polynomial in a smaller than n number of variables, is not constant, and we use induction.

In the remaining cases we can again replace one of the variables by zero and reduce the number of variables, if necessary, to use induction. In fact, we used h to make sure that when we replace a variable by zero we do not get a constant polynomial. This proves the existence of a zero for f and completes the proof of the Theorem. $\square$

We can generalize the last Theorem replacing $\mathbb{C}^n$ by its affine subvariety, X. The Theorem is again true. Although the previous Theorem is a particular case of the next Theorem, but their proofs are of different nature and we prefer to keep the previous proof.

Theorem 1.3.6 *Let X be an algebraic closed subset of $\mathbb{C}^n$ and let $f \in ({}^*\mathbb{C})[z_1, \ldots, z_n]$ be with limited coefficients. Then, there is a $g \in ({}^*\mathbb{C})[z_1, \ldots, z_n]$ with limited coefficients, satisfying the following:*

$$V(g) = V(f), \quad {}^{\circ}V(f) = V({}^{\circ}g).$$

where zeros of these polynomials are taken in ${}^*X$ and X correspondingly.

Proof If $V(f) = {}^*X$ then the Theorem is trivial. Hence, we may assume that $V(f) \neq {}^*X$. If ${}^{\circ}f$ is not identically zero on X we take $g = f$. Otherwise, let $\ddot{f}$ be f divided by one of its coefficients with maximum absolute value. If ${}^{o}\ddot{f}(X) \neq 0$ then put $g = \ddot{f}$. If ${}^{o}\ddot{f}({}^*X) = 0$ then $V(\ddot{f} - {}^{o}\ddot{f}) = V(\ddot{f}) = V(f)$. Now $\ddot{f} - {}^{o}\ddot{f}$ has a smaller number of monomials than f. By continuing this process eventually we get a polynomial g such that its shadow is not identically zero on X and $V(g) = V(f)$.

Now let $x \in {}^{\circ}V(g)$, then $x = {}^{o}\xi$ for some $\xi \in V(g)$. From $g(\xi) = 0$ we deduce ${}^{o}g({}^{o}\xi) = 0$, hence $x \in V({}^{o}g)$. Conversely, let $x \in V({}^{\circ}g)$. In this case, we want to prove that $h_X(x) \cap V(g) \neq \emptyset$. Let $Y \subseteq X$ be an irreducible curve containing x such that ${}^{o}g$ is not identically zero on Y. It is sufficient to prove that $h_Y(x) \cap V(g) \neq \emptyset$. Change the variables such that x is transferred to the origin and then consider ${}^*Y_{{}^*\mathbb{C}}$. First suppose that $V(g) \neq \emptyset$ on ${}^*Y$. So $V(g) \cap {}^*Y_{{}^*\mathbb{C}}$ is a finite set, that is, a zero dimensional subvariety, say $\mathcal{A} = \{\xi_1, \ldots, \xi_l\}$. Since ${}^*X \subseteq {}^*\mathbb{C}^n$, every point of ${}^*X$ is as $(b_1, \ldots, b_n)$, with n coordinates $b_1, \ldots, b_n$. If no point in $\mathcal{A}$ is infinitesimal, that is, with infinitesimal coordinates, then every ξ_i has at least a noninfinitesimal coordinate, say

a_{i_j}. The index j means that a_{i_j} appears in the j-th coordinate of ξ_i. Let $h_i = (z_j - a_{i_j})/a_{i_j}$ and let $h = h_1 \dots h_l$. Obviously, $\mathcal{A} \subseteq V(h)$ hence $h^t = eg$ on ${}^*Y$, for some polynomial e and natural number t. By construction h and g have limited coefficients. e also must have limited coefficients, otherwise $h^t/s = (e/s)g$ on ${}^*Y$ where s is a coefficient appearing in e with maximum absolute value. Then, ${}^o(h^t/s) = 0 = {}^o(e/s){}^og$ on Y. Since Y is irreducible, ${}^o(e/s) = 0$ on Y. Now we can use the method by which we constructed g and reduce the number of monomials appearing in e. Then, we get a new e with limited coefficients which satisfies ${}^oe \neq 0$, $h^t = eg$ and ${}^oh^t = {}^oe{}^og$ on Y. This is a contradiction because ${}^oh$ is not zero at origin.

If $V(g) = \emptyset$ on ${}^*Y$ then $V(g) \subseteq V(1)$ and so there is e such that $1 = eg$ on ${}^*Y$. Again e must have limited coefficients and so $1 = {}^o1 = {}^oe{}^og$ on Y. We get again a contradiction since ${}^og(0) \neq 1$. $\square$

Remark 1.3.7 It is not always possible to take g to be f itself. For example, let $X = V(z_1)$ in $\mathbb{C}^2$ and $f = z_1 + \varepsilon z_2$ in which ε is an infinitesimal hyperreal number. Then, ${}^of = z_1$ is identically zero on X. But the shadow of $V(f)$ is just a single point.

Infinitesimal deformation of plane curves

Let $C_1 = V(f_1)$ and $C_2 = V(f_2)$ be two affine plane curves. We may deform C_1, C_2 a "little" and investigate the relation between them. So let $\tilde{f}_1, \tilde{f}_2 \in ({}^*\mathbb{C})[x, y]$ be two internal polynomials such that ${}^o\tilde{f}_i = f_i$ and let $\mathcal{C}_i = V(\tilde{f}_i)$ be the corresponding internal plane curve. Let $P \in C_1 \cap C_2$. How $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect in $h_{\mathbb{A}^2}(P)$? Is the intersection number of C_1 and C_2

at P, the same as the intersection number of $\mathcal{C}_1$ and $\mathcal{C}_2$ in $h_{\mathbb{A}^2}(P)$? What happens if we choose $\mathcal{C}_1$ and $\mathcal{C}_2$ to be with simple normal crossings? Is the number of points in their intersection the same as the local intersection number $(C_1 \cdot C_2)_P$? We check this in some special cases. Suppose C_1 is a line and $\mathcal{C}_1 = {}^*C_1$. We can parameterize C_1 by a single parameter t, that is, C_1 can be given by $(\phi_1(t), \phi_2(t))$ and so $\mathcal{C}_1$ by $({}^*\phi_1(t), {}^*\phi_2(t))$. If we assume that P is the origin, then the intersection number of C_1 and C_2 is the smallest degree of t in $f_2(\phi_1(t), \phi_2(t))$. The polynomial $\tilde{f}_2(\phi_1(t), \phi_2(t))$ can be decomposed into a product of linear polynomials in t. Since ${}^o\tilde{f}_2(\phi_1(t), \phi_2(t)) = f_2(\phi_1(t), \phi_2(t))$, the number of linear components with an infinitesimal root, counting multiplicities, is the same as the smallest power of t in $f_2(\phi_1(t), \phi_2(t))$, that is equal to $(C_1 \cdot C_2)_P$.

More generally if C_1 is nonsingular at P, then it has a local parameter in this point so the above argument can be modified for this situation.

More properties of the * map

Suppose $X \subseteq \mathbb{A}^n$ is an affine variety over $\mathbb{C}$. Earlier, by giving some examples, we showed that the * map is not well behaved. But it turns out that it behaves quite well when we consider it as follows:

$$^{*_{\mathbb{C}}} : X \to {}^*X_{^*\mathbb{C}}$$

Theorem 1.3.8 *Let $\mathcal{Y}$ be a closed subset of ${}^*X_{^*\mathbb{C}}$. Then, ${}^{*}\overset{-1}{_{\mathbb{C}}}(\mathcal{Y})$ is a closed subset of X. Moreover, for any closed subset Z of X there is a hypersurface $\mathcal{Z}$ such that ${}^{*}\overset{-1}{_{\mathbb{C}}}(\mathcal{Z}) = Z$.*

Proof Suppose that $\mathcal{Y}$ is defined by $f_1 = 0, \ldots, f_l = 0$ in ${}^*X_{{}^*\mathbb{C}}$ and let $f_i = \sum_{j=1}^{m_i} \xi_{i,j} g_{i,j}$ where $\xi_{i,j} \in {}^*\mathbb{C}$ are linearly independent over $\mathbb{C}$. Hence, for $x \in X$, we have

$$f_i(x) = 0 \leftrightarrow \ \forall \ \ 1 \leq j \leq m_i, \ \ g_{(i,j)}(x) = 0$$

because of the linear independence that we assumed. So $x \in \mathcal{Y} \cap X$ if and only if $g_{(i,j)}(x) = 0$ for all (i, j). This gives explicit equations for the inverse image of $\mathcal{Y}$. That is

$$Y := {}^{*}\overset{-1}{\mathbb{C}} \ (\mathcal{Y}) = V(\{g_{(i,j)}\})$$

Now let $Z = V(\{h_k\}_{1 \leq k \leq K}) \subseteq X$. Define $\mathcal{Z}$ by $\sum_{k=1}^{K} h_k \xi_k = 0$ where $\xi_k \in {}^*\mathbb{C}$ are linearly independent over $\mathbb{C}$. This gives us the required subvariety. $\square$

Remark 1.3.9 $\mathcal{Y}$ and Y may not be of the same dimension. Moreover, Y is not unique for $\mathcal{Y}$ up to isomorphism. Let $\mathcal{Y} = V(1 + \epsilon x) \subseteq {}^*\mathbb{C}^1$ and $\mathcal{Y}' = V(1 + x) \subseteq {}^*\mathbb{C}^1$ where $\epsilon \notin \mathbb{C}$. So $\mathcal{Y} \simeq \mathcal{Y}'$ but $Y = \emptyset$ and $Y' = \{-1\}$.

1.4 Generic points for prime ideals

Let Γ be the ring of analytic functions at origin of $\mathbb{C}^n$. An important Theorem in complex analysis states that every prime ideal of Γ has a generic point in the halo of origin. We prove a similar Theorem in the algebraic context.

Theorem 1.4.1 *Let X be an irreducible affine variety and $x \in X$. Then, every prime ideal in the ring of regular functions at x has a generic point in the Zariski halo of x.*

Proof Let $\mathfrak{p}$ be a prime ideal in $\mathcal{O}_{X,x}$, the ring of regular functions at x. Define:

$A_{f,g,U} = \{y \in U : U \text{ is open in } X,\ g(y) \neq 0,\ f(y) = 0 \text{ and } f, g \text{ regular on } U\}$.

$A_{f,g,U} \neq \emptyset$ if $x \in U$, $f \in \mathfrak{p}$ and $g \notin \mathfrak{p}$. The collection $\{A_{f,g,U}\}_{x \in U, f \in \mathfrak{p}, g \notin \mathfrak{p}}$ has finite intersection property. Then, there is a ξ in the following set:

$$\bigcap_{x \in U, f \in \mathfrak{p}, g \notin \mathfrak{p}} {}^{*}A_{f,g,U}.$$

So ξ is a generic point for $\mathfrak{p}$ and $\xi \in zh_X(x)$. □

The previous Theorem shows that the map $\pi \colon zh_X(x) \longrightarrow \operatorname{Spec} \mathcal{O}_{X,x}$ is surjective where $\pi(\xi) = m_\xi$, the elements of $\mathcal{O}_{X,x}$ vanishing at ξ. This map demonstrates how close $zh_X(x)$ and $\operatorname{Spec} \mathcal{O}_{X,x}$ are.

Theorem 1.4.2 *With the hypotheses of the previous Theorem we get:*

$$\pi^{-1}(V_S(I)) = V_{zh}(I).$$

where I is an ideal of $\mathcal{O}_{X,x}$, $V_S(I)$ is the closed subset of $\operatorname{Spec} \mathcal{O}_{X,x}$ *defined by I and $V_{zh}(I)$ is the zeros of I in $zh_X(x)$.*

Proof Let $\xi \in zh_X(x)$ and $\pi(\xi) \in V_S(I)$. Then, obviously $I \subseteq \pi(\xi)$. In other words, every member of I vanishes at ξ. This shows that ξ is in the right side of the above equality.

Conversely, let ξ be in the right side of the equality, then every member of I vanishes at ξ. This implies that $I \subseteq \pi(\xi)$, that is, ξ is in the left side of the equality. □

In the analytic case the existence of generic points is used to prove the Nullstellensatz Theorem. That is, if $f, g_1, \ldots, g_l \in \Gamma$ and $V(g_1, \ldots, g_l) \subseteq V(f)$, then some power of f should be in the ideal generated by all g_i [4, sect. 4]. In [4, th.5.1] the existence of a generic point was proved for infinite dimensional spaces $\mathbb{C}^\Lambda$, in which Λ is an arbitrary infinite set. Robinson used generic points to deduce Nullstellensatz in this case [4, th.5.3]. Unfortunately, this is not correct. Now we indicate the gap.

Analysis of Robinson's proof. Let Γ be the set of cylindrical analytic functions in the origin of $\mathbb{C}^\Lambda$, each one depending only on a finite number of variables. Let $\mathcal{A} \subseteq \Gamma$ be such that $V(\mathcal{A}) \subseteq V(f)$ in a neighborhood of origin. If no power of f is in $< \mathcal{A} >$, then there is a prime ideal, say P containing $\mathcal{A}$ but not f. P has a generic point in the halo of origin, say ξ. Robinson concludes that f is zero at ξ because $V(\mathcal{A}) \subseteq V(f)$, in a neighborhood of origin, say U. But this is not true. Consider:

$$(\forall x \in U)((\forall h \in \mathcal{A}) h(x) = 0 \longrightarrow f(x) = 0).$$

and by transfer:

$$(\forall x \in {}^*U)((\forall h \in {}^*\mathcal{A}) h(x) = 0 \longrightarrow {}^*f(x) = 0).$$

This formula is true but it is different from:

$$(\forall x \in {}^*U)((\forall h \in {}^{\mathrm{im}}\mathcal{A}) h(x) = 0 \longrightarrow {}^*f(x) = 0).$$

which is a wrong formula Robinson applied to ξ.

Counter-Example 1.4.3 Let $\Lambda = \mathbb{C}$, $h_a = z_a(z_0 - a) - 1$, $\mathcal{A} = \{h_a : a \in \mathbb{C} \text{ and } a \neq 0\}$ and $f = z_0$ where z_a is a variable indexed by a. Then, $V(\mathcal{A}) \subseteq V(f)$ and no power of f is in $< \mathcal{A} >$.

Let $\xi \in V(\mathcal{A})$, then $z_0(\xi) = 0$ because for every nonzero $a \in \mathbb{C}$, $h_a(\xi) = z_a(\xi)(z_0(\xi) - a) - 1 = 0$ and then $z_0(\xi) - a$ is nonzero. Hence $z_a(\xi) = 1/(z_0(\xi) - a) = 1/(-a)$. This means that $V(\mathcal{A}) = \{\xi\}$. Clearly $\xi \in V(z_0)$. But if a power of z_0, say z_0^l, be in $< \mathcal{A} >$ then $z_0^l = \sum_{i=1}^{t} e_i h_{a_i}$ where $h_{a_i} \in \mathcal{A}$. Now we can find a point at which all h_{a_i}'s are zero and z_0 is not. But this is a contradiction. Then, no power of z_0 is in $< \mathcal{A} >$.

1.5 Varieties of infinite dimension

The previous section demonstrates some peculiar features of varieties of infinite dimension. In this section, at first we show that Nullstellensatz does not hold in infinite dimensional algebraic geometry as well as in infinite dimensional complex analysis.

Counter-Example 1.5.1 There is a set Λ and a proper ideal $\mathfrak{J}$ in S, the ring of polynomials over $\mathbb{C}$ in variables indexed by Λ, such that $V(\mathfrak{J}) = \emptyset$.

Let $\Lambda = \mathbb{C} \cup \{\mathbb{C}\}$, $h_a = z_a(z_0 - a) - 1$ for $a \neq 0$ in $\mathbb{C}$ and $h_{\mathbb{C}} = z_{\mathbb{C}} z_0 - 1$. Let $\mathfrak{J}$ be the ideal generated by all these functions in S. Then, $V(\mathfrak{J}) = \emptyset$. If $\mathfrak{J} = S$, then there are $a_1, \ldots, a_l$ (a_l can be $\mathbb{C}$) and $f_1, \ldots, f_l$ such that

$$\sum_{i=1}^{l} f_i h_{a_i} = 1.$$

Now consider all variables which occur in this formula and let R be the ring of polynomials in these variables over $\mathbb{C}$ and $\mathbb{C}^m$ the corresponding affine space. Then, the ideal generated by $h_{a_1}, \ldots, h_{a_l}$ in R is R itself. That is $V(\mathfrak{J}) = \emptyset$ in $\mathbb{C}^m$. This is not possible because we can find a point in $\mathbb{C}^m$ at which all h_{a_i}'s are zero. But the right side of the above equation would not be zero at that point.

Fortunately, this is not the end of the story. We prove a complete version of Nullstellensatz similar to the finite dimensions, in the particular case of $\Lambda = \mathbb{N}$. Let S be the ring $\mathbb{C}[z_1, z_2, \ldots]$.

Definition 1.5.2 Let $X \subseteq \mathbb{C}^{\mathbb{N}}$. We say X is an *affine variety* in $\mathbb{C}^{\mathbb{N}}$ if $X = V(\mathfrak{J})$ for some ideal $\mathfrak{J}$ of S. Moreover, we call $\mathbb{C}[X] = S/I(X)$ the *ring of regular functions* on X. Similarly, the field of fractions of $\mathbb{C}[X]$ denoted by $\mathbb{C}(X)$ is the *field of rational functions* on X.

Theorem 1.5.3 *Let $\mathfrak{M}$ be a maximal ideal of S. Then, $V(\mathfrak{M}) \neq \emptyset$.*

Proof If for every $n \in \mathbb{N}$ there is an $a_n \in \mathbb{C}$ such that $z_n - a_n \in \mathfrak{M}$ then $\mathfrak{M} =< z_n - a_n >_{n\in\mathbb{N}}$ because $< z_n - a_n >_{n\in\mathbb{N}}$ is a maximal ideal of S. Hence, $V(\mathfrak{M}) = \{(a_n)_{n\in\mathbb{N}}\}$. Now suppose there is an $n \in \mathbb{N}$ such that $z_n - a \notin \mathfrak{M}$ for any $a \in \mathbb{C}$. For simplicity we can take $n = 1$. Now let $S_i = \mathbb{C}[z_1, \ldots, z_i]$ and let $\mathfrak{M}_i$ be the contraction (inverse image) of $\mathfrak{M}$ to S_i. $\mathfrak{M}_i$ is a prime ideal in S_i and our goal is to prove that it is also a maximal ideal.

Let $Y_i = V(\mathfrak{M}_i)$ in $\mathbb{C}^i$. Then, by our hypothesis $Y_1 = \mathbb{C}$, that is, $\mathfrak{M}_1 = 0$. For every i, we have a projection:

$$\pi_i \colon Y_i \longrightarrow \mathbb{C}.$$

where $\pi_i(z_1, \ldots, z_i) = z_1$. Every member of S is a polynomial with a finite number of variables. Then, $\bigcup \mathfrak{M}_i = \mathfrak{M}$. By a Theorem in algebraic geometry [7, ch. I,§5,th.6] $\pi_i(Y_i)$ is an open subset of $\mathbb{C}$ or just a single point. If $\pi_i(Y_i) = \{b\}$ for some i, then $z_1 - b \in \mathfrak{M}_i$ which is a contradiction. If all $\pi_i(Y_i)$ are open, then consider $x \in \mathbb{C}$. Hence, there is an $h \in S$ such that $1 - h(z_1 - x) \in \mathfrak{M}$, so $1 - h(z_1 - x) \in \mathfrak{M}_j$ for some j. x cannot be in $\pi_j(Y_j)$

because $1 - h(z_1 - x)$ does not vanish at any point where its first coordinate is x. This proves the following equality:

$$\mathbb{C} = \bigcup_{i=1}^{\infty} \mathbb{C} \setminus \pi_i(Y_i).$$

which is impossible. $\square$

Corollary 1.5.4 *An ideal $\mathfrak{M}$ in S is maximal iff it is as $< z_i - a_i >_{i \in \mathbb{N}}$ for some $a_i \in \mathbb{C}$.*

$\square$

In the proof of the previous Theorem we have not used any specific property of $\mathbb{C}$, we just used the properties that it is algebraically closed and uncountable. So, we have the following.

Corollary 1.5.5 *The Theorem holds if we replace $\mathbb{C}$ by any uncountable algebraically closed field k.*

$\square$

Now we look at other parts of Nullstellensatz.

Theorem 1.5.6 *Let $\mathfrak{J}$ be an ideal in S, then $I(V(\mathfrak{J})) = \sqrt{\mathfrak{J}}$.*

Proof One inclusion is obvious. Let $T = \mathbb{C}^{\mathbb{N}}$ and let $V(\mathfrak{J}) \subseteq V(g)$ where $g \in S$. Now we consider a new space of the same type, namely $W = \mathbb{C} \times T$. We will have a new variable like z_0 and a new coordinate corresponding to this variable (note that $0 \notin \mathbb{N}$ in this thesis). Consider the ideal $\mathfrak{J}^+ = \mathfrak{J} + < 1 - z_0 g >$ in the ring $S[z_0]$. $\mathfrak{J}^+$ has no zero in W, so $\mathfrak{J}^+ = S[z_0]$. Hence, there are $h_0, h_1, \dots, h_l$ in $S[z_0]$ and $f_1, \dots, f_l$ in $\mathfrak{J}$ for which we have:

$$\sum_{i=1}^{l} h_i f_i + h_0(1 - z_0 g) = 1.$$

Now we can put $z_0 = 1/g$ and conclude that either $\mathfrak{J} = S$ or some power of g is in $\mathfrak{J}$. □

Corollary 1.5.7 *Let $\mathfrak{J}_1, \mathfrak{J}_2$ be ideals in S, then we have the following:*

(i) $V(\mathfrak{J}_1\mathfrak{J}_2) = V(\mathfrak{J}_1 \cap \mathfrak{J}_2) = V(\mathfrak{J}_1) \cup V(\mathfrak{J}_2)$*;*

(ii) $V(\mathfrak{J}_1 + \mathfrak{J}_2) = V(\mathfrak{J}_1) \cap V(\mathfrak{J}_2)$*;*

(iii) $\sqrt{\mathfrak{J}_1}$ *is prime iff* $V(\mathfrak{J}_1)$ *is irreducible.*

Proof Standard. □

It is not obvious that every rational map of affinae varieties of infinite dimension has a nonempty domain (points where the rational map is defined).

Theorem 1.5.8 $\mathrm{dom}(\phi) \neq \emptyset$ *for any rational map* $\phi\colon X \longrightarrow Y$*.*

Proof Let $\phi = (\phi_1, \phi_2, \dots)$, $\phi_i = g_i/f_i$ and $T = \mathbb{C}^{\mathbb{N}}$. It is sufficient to prove that there is a point at which none of f_i's vanishes. Suppose there is no such point, that is,

$$\bigcup_{i=1}^{\infty} V(f_i) = \mathbb{C}^{\mathbb{N}}.$$

Now let $W = \mathbb{C}^{\mathbb{N}}$. We define a coordinate system on W such that the $(2i-1)$th component in it is the same as the ith component of T, that is, we associate the variable z_i to the component with number $2i-1$, and the variable w_i to the $2i$th component.

Now consider the set:

$$\mathcal{A} = \{1 - w_i f_i \colon i \in \mathbb{N}\}.$$

This set has no zero in W. Then, by Theorem 1.5.3, $< \mathcal{A} >= \mathbb{C}[z_1, w_1, z_2, w_2, \dots]$, hence there are $h_1, \dots, h_l$ in $\mathbb{C}[z_1, w_1, z_2, w_2, \dots]$ such that:

$$\sum_{j=1}^{l} h_j (1 - w_{i_j} f_{i_j}) = 1.$$

But this is a contradiction because we know that there is some $\xi \in T$ such that $f_{i_j}(\xi) \neq 0$ for $1 \leq j \leq l$. By putting $w_{i_j}(\xi) = 1/f_{i_j}(\xi)$ we get a point in W at which all $(1 - w_{i_j} f_{i_j})$ vanish. □

Corollary 1.5.9 *Neither $\mathbb{C}^{\mathbb{N}}$ nor $\mathbb{C}^n$ (n is finite) is the union of a countable set of proper subvarieties.*

Proof We just proved this for $\mathbb{C}^{\mathbb{N}}$. Suppose that $\mathbb{C}^n = \bigcup_{i=1}^{\infty} V(f_i)$ in which f_i is in $\mathbb{C}[z_1, \dots, z_n]$. Now we extend it to $\mathbb{C}^{\mathbb{N}}$ and we get the result. □

Let $S_{\mathbb{N}} = \mathbb{C}[z_1, z_2, \dots]$ and $S_i = \mathbb{C}[z_1, \dots, z_i]$. We have the following inclusions when $n < m$:

$$S_n \longrightarrow S_{\mathbb{N}}$$

$$S_n \longrightarrow S_m$$

and by transfer we have

$$^*S_n \longrightarrow {}^*S_{\mathbb{N}}$$

$$^*S_n \longrightarrow S_N$$

in which $S_N = \mathbb{C}[z_1, \ldots, z_N]$ is the set of internal polynomials over $^*\mathbb{C}$ in variables $z_1, \ldots, z_N$ with an unlimited hypernatural number N.

Now let $\mathfrak{J}$ be an ideal in $S_{\mathbb{N}}$, $\mathfrak{J}_n$ its contraction to S_n and $\mathfrak{J}_N$ the corresponding internal ideal in S_N. We have the following diagram:

$$\begin{array}{ccc} S_n & \xrightarrow{\alpha_{n,\mathbb{N}}} & S_{\mathbb{N}} \\ \downarrow & & \downarrow \\ ^*S_n & \xrightarrow{\alpha_{n,N}} & S_N \\ & & \downarrow \\ & & ^*S_{\mathbb{N}} \end{array}$$

Using transfer we can see that $\alpha_{n,N}^{-1}(\mathfrak{J}_N) = {}^*\mathfrak{J}_n$, for all $n \in \mathbb{N}$. Hence, $\alpha_{\mathbb{N},N}^{-1}(\mathfrak{J}_N) = \mathfrak{J}$.

1.6 Enlargement of commutative rings

In this section, we study the enlargement of commutative rings, especially Noetherian rings. In the theory of commutative rings, localization and completion of rings and modules have some typical properties like preserving exactness of sequences and behaving well with tensor product. That is, if R is a Noetherian ring, $\mathfrak{p}$ a prime ideal and M is a finitely generated R-module, then we have:

$$M_{\mathfrak{p}} \simeq R_{\mathfrak{p}} \otimes_R M.$$

$$\widehat{M} \simeq \widehat{R} \otimes_R M.$$

We prove similar properties of enlargement of modules. As usual, we denote the enlargement of R and M as $^*R$ and $^*M$. For any ideal $\mathfrak{J}$ of R we have two notions of radical of ${}^*\mathfrak{I}$ in the ring ${}^*R$. One is the usual $\sqrt{{}^*\mathfrak{I}}$ when we consider ${}^*R$ as a ring. The other is the internal notion of radical of idelas, namely ${}^{\text{int}}\sqrt{{}^*\mathfrak{I}}$ which is the enlargement of $\sqrt{\mathfrak{I}}$, that is,

$$ {}^{\text{int}}\sqrt{{}^*\mathfrak{I}} = {}^*\sqrt{\mathfrak{I}}. $$

From now on we work with a Noetherian commutative ring R.

Theorem 1.6.1 *For any ideal $\mathfrak{I}$ in R, we have:*

$$ {}^*\min(\mathfrak{I}) = \min_{\text{int}}({}^*\mathfrak{I}) = \min({}^*\mathfrak{I}). $$

where min *of an ideal is the set of minimal prime ideals over the corresponding ideal.*

Proof Since R is Noetherian, $\min(\mathfrak{I})$ is a finite set, say $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_l\}$. Then, ${}^*\min(\mathfrak{I}) = \min(\mathfrak{I})$. Now let $\mathfrak{q}$ be a prime ideal of ${}^*R$ containing ${}^*\mathfrak{I}$. Hence, its contraction $\mathfrak{q}^c$ in R is a prime ideal containing $\mathfrak{I}$. There is some j such that $\mathfrak{p}_\mathfrak{j} \subseteq \mathfrak{q}^\mathfrak{c}$. Since R is Noetherian, each ideal in R is generated by a finite number of elements, the same is true for the enlargement of any ideal. Therefore, ${}^*\mathfrak{p}_\mathfrak{j} \subseteq \mathfrak{q}$ which in turn implies the equalities. $\square$

We also can prove that ${}^*J(R) = J({}^*R)$ where $J(R)$ is the Jacobson radical of R and similarly $J({}^*R)$ is the Jacobson radical of ${}^*R$.

Corollary 1.6.2 *For $\mathfrak{I}$ as in Theorem 1.6.1, we have the following:*

(i) ${}^{\text{int}}\sqrt{{}^*\mathfrak{I}} = \sqrt{{}^*\mathfrak{I}}$ *and* $\text{nil}_{\text{int}}({}^*R) = {}^*\text{nil}(R) = \text{nil}({}^*R)$*;*

(ii) $\mathfrak{q}$ *is* $\mathfrak{p}$*-primary iff* ${}^*\mathfrak{q}$ *is* ${}^*\mathfrak{p}$*-primary iff* ${}^*\mathfrak{q}$ *is internally* ${}^*\mathfrak{p}$*-primary.*

Lemma 1.6.3 *Let $\phi\colon M \longrightarrow N$ be a homeomorphism of R-modules. Then,*

(i) $\ker {}^*\phi = {}^*\ker\phi$*;*

(ii) $\mathrm{im}\,{}^*\phi = {}^*\,\mathrm{im}\phi$*.*

Proof (i)

$$(\forall m \in M)(m \in \ker\phi \longleftrightarrow \phi(m) = 0).$$

and by transfer:

$$(\forall m \in {}^*M)(m \in {}^*\ker\phi \longleftrightarrow {}^*\phi(m) = 0).$$

(ii) Use a similar formula. □

Remark 1.6.4 Let M, N, L and K be R-modules. Then, the above Lemma shows that

(i) $0 \longrightarrow N \longrightarrow M \longrightarrow K \longrightarrow 0$ is exact iff

$$0 \longrightarrow {}^*N \longrightarrow {}^*M \longrightarrow {}^*K \longrightarrow 0$$

is exact.

(ii) ${}^*M/{}^*N = {}^*(M/N)$.

Lemma 1.6.5 *${}^*R$ is a faithfully flat R-algebra.*

Proof By [2, ch. I,§2,n^o11] ${}^*R$ is a faithfully flat R-algebra iff for any maximal ideal $\mathfrak{m}$ in ${}^*R$, $\mathfrak{m}{}^*R \neq {}^*R$ and any solution of an R-homogeneous linear equation $\sum_{i=1}^{l} a_i Y_i = 0$ in ${}^*R^l$ is a ${}^*R$ linear combination of solutions in R^l.

Let $\mathfrak{m}$ be any maximal ideal of R. Since R is Noetherian, $\mathfrak{m}^*R = {}^*\mathfrak{m}$, $\mathfrak{m}^*R \neq {}^*R$.

Now let $f = \sum_{i=1}^{l} a_i Y_i = 0$ be an R-homogeneous linear equation. Let $\mathcal{A}$ be the module of solutions of f in R^l. $\mathcal{A}$ is an R-submodule of R^l. Since R is Noetherian, then $\mathcal{A}$ is finitely generated, say $\mathcal{A} =< \beta_1, \ldots, \beta_c >$. Then, we have:

$$(\forall x_1, \ldots, x_l \in R)\big(\sum_{i=1}^{l} a_i x_i = 0 \longleftrightarrow (\exists r_1, \ldots, r_c \in R)(x_1, \ldots, x_l) = \sum_{i=1}^{c} r_i \beta_i\big).$$

and using transfer:

$$(\forall x_1, \ldots, x_l \in {}^*R)\big(\sum_{i=1}^{l} a_i x_i = 0 \longleftrightarrow (\exists r_1, \ldots, r_c \in {}^*R)(x_1, \ldots, x_l) = \sum_{i=1}^{c} r_i \beta_i\big).$$

This proves that ${}^*R$ is R-flat, hence faithfully flat R-algebra. $\Box$

Let M be a finitely generated R-module. Define a bilinear function

$$\omega \colon M \times {}^*R \longrightarrow {}^*M$$

such that $\omega(m, r) = rm$. This induces a unique R-homomorphism

$$\Omega_M \colon M \otimes_R {}^*R \longrightarrow {}^*M, \quad \Omega_M(\sum_{i=1}^{t} a_i(m_i \otimes r_i)) = \sum_{i=1}^{t} a_i r_i m_i.$$

where $a_i \in R$, $m_i \in M$ and $r_i \in {}^*R$. Clearly, Ω is surjective.

Theorem 1.6.6 *Ω_M is an isomorphism.*

Proof We first assume that M is a free module, say $M = R^s$. Let $\{e_1, \ldots, e_s\}$ be a basis for M over R. Then, every element of $M \otimes_R {}^*R$ can be written as $\sum_{i=1}^{s} a_i(e_i \otimes r_i)$ and its image under Ω_M as $\sum_{i=1}^{s} a_i r_i e_i$. Now assume that

$\sum_{i=1}^{s} a_i r_i e_i = 0$. By transfer all $a_i r_i$ must be zero. This proves the Theorem when M is free.

Now in the general case, there is an l and a surjective homomorphism from R^l to M. Let K be the kernel of this homomorphism. Then, we get an exact sequence of R-modules:

$$0 \longrightarrow K \longrightarrow R^l \longrightarrow M \longrightarrow 0.$$

and so

$$0 \longrightarrow {}^*K \longrightarrow {}^*R^l \longrightarrow {}^*M \longrightarrow 0.$$

Also by the flatness of ${}^*R$ we have:

$$0 \longrightarrow K \otimes_R {}^*R \longrightarrow R^l \otimes_R {}^*R \longrightarrow M \otimes_R {}^*R \longrightarrow 0.$$

The maps Ω_K, Ω_{R^l} and Ω_M give us the vertical homomorphisms between the two exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K \otimes_R {}^*R & \xrightarrow{\lambda} & R^l \otimes_R {}^*R & \xrightarrow{\gamma} & M \otimes_R {}^*R & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & {}^*K & \xrightarrow{\alpha} & {}^*R^l & \xrightarrow{\beta} & {}^*M & \longrightarrow & 0 \end{array}$$

Suppose $\Omega_M(a) = 0$. There is b such that $\gamma(b) = a$. Let $\Omega_{R^l}(b) = c$. By commutativity of the diagram $\beta(c) = 0$. Hence, there is d such that $\alpha(d) = c$. Since Ω_K is surjective, there is e such that $\Omega_K(e) = d$. Then, $\Omega_{R^l}\lambda(e) = c$. On ther hand, Ω_{R^l} is an isomorphism, then $\lambda(e) = b$. By the exactness of the sequence $\gamma(b) = \gamma\lambda(e) = 0$. This shows that Ω_M is an isomorphism of R-modules. This completes the proof. $\square$

We can consider $M \otimes_R {}^*R$ as a ${}^*R$-module. Ω_M also can be considered as a ${}^*R$-homomorphism, so a ${}^*R$-isomorphism.

By [2, ch. IV,§2.6,th.2], we have

$$\mathrm{Ass}_{{}^*R}\, {}^*M = \mathrm{Ass}_{{}^*R}(M \otimes_R {}^*R) = \{{}^*\mathfrak{p} \colon \mathfrak{p} \in \mathrm{Ass}_R\, M\}.$$

Corollary 1.6.7 $\mathrm{Ass}_{{}^*R}\, {}^*M = {}^*\,\mathrm{Ass}_R\, M.$

□ By [8,th1.1], we can say that (as a particular case) $T = {}^*\mathbb{C}[z_1, \ldots, z_m]$ is a faithfully flat $S = ({}^*\mathbb{C})[z_1, \ldots, z_m]$-algebra. By Lemma 1.6.5, T is also a faithfully flat $R = \mathbb{C}[z_1, \ldots, z_m]$-algebra:

$$\begin{array}{ccc} S & \xrightarrow{\beta} & T \\ {\scriptstyle\gamma}\uparrow & \nearrow{\scriptstyle\alpha} & \\ R & & \end{array}$$

Now let $\mathfrak{J}$ be an ideal of R. Hence $(\mathfrak{J}S)T = {}^*\mathfrak{J}$. Let $\mathfrak{J}_1 = \gamma^{-1}(\mathfrak{J}S)$. By flatness of β, $\beta^{-1}({}^*\mathfrak{J}) = \mathfrak{J}S$, hence $\gamma^{-1}[\beta^{-1}({}^*\mathfrak{J})] = \mathfrak{J}_1$. On the other, hand by flatness of α, $\alpha^{-1}({}^*\mathfrak{J}) = \mathfrak{J}$. Then, we conclude that $\gamma^{-1}(\mathfrak{J}S) = \mathfrak{J}$. Therefore, we get another diagram:

$$\begin{array}{ccc} S/\mathfrak{J}S & \xrightarrow{\beta} & T/{}^*\mathfrak{J} \\ {\scriptstyle\gamma}\uparrow & \nearrow{\scriptstyle\alpha} & \\ R/\mathfrak{J} & & \end{array}$$

Corollary 1.6.8 *$\mathfrak{J}$ is prime iff $\mathfrak{J}S$ is prime.*

$\square$ If $\mathfrak{J}$ be a radical ideal. Then, $\mathfrak{J}S$ and ${}^*\mathfrak{J}$ are also radical.

These ideals, respectively, define closed subsets Y (in $\mathbb{C}^m$), ${}^*Y_{{}^*\mathbb{C}}$ (in ${}^*\mathbb{C}^m_{{}^*\mathbb{C}}$) and ${}^*Y$ (in ${}^*\mathbb{C}^m$). Moreover, $R/\sqrt{\mathfrak{J}}, S/\sqrt{\mathfrak{J}S}$ and $T/\sqrt{{}^*\mathfrak{J}}$ are their coordinate rings, respectively. Now using the previous Corollary we get

Corollary 1.6.9 *${}^*Y$ is irreducible iff ${}^*Y_{{}^*\mathbb{C}}$ is irreducible iff ${}^*Y$ is internally irreducible.*

$\square$

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2 Shokurov's log flips

2.1 Introduction

In this section we give a survey of the paper [Sh4] by Shokurov on the existence of log flips in the three dimensional case. There is no result of mine and no rigorous proof in this section. Moreover, it is independent of other sections.

The birational classification of algebraic varieties in dimensions more than 2 has fundamental differences from the classification of curves and surfaces. In the case of curves, essentially we do not have birational classification because any two birational normal projective curves are isomorphic. The case of surfaces is more complicated but still we do not face much difficulties. The exceptional locus is always a bunch of rational curves and we always deal with nonsingular surfaces as far as we are concerned about the classification of nonsingular surfaces. But in the case of 3-folds or higher dimensions the exceptional locus can be of codimension more than 1 and this creates fundamental difficulties. It creates rough sorts of singularities (even non-$\mathbb{Q}$-Gorenstein), then we have to do an operation to get rid of this singularities, which is called *flip* (see below for the definition). The flip operation proved by Reid [R3] for toric varieties, turned out to be extremely difficult in the general case. The general case was proved by Mori [M] in dimension 3 with terminal singularities. In more general settings and using quite different methods, Shokurov [Sh2] solved the flip problem in dimension 3 with log terminal singularities. Recently, Shokurov in his fundamental paper [Sh4] created new powerful methods which are able to prove the flip

problem shortly in dimension 3 and more complicated in dimension 4. The paper is very technical and still not digested by algebraic geometers.
This note is intended for two group of people. Those who may read this before starting Shokurov's Marathon. And those who do not want to get into technical details.

2.2 Flips

For basic definitions, we refer you to [KM]. We some times write ${}^*D$ and ${}_*D$ for the pullback and pushdown of divisors.

Let (X, B) be a Klt pair and $f\colon X \to Z$ a birational contraction. In this section, we assume that $\rho(X/Z) = 1$ and we also assume that X is $\mathbb{Q}$-factorial throughout this chapter.

Definition 2.2.1 f is a *flipping contraction* if the followings hold:

Klt $K_X + B$ is Klt.

small f is a small contraction, i.e. codim $\operatorname{exc}(f) > 1$.

Fano $-(K_X + B)$ is f-ample.

Now (Z, f_*B) does not have Klt singularities (actually, it is not even $\mathbb{Q}$-Gorenstein), so we have to replace it with some other varieties with Klt singularities, in hope of getting a better model for (X, B). The nominated variety is a pair (X^+, B^+) and a map $f^+\colon X^+ \to Z$ such that:

small f^+ is a small contraction.

$\mathbb{Q}$-Gorenstein $K_{X^+} + B^+$ is $\mathbb{Q}$-Cartier.

Compatible B^+ is the strict transform of B.

Ample $K_{X^+} + B^+$ is f^+-ample.

We have not assumed much about the singularities of (X^+, B^+), but it turns out that its singularities are at least as good as (X, B). The induced birational map $X \dashrightarrow X^+$ is called a K_X+B-flip. Now one might ask a stupid question: is this really the only choice? At least, it looks very natural because we make the log canonical divisor "more" nef and we improve singularities.

It is well known that the existence of a the $K_X + B$-flip is equivalent to the finite generation of the following sheaf of graded $\mathcal{O}_Z$-algebras:

$$\mathcal{R} = \mathcal{R}_{X/Z}(K_X + B) = \mathcal{R}(X/Z, K_X + B) = \bigoplus_{i=0}^{\infty} f_*\mathcal{O}_X(i(K_X + B))$$

If this algebra is finitely generated, then we take $X^+ = \operatorname{Proj} \mathcal{R}$. This is the first step towards the algebraisation of the problem. Algebraic methods are usually much more powerful and work better in higher dimension. Shokurov's idea is to reduce the problem to lower dimensions, that is to use induction. He reduces the problem of the existence of flip to the existence of *pl flips* (definition 2.2.2) where the reduced part of the boundary is not zero. This enables him to use adjunction and good properties of components of the reduced part and then restrict the above algebra to the intersection of these components.

Definition 2.2.2 (Pl Flip) Let $1 \leq s$ and $S = \sum_{i=1}^{s} S_i$ be a sum of reduced Weil divisors on X. A birational contraction $f\colon X \to Z$ is a *pl contraction* if:

- $K_X + B + S$ is dlt and plt if $s = 1$.
- $-(K_X + B + S)$ is f-ample.
- each S_i is $\mathbb{Q}$-Cartier and $S_i \sim_{\mathbb{Q}} r_{i,j} S_j$ for rational numbers $r_{i,j} > 0$.

Moreover, f is an *elementary pl contraction* if in addition the following hold:

- f is extremal, that is the relative Picard number $\rho(X/Z) = 1$.
- $-S$ is f-ample.
- f is small.
- X is $\mathbb{Q}$-factorial and projective/Z.

The S-flip for this contraction is a *pl flip* if it exists. The S-flip is as in definition 2.2.1 replacing $K_X + B$ and $K_{X^+} + B^+$ by S and S^+ respectively.

Remark 2.2.3 *Special termination* claims that in any sequence of flips, after a finite number of steps the flipping locus (the locus of those curves contracted by the flipping contraction) does not intersect the reduced part of the boundary. More generally, it does not intersect any log canonical centre on X.

The following theorem indicates why we are interested in pl flips.

Reduction Theorem 2.2.4 *Log flips exist in dimension n if the followings hold:*

Pl Flips *Pl flips exist in dimension n.*

Special Termination *special termination holds in dimension n.*

See Shokurov [Sh4]. Sketch of proof: The main idea is to choose a reduced Cartier divisor H on Z such that it contains all singularities of Z and singularities of the push down of the boundary on Z. Moreover, that the components of ${}^*H$, on any model W/Z, generate the Neron-Severi group of W. Then, we take $R \to X$ to be a log resolution for the pair (X, B) and put $D = B^\sim + H^\sim + \sum E_i$ where the superscript $^\sim$ stands for the strict birational transform and E_i are all exceptional divisors of the resolution. Now we start running the LMMP for the pair (R, D). In each step, discarding the relatively ample components of D, we face a pl contraction or a divisorial contraction if we choose our contractions to be extremal. So, we are fine by the assumptions on pl flips. Moreover, at each step some component of the reduced part of the boundary is relatively negative (by the assumptions on H) so, the special termination applies to this case.

Remark 2.2.5 LMMP in dimensions less than n implies special termination in dimension n [Sh4, 2.3], so we do not have to worry about special termination in dimension four. The important thing is to prove the existence of pl flips.

To deal with special termination using LMMP in lower dimensions, we note that log canonical centres on a dlt pair (R, B) (in particular irreducible components of the reduced part of B) on R are located in the local intersection of irreducible components of the reduced part of B. Let $(R_i, B_i) \dashrightarrow (R_{i+1}, B_{i+1})/T_i$ be a sequence of flips starting from $(R_0, B_0) = (R, B)$ and suppose that ω is a log canonical centre on (R_0, B_0) such that its birational transform on R_i and T_i is ω_i and γ_i respectively. Then, using adjunction, we get a sequence of birational maps $(\omega_i, B_{\omega_i}) \dashrightarrow \omega_{i+1}, B_{\omega_{i+1}})/\gamma_i$ where we may have both divisorial and small contractions $/\gamma_i$. We have to get rid of the divisorial ones (using versions of *difficulty* introduced by Shokurov) and get a sequence of log flips for (ω, B_ω) and use the LMMP to conclude that the original sequence of flips induces isomorphisms on (ω, B_ω). Then, we can get special termination from this. Because if any flipping curve C_i intersects ω_i, then $C_i \cdot \omega_i > 0$ (note by the above ω does not contain any flipping curve). So, we have $C_{i+1} \cdot \omega_{i+1} < 0$ for some flipped curve C_{i+1}, that is, ω_{i+1} contains C_{i+1} which is a contradiction.

2.3 Reduction to Lower Dimensions and b-divisors

To prove the existence of pl flips, now we know how to use induction (of course after Shokurov!). The targeted lower dimensional variety is the intersection of all S_i given in the definition of pl flips (definition 2.2.2). $Y = \bigcap_{i=1}^{s} S_i$ is called the *core* of f and its dimension d is called the *core dimension*. The smaller d, the easier life is. Y is normal because $(X, K_X + B + S)$ is a dlt pair. It is irreducible near the fibres of a point $P \in Z$ (we can shrink Z as our problem is local with respect to Z). Using adjunction, we also know that the new pair $(Y/T, B_Y)$ is Klt where $T = f(Y)$.

Moreover, special termination is proved up to dimension 4, so the existence of pl flips in dimension 4 implies the existence of all log flips in dimension 4.

The existence of pl flips is equivalent to the finite generation of a graded sheaf of algebras, namely $\mathcal{R}_{X/Z}(D)$ for a suitable $D \sim_{\mathbb{Q}} S$. Now we can restrict this algebra naturally to Y via maps $r_i\colon \mathcal{O}_X(iD) \to \mathcal{O}_Y(iD|_Y)$ and denote it by $\mathcal{R}|_Y$. Unfortunately, the resulting algebra is not divisorial. That is, it is not of the form $\mathcal{R}_{Y/T}(D') = \bigoplus_{i=0}^{\infty} f_*\mathcal{O}_Y(i(D'))$ for a divisor D' on Y. This difficulty is remedied by another beautiful idea of Shokurov [Sh4,], the notion of *b-divisor* or *birational divisor*.

A prime b-divisor $\mathcal{P}$ over Y is a valuation of the function field $K(Y)$ which corresponds to a prime divisor $\mathcal{P}_W$ (possibly zero) on any birational model W. A b-divisor is a formal sum $\mathcal{D} = \sum_{i=1}^{\infty} d_i\mathcal{P}_i$ where $d_i \in \mathbb{Z}$ and $\mathcal{P}_i$ is a prime b-divisor, such that the *trace* $\mathcal{D}_W := \sum_i d_i\mathcal{P}_{iW}$ is a finite sum on any birational model W. And also, it should be compatible with pushdown of divisors for any birational morphism $W \to W'$. Similarly, b-divisors can be defined with rational or real coefficients.

For any Cartier divisor D on Y, we can naturally define a b-divisor $\overline{D}$ which has trace f^*D on any birational model $f\colon W \to Y$. This b-divisor is called the Cartier closure of D.

Lets define a b-divisor $\mathcal{M}_i$ as

$$\mathcal{M}_i = \limsup\{-\overline{(s)}\colon s \in \mathcal{R}_i\}$$

where $\mathcal{R}|_Y = \bigoplus_{i=0}^{\infty} \mathcal{R}_i$. Put

$$\overline{\mathcal{R}|_Y} := \bigoplus_{i=0}^{\infty} f_*\mathcal{O}_Y(\mathcal{M}_i)$$

Theorem 2.3.1 *$\mathcal{R}$ is f.g. if and only if $\mathcal{R}|_Y$ is f.g. if and only if $\overline{\mathcal{R}|_Y}$ is f.g.*

Proof See [Sh4, 3.43] and [Sh4, 4.15].

We now convert our problem to another about b-divisors. Put $\mathcal{D}_i = \mathcal{M}_i/i$.

Theorem 2.3.2 (Limiting Criterion) *$\overline{\mathcal{R}|_Y}$ is f.g. if and only if the system $\{\mathcal{D}_i\}_{i=0}^{\infty}$ stabilises, that is, $\mathcal{D}_i = \mathcal{D}$ for all large i where $\mathcal{D} = \lim_{i\to\infty} \mathcal{D}_i$.*

Proof See [Sh4, 4.28].

In practise, first we always try to prove that $\mathcal{D}$ is a b-divisor over $\mathbb{Q}$ and then, prove that the system stabilises. To prove this rationality condition, Shokurov introduced the notion of (asymptotic) saturation of linear systems and proved that our system has this property [Sh4, section 4]:

Log canonical asymptotic saturation There is an integer I, *the index of the saturation*, such that for any i and j which satisfies I| i,j we have

$$\mathrm{Mov}\ \ulcorner j\mathcal{D}_i + \mathcal{A}\urcorner_W \leq (j\mathcal{D}_j)_W$$

on any high resolution W, where $\mathcal{A}$ is the discrepancy b-divisor, that is, $\mathcal{A}_W = K_W - {}^*(K_Y + B_Y)$.

Remark 2.3.3 (Truncation principle) It is easy to prove that $\bigoplus_{i=0}^{\infty} \mathcal{R}_i$ is f.g. if and only if $\bigoplus_{i=0}^{\infty} \mathcal{R}_{il}$ is f.g. for a natural number l. So we may replace our sequence $\mathcal{D}_i$ with $l\mathcal{D}_{il}$ without mention.

2.4 The FGA Conjecture

The following conjecture implies our original conjecture and Shokurov proves that it holds in dimension two:

The FGA Conjecture 2.4.1 *Let $(Y/T, B)$ be a Klt weak log Fano contraction. Then, any system of b-divisors $\{\mathcal{D}_i\}_1^\infty$ which satisfies the following, stabilises.*

- $_*\mathcal{O}_Y(i\mathcal{D}_i)$ *is a coherent sheaf on T for all i.*
- *Log canonical asymptotic saturation.*
- $i\mathcal{D}_i + j\mathcal{D}_j \leq (i+j)\mathcal{D}_{i+j}$ *for all i, j.*

First, we consider the 1-dimensional case of this conjecture. On curves, b-divisors are the usual divisors. Let $(C/pt., B)$ be a 1-dimensional klt pair, $P \in C$ and b the coefficient of P in B. Then, the saturation (at P) looks like the following

$$\ulcorner jd_i + a \urcorner \leq jd_j$$

where $a = -b$, $b < 1$ and d_i is the coefficient of P in $\mathcal{D}_i$. Note that divisors with high degree have no fixed part, so they are movable. Let $d = \lim_{i\to\infty} d_i$, hence

$$\ulcorner jd - b \urcorner \leq jd$$

This inequality implies that d is a rational number. If d is not rational, then the set $\{< jd >: j \in \mathbb{N}\}$ is dense in $[0, 1]$, where $< x >$ stands for the fractional part of x. This fact and the fact that $b < 1$ imply that d must be rational. So, for some j, we have $jd + \ulcorner -b \urcorner \leq jd_j$, hence $d \leq d_j$ and then $d = d_j$ (for infinitely many j). This approximation procedure is an essential part of the proof of the limiting criterion also in higher dimension. To get this approximation, we use the fact that semiampleness is an open condition. But we must first prove that $\mathcal{D}_Y$ is semiample on certain models. We can make $\mathcal{D}_Y$ nef using LMMP and then the weak log Fano condition plays its role: in this case, nef divisors are semiample $(/T)$.

2.5 Finding Good Models

The b-divisors $i\mathcal{D}_i$ appearing in section 2.3 in the restriction algebra have many good properties. I mentioned the log canonical asymptotic saturation property. $i\mathcal{D}_i$ are b-free (and so b-nef), which means that there is a model W such that $i\mathcal{D}_i = \overline{i\mathcal{D}_{iW}}$ and $i\mathcal{D}_{iW}$ is a free divisor and in particular a nef divisor, but for different $\mathcal{D}_i$ we have different W (the ultimate goal is to prove that many of them share the same model). $\mathcal{D}_i$ also have all properties listed in the statement of the FGA conjecture (2.4.1). In the last section, I mentioned the approximation procedure and used it to prove the FGA conjecture in the 1-dimensional case. But in higher dimensions it is more complicated. To use this method, we first make infinitely many $\mathcal{D}_i$ nef on a single model$/T$ of (Y, B) without losing the weak log Fano condition and the other mentioned properties of $\mathcal{D}_i$. We replace (Y, B) by this model, denote it again by (Y, B). In the course of obtaining this model, the boundary may increase (see [Sh4, Example 5.27] for full details). Now all $\mathcal{D}_{iY}$ nef$/T$ implies that $\mathcal{D}_Y$ is also nef$/T$ and so semiample$/T$. The difficulty is that we do not know if $D = \mathcal{D}_Y$ is a $\mathbb{Q}$-divisor so we cannot simply say that a multiple of it is free$/T$. But we know that $\mathbb{Q}$-divisors very close to it are eventually free. Assuming that D is not a $\mathbb{Q}$-divisor and using Diophantine approximation, we can get $\mathbb{Q}$-divisors $\{D_\alpha\}_{\alpha\in\mathbb{N}}$ such that:

- $D_\alpha \nleq D$ for any α.

- αD_α is free.
- for any ϵ there is an N such that $|\alpha D_\alpha - \alpha D| \leq \epsilon$ if $N \leq \alpha$.

To prove our stabilisation it is enough to prove that for a crepant model $(U, B_U)/T$ of $(Y, B)/T$, we have the following:

1. $\mathcal{D}_i = \overline{(\mathcal{D}_i)}_U$.
2. $\mathcal{D}_U = (\mathcal{D}_i)_U$ for infinitely many i.

Now let us consider how we can solve this problem using the above informations assuming that the approximation has been carried out on a crepant model of $(Y, B)/T$, say (Y', B'):

$$\begin{aligned}
&\operatorname{Mov} \ulcorner j\mathcal{D}_i + \mathcal{A} \urcorner_W \\
&\qquad = \ulcorner (\mathcal{A} - j\overline{D}_{iY'} + j\mathcal{D}_i) + (j\overline{D}_{iY} - j\overline{D}) + (j\overline{D} - j\overline{D_\alpha}) + j\overline{D_\alpha} \urcorner_W \\
&\qquad \leq (j\mathcal{D}_j)_W \leq (j\mathcal{D})_W
\end{aligned}$$

Denote $\overline{\mathcal{N}}_{Y'} - \mathcal{N}$ by $\mathcal{E}_{\mathcal{N},Y'}$ for any b-divisor $\mathcal{N}$. By negativity lemma $0 \leq \overline{\mathcal{N}_{Y'}} - \mathcal{N}$ if $\mathcal{N}$ is b-nef. To get a contradiction, the only problematic term in the above formula is $-j(\overline{\mathcal{D}}_{iY'} - \mathcal{D}_i) + \mathcal{A} = -j\mathcal{E}_{\mathcal{D}_i,Y'} + \mathcal{A}$. If we can prove that $0 \leq -r_i\mathcal{E}_{\mathcal{D}_i,Y'} + \mathcal{A}$ for all i for a subsequence $r_i \to \infty$, we get a contradiction. It is even enough to prove that $0 \leq \ulcorner -r_i\mathcal{E}_{\mathcal{D}_i} + \mathcal{A} \urcorner$. This inequality is one of the most important things that Shokurov tries to prove and this leads to the CCS conjecture. It is important to have the following conditions on a crepant model $(Y', B')/T$ of $(Y, B)/T$:

Semiampleness $\mathcal{D}_{Y'}$ is semiample.

Canonical asymptotic confinement There are positive real numbers r_i such that $r_i \to \infty$ and $0 \leq -r_i\mathcal{E}_{\mathcal{D}_i,Y'} + \mathcal{A}$ holds for any i.

I discussed the first condition and the second one will be discussed in the next section. Note that the second one implies that $0 \leq \mathcal{A}(Y', B')$ since $\mathcal{D}_i$ are b-nef. This means that (Y', B') is canonical. So this predicts that to get the asymptotic confinement it is better to work on a terminal crepant model of $(Y, B)/T$.

2.6 The CCS Conjecture

The conditions at the end of the last section are sufficient to solve our problem. The second condition has a very important advantage, that is, $\mathcal{E}_N = \mathcal{E}_{\mathcal{N}'}$ if $\mathcal{N} \sim \mathcal{N}'$ or even if $\mathcal{N} \equiv \mathcal{N}'$. We know that the saturation condition is preserved under linear equivalence. So, we may move our divisors linearly and use their freeness properties.

Canonical confinement of singularities(CCS) Let $\{D_\alpha\}_{\alpha \in A}$ be a set of divisors on (Y', B'). We say that singularities of these divisors is confined up to linear equivalence, if there is $c > 0$ such that for any α there is $D'_\alpha \in |D_\alpha|$ s.t. the pair $(Y', B' + cD'_\alpha)$ is canonical.

Note that the general member of a free linear system is reduced and irreducible. So, any such divisor can be confined by a c which just depends on the model and not on the free divisor. Actually, all but a bounded family of the b-divisors in the bellow conjecture are free on the terminal model. The bounded family in the conjecture corresponds to this bounded family of divisors such that each of these divisors is free on a model depending on the divisor.

Back to our set of b-divisors $\{\mathcal{D}_i\}$. Suppose there is a c which confines the singularities of these divisors on $(Y', B_{Y'})$ and $\mathcal{D}'_{iY'} \in |\mathcal{D}_{iY'}|$ (see 6.1). Then, for any model W over Y' we have:

$$\begin{aligned}(\mathcal{A}(Y', B_{Y'}) - ic\mathcal{E}_{\mathcal{D}_i, Y'})_W &= (\mathcal{A}(Y', B_{Y'}) - ic\mathcal{E}_{\mathcal{D}'_i, Y'})_W \\ &= K_W -^* (K_{Y'} + B_{Y'}) - ic(\mathcal{E}_{\mathcal{D}'_i, Y'})_W \\ &= K_W -^* (K_{Y'} + B_{Y'} + ci\mathcal{D}'_{iY'}) + ci(\mathcal{D}'_i)_W \\ &\geq ci(\mathcal{D}'_i)_W \geq 0.\end{aligned}$$

In other words, the b-divisors $\mathcal{D}_i$ over $(Y', B_{Y'})$ satisfy the asymptotic confinement. Now, can we find such a model? This is what CCS conjecture is about.

Roughly speaking the CCS conjecture is as follows:

The CCS Conjecture 2.6.1 *Let $\mathfrak{M}(Y', B_{Y'})$ be the set of b-free b-divisors which are log canonically saturated (i.e.* $\mathrm{Mov}\ulcorner \mathcal{M} + \mathcal{A} \urcorner \leq \mathcal{M}$*. Then, there is a bounded family of models on which $\mathfrak{M}(Y', B_{Y'})$ has canonically confined singularities. In other words, there is $c > 0$ s.t.for each $\mathcal{M} \in \mathfrak{M}(Y', B_{Y'})$ there is a crepant terminal model $(Y'_{\mathcal{M}}, B_{\mathcal{M}})$ and $\mathcal{M}' \in |\mathcal{M}|$ in which $B_{Y_{\mathcal{M}}} + c\mathcal{M}'_{Y'_{\mathcal{M}}}$ is canonical. Moreover, if $(Y, B_Y)/T$ is birational, then this family can be taken finite (for our problem we can take just one model).*

See [Sh4, 6.14]. Now, by asymptotic saturation for $\{\mathcal{D}_i\}$, we get canonical saturation for $\mathcal{M}_j = j\mathcal{D}_j$:

$$\text{Mov } \ulcorner j\mathcal{D}_j + \mathcal{A} \urcorner_W \leq (j\mathcal{D}_j)_W$$

where we take $i = j$. So, we may apply the above conjecture to prove the canonical confinement of singularities. Also on this model we have the semiampleness property because it is a crepant model of (Y, B), so this gives a solution to our problem. This conjecture has been proved up to dimension 2 [Sh4, 6.25 and 6.26].

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3 Boundedness of epsilon-log canonical complements on surfaces

3.1 Introduction

The concept of ***complement*** was introduced and studied by Shokurov [Sh1, Sh2]. He used complements as a tool in the construction of 3-fold log flips [Sh1] and in the classification of singularities and contractions [Sh2]. Roughly speaking, a complement is a "good member" of the anti-pluricanonical linear system i.e. a general member of $|-nK_X|$ for some $n > 0$. The existence of such a good member and the behaviour of the index n are the most important problems in the theory of complements. Below we give the precise definition of "good member".

Throughout this chapter and chapter four, we assume that the varieties involved are algebraic varieties over $\mathbb{C}$. In this chapter, the varieties are all surfaces unless otherwise stated. By a *log pair* (X, B), we mean a normal variety X and an $\mathbb{R}$-boundary B [Sh5]. A log pair $(X/Z, B)$ consists normal algebraic varieties X and Z equipped with a projective contraction $X \to Z$ (we often use the notation $X \to Z$ instead of $f\colon X \to Z$) and B is an $\mathbb{R}$-boundary on X. When we write $(X/P \in Z, B)$, we mean a log pair $(X/Z, B)$ with a fixed point $P \in Z$; in this situation, we may shrink Z around P in the Zariski topology without mention. If $Z = X$ and the morphism $X \to Z$ is the identity, then we may use $(P \in X, B)$ instead of $(X/P \in X, B)$. We denote the *log discrepancy* [Sh1, §1] of (X, B) at a prime divisor E as $a(E, X, B)$. We use the definition of terminal, canonical, Kawamata log terminal (Klt), divisorial log terminal (dlt), purely log terminal (plt) and log canonical (lc) singularities as in [Sh5]. In particular, $K_X + B$ must be $\mathbb{R}$-Cartier. The pair $(X/Z, B)$ is *weak log Fano* (WLF) if (X, B) is lc and $-(K_X + B)$ is nef and big$/Z$ and X is $\mathbb{Q}$-factorial.

When we say that a property holds$/P$, we mean that that property holds in U^{-1} where U is an open subset of Z containing P and $U^{-1} \subseteq X$ is the set-theoretic pullback of U.

For the basic definitions of the Log Minimal Model Program (LMMP), the main references are [KMM] and [KM]. And to learn more about the theory of complements [Sh2] and [Pr] are the best.

Definition 3.1.1 Let $\epsilon \in \mathbb{R}$. A log pair $\big(X, B = \sum_i b_i B_i\big)$ is *ϵ-lc in codimension* 2 (codim 2) if for any exceptional$/X$ divisor E on any log resolution $W \to X$ the log discrepancy satisfies the inequality $a(E, X, B) \geq \epsilon$. Moreover, (X, B) is *ϵ-lc* if it is ϵ-lc in codim 2 and every $b_i \leq 1 - \epsilon$.

Definition 3.1.2 Let (X, B) be a log pair of dimension (dim) d. A log divisor $K_X + B^+$ is an $(\epsilon, \mathbb{R})$-*complement*$/P \in Z$ for $K_X + B$ if $(X, K_X + B^+)$ is ϵ-lc$/P \in Z$, $K_X + B^+ \sim_{\mathbb{R}} 0/P \in Z$ and $B^+ \geq B$. An $(\epsilon, \mathbb{Q})$-*complement*$/P \in Z$ can be similarly defined where $\sim_{\mathbb{R}}$ is replaced by $\sim_{\mathbb{Q}}$.

Definition 3.1.3 (ϵ-lc complements) Let $\big(X/Z, B = \sum_i b_i B_i\big)$ be a pair of dim d. Then, $K_X + B^+$ is called an (ϵ, n)-*complement*$/P \in Z$ for $K_X + B$, where $B^+ = \sum_i b_i^+ B_i$, if the following properties hold:

- ◇ $(X, K_X + B^+)$ is an ϵ-lc pair$/P \in Z$ and $n(K_X + B^+) \sim 0/P \in Z$.

- ◇ $\llcorner (n+1) b_i \lrcorner \leq n b_i^+$.

We say $(X/P \in Z, B)$ is (ϵ, n)-*complementary*/P if there exists an (ϵ, n)-complement/P for $K_X + B$.

Definition 3.1.4 (ϵ-lc complements in codim 2) An (ϵ, n)-complement, $(\epsilon, \mathbb{R})$-complement or $(\epsilon, \mathbb{Q})$-complement in codim 2 is defined as in Definition 3.1.3 and Definition 3.1.2 replacing the ϵ-lc condition by ϵ-lc in codim 2.

Remark 3.1.5 In Definitions 3.1.2 and 3.1.3, if we take $\epsilon = \delta = 0$ then we have the usual notion of complement as defined in [Sh2].

Despite the somewhat tricky definition above, complements have very good birational and inductive properties which make the theory a powerful tool to apply to the LMMP. Complements do not always exist even with strong conditions such as $-(K_X + B)$ nef [Sh2, 1.1]. But they certainly do exist when $(X/Z, B)$ is a Klt WLF and B is a $\mathbb{Q}$-divisor [Example 3.1.29]. In this thesis, complements usually exist. Therefore, we concentrate on the second main problem about complements, namely boundedness. This relates to several open problems in the LMMP. We state Shokurov's conjectures on the boundedness of complements.

Definition 3.1.6 Let $\Gamma \subseteq [0, 1]$. For a divisor $B = \sum_i b_i B_i$, we write $B \in \Gamma$ if all $b_i \in \Gamma$. The set $\Phi_{\rm sm} = \{\frac{k-1}{k} | k \in \mathbb{N}\} \cup \{1\}$ is called the set of *standard* boundary multiplicities. Γ_f denotes a finite subset of $[0, 1]$.

Conjecture 3.1.7 (Weak ϵ-lc complements) *Let $\Gamma \subset [0, 1]$ be a set of real numbers which satisfies the descending chain condition (DCC). Then, for any $\delta > 0$ and d there exist a finite set $\mathcal{N}_{\delta,d,\Gamma}$ of positive integers and*

$\epsilon > 0$ *such that any d-dimensional δ-lc weak log Fano pair $(X/P \in Z, B)$, where $B \in \Gamma$, is (ϵ, n)-complementary/$P \in Z$ for some $n \in \mathcal{N}_{\delta,d,\Gamma}$.*

We refer to this conjecture as $\mathrm{WC}_{\delta,d,\Gamma}$. We prove $\mathrm{WC}_{\delta,2,\{0\}}$ when $Z = \mathrm{pt}$. (Theorem 3.7.1) and $\mathrm{WC}_{\delta,2,\Phi_{\mathrm{sm}}}$ when $\dim Z \geq 1$ (Theorem 3.10.1) for any $\delta > 0$.

Conjecture 3.1.8 (Weak ϵ-lc complements in codim 2) *Let $\Gamma \subset [0, 1]$ be a set of real numbers which satisfies the DCC. Then, for any $\delta > 0$ and d there exist a finite set $\mathcal{N}_{\delta,d,\Gamma,\mathrm{codim}\,2}$ of positive integers and $\epsilon > 0$ such that any d-dimensional δ-lc in codim 2 weak log Fano pair $(X/P \in Z, B)$, where $B \in \Gamma$, is (ϵ, n)-complementary/$P \in Z$ in codim 2 for some $n \in \mathcal{N}_{\delta,d,\Gamma,\mathrm{codim}\,2}$.*

We refer to this conjecture as $\mathrm{WC}_{\delta,d,\Gamma,\mathrm{codim}\,2}$.

Conjecture 3.1.9 (Strong ϵ-lc complements) *For any $\epsilon > 0$ and d there exists a finite set $\mathcal{N}_{\epsilon,d}$ of positive integers such that any d-dimensional ϵ-lc weak log Fano pair $(X/P \in Z, B)$ has an (ϵ, n)-complement/$P \in Z$ for some $n \in \mathcal{N}_{\epsilon,d}$.*

We refer to this conjecture as $\mathrm{SC}_{\epsilon,d}$. If we replace $\epsilon > 0$ with $\epsilon = 0$ in the above conjecture (it makes a big difference), we get the usual conjecture on the boundedness of lc complements which has been studied by Shokurov, Prokhorov and others [Sh2, PSh, PSh1, Pr]. It is proved in dim 2 [Sh2] with some restrictions on the coefficients of B.

Conjecture 3.1.10 (Strong ϵ-lc complements in codim 2) *For any $\epsilon > 0$ and d there exists a finite set $\mathcal{N}_{\epsilon,d,\mathrm{codim}\,2}$ of positive integers such that any* *d-dimensional ϵ-lc in codim 2 weak log Fano pair $(X/P \in Z, B)$ has an (ϵ, n)-complement/$P \in Z$ in codim 2 for some $n \in \mathcal{N}_{\epsilon,d,\mathrm{codim}\,2}$.*

We refer to this conjecture as $\mathrm{SC}_{\epsilon,d,\mathrm{codim}\,2}$.

The following important conjecture, due to Alexeev and the Borisov brothers, is related to the above conjectures [Mc, A1, PSh, MP].

Conjecture 3.1.11 (BAB) *Let $\delta > 0$ be a real number, $d > 0$ and $\Gamma \subset [0, 1]$. Then, varieties X for which $(X/\mathrm{pt.}, B)$ is a d-dimensional δ-lc WLF pair for a boundary $B \in \Gamma$ form a bounded family (Definition 3.2.1).*

BAB stands for Borisov-Alexeev-Borisov. We refer to this conjecture as $\mathrm{BAB}_{\delta,d,\Gamma}$. Alexeev [A1] proved $\mathrm{BAB}_{\delta,2,\Gamma}$ for any $\delta > 0$ and Γ. This conjecture was proved by Kawamata [K1] for terminal singularities in dim 3 and $\mathrm{BAB}_{1,3,\{0\}}$ was proved by Kollar, Mori, Miyaoka and Takagi [KMMT]. The smooth case was proved by Kollar, Mori and Miyaoka in any dimension. The conjecture is open even in dim 3 when $\delta < 1$. In any case, in many interesting applications $\delta < 1$.

Definition 3.1.12 The *index* of a $\mathbb{Q}$-divisor D at $Q \in X$ is the Cartier index of D at Q, that is, the smallest natural number I such that ID is Cartier at Q. We denote the index of D at Q as $\mathrm{index}_Q(D)$. The index of D is the smallest natural number I such that ID is Cartier. We denote it as $\mathrm{index}(D)$. The index of a pair (X, B) is the index of $K_X + B$.

The following special case of Conjecture 3.1.11 was proved by Borisov in dim 3 [B] and by M$^{\mathrm{c}}$Kernan in any dimension [Mc].

Theorem 3.1.13 (Borisov-M$^{\text{c}}$Kernan) *The set of all Klt WLF pairs* $(X/\text{pt.}, B)$ *with a fixed given index form a bounded family of pairs (Definition 3.2.2).*

Definition 3.1.14 Let (X, B) be a lc pair and $\eta \in X$ where $\operatorname{codim} \eta \geq 2$. The *minimal log discrepancy* (mld) of (X, B) at $\eta \in X$ denoted as $\text{mld}(\eta, X, B)$ is the minimum of $\{a(E, X, B)\}$ where E runs over all exceptional divisors/η on all log resolution $W \to X$.

The following conjecture is due to Shokurov.

Conjecture 3.1.15 (ACC for mlds) *Suppose* $\Gamma \subseteq [0, 1]$ *satisfies the DCC. Then, the following set satisfies the ascending chain condition (ACC):*

$$\left\{ \text{mld}(\eta, X, B) \middle| \ (X, B) \textit{ is lc of dim } d, \ \eta \in X \textit{ and } B \in \Gamma \right\}$$

We refer to this conjecture as $\text{ACC}_{d,\Gamma}$. Alexeev [A2] proved $ACC_{2,\Gamma}$ for any DCC set $\Gamma \subseteq [0, 1]$. This conjecture is open in higher dimension except in some special cases.

Conjecture 3.1.16 (Log termination) *Let* (X, B) *be a* d*-dimensional Klt pair. Then, any sequence of* $(K_X + B)$*-flips terminates.*

This conjecture guarantees that LMMP_d terminates after finitely many steps. We refer to it as LT_d. Kawamata [K2] proved LT_3 and the 4-dimensional case with terminal singularities [KMM]. Actually, LT_4 is the main missing component of LMMP_4, without which we cannot apply the powerful LMMP to problems in algebraic geometry. At first sight, this conjecture does not seem to be that difficult at least because of the short proof of Kawamata to LT_3 where he uses the classification of terminal singularities. The latter classification is not known in higher dimension (probably intractable). Recent attempts by Kawamata and others to solve LT_4 showed that this problem is much deeper than expected. There is speculation that it may be even more difficult than the flip problem.

We listed several important conjectures with no obvious relation. It is Shokurov's amazing idea to put all these conjectures in a single framework that we refer to as **Shokurov's Program**:

(3.1.16.1)

$\mathbf{ACC}_d \to \mathbf{LT}_d$ Shokurov proved that LT_d follows from ACC (Conjecture 3.1.15) up to dim d and the following problem up to dim d [Sh4]:

Conjecture 3.1.17 (Lower semicontinuity) *For any Klt pair (X, B) of dim d and any $c \in \{0, 1, \ldots, d-1\}$ the function*

$$\mathrm{mld}_c(\mu, X, B) \colon \{\textit{codim } c \textit{ points of } X\} \longrightarrow \mathbb{R}$$

is lower semicontinuous.

This conjecture is proved up to dim 3 by Ambro [Am]. This conjecture does not seem to be as tough as the previous conjectures. Shokurov [Sh4, Lemma 2] solved this problem in dim 4 for mlds in $[0, 2]$. Thus, ACC in dim 4 is enough for log termination in dim 4 [Sh4, Corollary 5]. Actually ACC for mlds in [0,1] for closed points is enough [Sh4, Corollary 5].

$\mathbf{BAB}_{d-1} \to \mathbf{ACC}_d$ Shokurov [Sh2, 7.11] defines a new invariant $\mathrm{reg}(P \in X, B^+) \in \{0, \ldots, d-1\}$ for any d-dimensional lc singularity $(P \in X, B)$

and proves (see [Sh2, 7.16] for 3-dimensional case and [PSh, 4.4] or [Sh2, 7.17] for general case when $B \in \Phi_{\rm sm}$) that $\mathrm{ACC}_{d,\Gamma}$ (Conjecture 3.1.15) for pairs with $\mathrm{reg}(P \in X, B^+) = 0$ follows from BAB_{d-1} (Conjecture 3.1.11). If $\mathrm{reg}(P \in X, B^+) = 0$, then the singularity is exceptional (see Definition 3.2.9). Also if $\mathrm{reg}(P \in X, B^+) \in \{1, \ldots, d-2\}$, then $\mathrm{ACC}_{d,\Gamma}$ can be reduced to lower dimensions ([Sh2, 7.16 and 7.17] and Shokurov's unpublished work). Thus, the only remaining part of $\mathrm{ACC}_{d,\Gamma}$ is when $\mathrm{reg}(P \in X, B^+) = d - 1$. This case is expected to be proved using different methods (similar to toric singularities [Sh2, 7.16]). So *in particular* $\mathrm{ACC}_{4,\Gamma}$ follows from the BAB_3 and the $\mathrm{reg}(P \in X, B^+) = 3$ case. Moreover, $\mathrm{ACC}_{3,\Gamma}$ follows from the $\mathrm{reg}(P \in X, B^+) = 2$ case.

$\mathbf{WC}_{d-1} \to \mathbf{BAB}_{d-1}$ For us, this is possibly the most important application of the theory of complements: $\mathrm{WC}_{\delta,d-1,\{0\}}$ (Conjecture 3.1.7) "implies" $\mathrm{BAB}_{\delta,d-1,[0,1]}$ (Conjecture 3.1.11). More precisely, these two problems can be solved at the same time. In other words, in those situations where boundedness of varieties is difficult to prove, boundedness of complements is easier to prove. And that is exactly what we do in this thesis for the 2-dimensional case: we prove $\mathrm{WC}_{\delta,2,\{0\}}$ and $\mathrm{BAB}_{\delta,2,[0,1]}$. Our main objective is to find a proof of $\mathrm{WC}_{\delta,2,\{0\}}$ and $\mathrm{BAB}_{\delta,2,[0,1]}$ using as little of the geometry of the algebraic surfaces as possible, so that it can be generalised to higher dimension. In other words, the methods used in the proof of these results are the most important for us. After finishing this work, we expect to finish the proof that $\mathrm{WC}_{\delta,3,\{0\}}$ "implies" $\mathrm{BAB}_{\delta,3,[0,1]}$ in the not-too-distant future!

The program in dim 4 Let us mention that by carrying out Shokurov's program in dim 4, in which the main ingredient is $\mathrm{WC}_{\delta,3,\{0\}}$ (Conjecture 3.1.7) i.e. boundedness of ϵ-lc complements in dim 3, one would prove the following conjectures:

- ACC for mlds in dim 3 (Conjecture 3.1.15).
- Boundedness of δ-lc 3-fold log Fanos BAB_3 (Conjecture 3.1.11).
- ACC for mlds in dim 4.
- Lower semicontinuity for mlds in dim 4 (Conjecture 3.1.17).
- Log termination in dim 4 and then LMMP_4 (Conjecture 3.1.16).

Plan of remaining sections:

1. Chapter 3 (current chapter) studies complements on log surfaces.

2. In 3.2, we recall some definitions and lemmas.

3. In 3.3, we prove $\mathrm{WC}_{\delta,1,[0,1]}$, that is, the boundedness of ϵ-lc complements in dim 1 (**Theorem 3.3.1**).

4. In 3.5, we prove $\mathrm{WC}_{\delta,2,\{0\}}$ for the case $X = Z$ i.e. the boundedness of ϵ-lc complements in dim 2, locally, for points on surfaces with $B = 0$ (**Theorem 3.5.1**).

5. In 3.6, we prove $\mathrm{WC}_{\delta,2,\{0\}}$ when X/Z is a birational equivalence i.e. the boundedness of ϵ-lc complements in dim 2, locally, for birational contractions of surfaces with $B = 0$ (**Theorem 3.6.1**). This proof is a surface proof i.e. we make heavy use of the geometry of algebraic

surfaces, so it seems unlikely that it can be generalized to higher dimensions. A second proof of the birational case is given in 3.10 (**Theorem 3.10.1**).

6. In 3.7, we prove $\mathrm{WC}_{\delta,2,\{0\}}$ when $Z = pt.$, that is, boundedness of ϵ-lc complements on surfaces, globally, with $B = 0$ (**Theorem 3.7.1**). The proof is based on the LMMP and we expect it to generalise to higher dimension. As a corollary, we give a completely new proof to the boundedness of ϵ-lc log del Pezzo surfaces (**Corollary 3.7.9**). Another application of our theorem is a proof of boundedness of lc ($\epsilon = 0$) complements only using the theory of complements (**Theorem 3.1.24**). The latter boundedness was proved by Shokurov [Sh2].

7. In 3.8, we give a second proof of $\mathrm{WC}_{\delta,2,\{0\}}$ in the global case, that is, when $Z = \mathrm{p}t$. (**Theorem 3.8.1**). This proof also uses surface geometry and we do not expect it to generalise to higher dimension.

8. In 3.10 we give a proof of $\mathrm{WC}_{\delta,2,\Phi_{\mathrm{sm}}}$ in all local cases, in particular, the case where X/Z is a fibration over a curve (**Theorem 3.10.1**). This proof is also based on the LMMP.

9. Chapter 4 is about higher dimensional ϵ-lc complements. We discuss our joint work in progress with Shokurov.

10. In 4.1-2, we discuss **Plans** for attacking the boundedness of ϵ-lc complements in dimension 3, one due to myself and the second suggested by Shokurov.

We summarize the main results of chapter 3:

Theorem 3.1.18 *Conjecture 3.1.7 holds in dim one for* $\Gamma = [0, 1]$.

See 3.3.1 for the proof.

Theorem 3.1.19 *Conjecture 3.1.7 holds in dim two in the global case, that is, when* $Z = \text{pt.}$ *and* $\Gamma = \{0\}$.

See 3.7.1 and 3.8.1 for proofs.

Theorem 3.1.20 *Conjecture 3.1.7 holds in dim two in the local cases, that is, when* $\dim Z > 0$ *and* $\Gamma = \Phi_{\text{sm}}$.

See 3.10.1 , 3.5.1 and 3.6.1 for proofs.

Corollary 3.1.21 *Conjecture 3.1.11 holds in dim two.*

See 3.7.9 for proof. Conjecture 3.1.11 in dim 2 was first proved by Alexeev using different methods [A1].

Corollary 3.1.22 *Theorem 3.1.24 can be proved using only the theory of complements.*

See the discussion following Theorem 3.1.24.

Remark 3.1.23 (ϵ-lc complements method) Though formally speaking the list above are the main results in chapter 3, we believe that the method used to prove 3.7.1 and 3.10.1 is the most important result of this chapter.

Here we mention some developments in the theory of complements. The following theorem was proved by Shokurov [Sh2] for surfaces.

Theorem 3.1.24 *There exists a finite set $\mathcal{N}_2$ of positive integers such that any 2-dim lc weak log Fano pair $(X/P \in Z, B)$ has a $(0,n)$-complement$/P \in Z$ for some $n \in \mathcal{N}_2$ if B is semistandard, that is, each coefficient b of B, satisfies $b \geq \frac{6}{7}$ or $b = \frac{m-1}{m}$ for some natural number m. Moreover if* $\dim Z > 0$ *then the theorem holds for a general boundary.*

Shokurov uses BAB_2 (3.1.11) in the proof of the above theorem. As mentioned before, the results of this thesis imply the BAB_2 (Corollary 3.7.9). So, the above theorem can be proved only based on the theory of complements. A similar theorem is proved by Prokhorov and Shokurov in dim 3 modulo BAB_3 and the *effective adjunction* in dim 3 (Conjecture 4.2.2). However, the local case does not need the latter assumptions as the following theorem shows [PSh].

Theorem 3.1.25 *Let $(X/P \in Z, B)$ be a Klt WLF 3-fold pair where* $\dim Z \geq 1$ *and $B \in \Phi_{\mathrm{sm}}$. Then, $K_X + B$ is $(0,n)$-complementary$/P \in Z$ for some $n \in \mathcal{N}_2$.*

Complements have good inductive properties as Theorem 3.1.25 shows. This theorem is stated and proved in higher dimensions in more general settings (see [PSh]). To avoid some exotic definitions, we stated only the 3-fold version.

Finally, we give some examples of complements and their boundedness. More examples can be found in [Sh1, Sh2, Pr, PSh, PSh1].

Example 3.1.26 Let $(X/Z, B) = (\mathbb{P}^1/\mathrm{pt.}, 0)$ and let P_1, P_2, P_3 be distinct points on $\mathbb{P}^1$. Then, $K_X + P_1 + P_2$ is a $(0,1)$-complement for K_X but it is not an (ϵ, n)-complement for any $\epsilon > 0$ since $K_X + P_1 + P_2$ is not Klt. On the other hand, $K_X + \frac{2}{3}P_1 + \frac{2}{3}P_2 + \frac{2}{3}P_3$ is a $(\frac{1}{3}, 3)$-complement for K_X.

Example 3.1.27 Let $(X_1/Z_1, B_1) = (\mathbb{P}^2/\mathrm{p}t., 0)$ and $(X_2/Z_2, B_2) = (\mathbb{P}^2/\mathbb{P}^2, 0)$. Then, K_{X_2} is a $(2,1)$-complement$/Z_2$ at any point $P \in Z_2$, but obviously K_{X_1} is not even numerically zero$/Z_1$ though $K_{X_1} = K_{X_2}$.

Example 3.1.28 Let $(X/Z, B) = (X/X, 0)$ where X is a surface with canonical singularities. Then, the index of K_X is 1 at any point $P \in X$. So we can take $B^+ = 0$ and K_X is a $(1,1)$-complement$/X$ for K_X at any $P \in X$.

Example 3.1.29 Let $(X/P \in Z, B)$ be a δ-lc WLF pair of dim d where $\delta > 0$ and $B \in \mathbb{Q}$. Then, there exists a (δ, n)-complement$/P$ for some $n \in \mathbb{N}$. By the base point free theorem, $-l(K_X + B)$ is free$/P$ for all large enough $l \in \mathbb{N}$. There is an l and a general member H in the free$/P$ linear system $|-l(K_X + B)|$ such that $K_X + B + \frac{1}{l}H$ is δ-lc$/P$. $K_X + B + \frac{1}{l}H$ is a (δ, n)-complement$/P$ for $n = l$. Obviously, $l(K_X + B + \frac{1}{l}H) \sim 0/P$ by the construction. Moreover, $\llcorner (l+1)\frac{t}{l} \lrcorner = \llcorner t + \frac{t}{l} \lrcorner = t$ for any coefficient $b = \frac{t}{l}$ of B.

In particular, if $Z = X$ and $n(K_X + B)$ is Cartier at P then $K_X + B^+ = K_X + B$ is a (δ, n)-complement$/P$. Thus, the existence of complements for Klt singularities $(P \in X, B)$, where $B \in \mathbb{Q}$, is not a problem. We are interested in the boundedness of such complements (as in Conjecture 3.1.7. See section 3.5).

Example 3.1.30 Let $(P \in X, 0)$ be a surface δ-lc singularity for some

$\delta > 0$. By Example 3.1.29, there is a (δ, n)-complement for K_X where n is the index of K_X.

The singularity at P is either of type A_r, D_r, E_6, E_7 or E_8 [Pr, 6.1.2]. In section 3.5, we prove that if the singularity at P is of type D_r, E_6, E_7 or E_8 then the index of K_X at P is bounded. In other words, complements are bounded in the sense of Conjecture 3.1.7. However, if the singularity at P is of type A_r, then the index of K_X at P is not bounded. In section 3.5, we construct a (ϵ, n)-complement for K_X with a bounded n and $\epsilon > 0$. A_r type singularities are always $(0, 1)$-complementary (3.5.3).

Example 3.1.31 Let $\mathcal{X}$ be a bounded set of WLF pairs $(X/\mathrm{p}t., B)$ (Definition 3.2.2). Then, there is an n such that $-n(K_X + B)$ is a free divisor for any $(X/\mathrm{p}t., B) \in \mathcal{X}$. Hence, by Example 3.1.29 and the boundedness of $\mathcal{X}$ there is an $\epsilon > 0$ such that $K_X + B$ is (ϵ, n)-complementary/pt. for any $(X/\mathrm{p}t., B) \in \mathcal{X}$.

To prove the boundedness of complements (Conjecture 3.1.7), in some situations, we first prove the boundedness of pairs. However, we are more interested in the other way around. In other words, we try to use the boundedness of complements to get the boundedness of pairs (Corollary 3.7.9).

Example 3.1.32 Let $I \in \mathbb{N}$. Suppose $(X/Z, D)$ is a 2-dimensional Klt pair with index I where X is projective and Z is a curve. Moreover, assume that $-(K_X+D)$ is nef/pt.. If $-(K_X+D)$ is big/pt. then, we can use Borisov–McKernan (Theorem 3.1.13) to get the boundedness of (X, D) and also the boundedness of complements for $K_X + D$ (Example 3.1.31). However, if $K_X + D \equiv 0/Z$ then we cannot use Theorem 3.1.13. In this case, there is a Klt pair $(X_1/Z, D_1)$ with index I such that $X_1 \to Z$ is an extremal contraction (we get X_1 by contracting some curves on X). Moreover, assume that $Z = \mathbb{P}^1$ (for example when X is pseudo-WLF/pt. as in Definition 3.2.6) and $K_{X_1} + D_1^+$ is a $(0, n)$-complement/pt. for $K_{X_1} + D_1$ where n is bounded. Let $D_1^+ - D_1 = \sum_i f_i F_i \geq 0$ where F_i are fibres over Z. The index of $D_1^+ - D_1$ is bounded by assumptions. Thus, we can replace $D_1^+ - D_1$ by $\sum_j f'_j F'_j$ where F'_j are general fibres such that $K_X + D + \sum_j f'_j F'_j$ is a $(\frac{1}{I}, m)$-complement/pt. for a bounded m.

This example shows how one can use boundedness of $(0, n)$-complements to get boundedness of (ϵ, m)-complements. This method is used in the proof of Theorem 3.7.1.

3.2 Preliminaries

In this section, we discuss some basic definitions and constructions.

Definition 3.2.1 A set $\mathcal{X}$ of varieties of the same dimension is *bounded* if there are schemes $\mathbb{X}$ and S of finite type and a morphism $\phi\colon \mathbb{X} \to S$ such that every geometric fibre of ϕ is a variety in $\mathcal{X}$, and every $X \in \mathcal{X}$ is isomorphic to a geometric fibre of ϕ.

Definition 3.2.2 Let $\mathcal{X}$ be a set of pairs (X, B_X) of the same dimension. $\mathcal{X}$ is *bounded* if there are schemes $\mathbb{X}$ and S of finite type, a divisor $\mathbb{B}$ on $\mathbb{X}$, a morphism $\phi\colon \mathbb{X} \to S$ and a finite set $\Gamma_f \subset [0,1]$ such that for every $(X, B_X) \in \mathcal{X}$, the variety X is isomorphic to a geometric fibre $\mathbb{X}_s$ for some $s \in S$, $\operatorname{Supp} B_X = \operatorname{Supp} \mathbb{B}|_{\mathbb{X}_s}$ and $B_X \in \Gamma_f$. In addition, every $(\mathbb{X}_s, \operatorname{Supp} \mathbb{X}|_s)$, with $\mathbb{X}_s$ a geometric fibre, must be isomorphic to some $(X, \operatorname{Supp} B_X) \in \mathcal{X}$.

Remark 3.2.3 For a morpishm $f\colon X \to Z$ and divisors A and B on X and Z respectively, we usually use $_*A$ instead of f_*A and use $^*B$ instead of f^*B (if B is $\mathbb{R}$-Cartier). This is especially useful when we have a morphism $X \to Z$ with no name.

Definition 3.2.4 Let (X, B) be a lc pair of dim d. Let $(Y/X, B_Y)$ be a log pair such that $K_Y + B_Y := {}^*(K_X + B)$. Then, Y is called a *partial resolution* of (X, B).

Lemma 3.2.5 *Let $\mathcal{X} = \{X\}$ be a bounded set of projective Klt varieties of dim d such that $-K_X$ is nef and big. Then, the set of partial resolutions for all $X \in \mathcal{X}$ is bounded.*

Proof Let $\mathcal{Y}$ be the set of partial resolutions for all $X \in \mathcal{X}$. Since $\mathcal{X}$ is bounded, there is a natural number I such that IK_X is Cartier for every $X \in \mathcal{X}$. Let $Y \in \mathcal{Y}$. There are $X \in \mathcal{X}$ and a boundary B_Y such that $K_Y + B_Y = {}^*K_X$. Thus $I(K_Y + B_Y)$ is Cartier. So the index of (Y, B_Y) is bounded. (X, B) is Klt WLF so will be (Y, B_Y). Now by Borisov–M^{c}Kernan (Theorem 3.1.13) the set $\mathcal{Y}$ is bounded. □

Definition 3.2.6 A variety X/Z of dim d, is *pseudo*-WLF/Z if there exists a boundary B such that $(X/Z, B)$ is WLF. Moreover, we say that X is Klt pseudo-WLF/Z if there is a Klt WLF $(X/Z, B)$.

Remark 3.2.7 Pseudo-WLF varieties have good properties. For example, $\overline{\text{NE}}(X/Z)$ (see [KM] for definition), the closure of the cone of effective curves, is a finite rational polyhedral cone ([Sh2] or [Sh3]). Moreover, each extremal face of the cone is contractible [Sh3]. In addition, any nef divisor on a Klt pseudo-WLF variety is semiample [Sh3].

Lemma 3.2.8 *The Klt pseudo-WLF/Z property is preserved under extremal flips and divisorial contractions/Z with respect to any log divisor.*

Proof Let X be a Klt pseudo-WLF/Z and B a boundary such that $(X/Z, B)$ is a Klt WLF. Now let $X \dashrightarrow X'/Z$ be an extremal flip corresponding to an extremal ray R on X. Since $(X/Z, B)$ is a Klt WLF, there is a boundary $D \in \mathbb{Q}$ such that $K_X + D$ is antiample/Z and Klt (remember that X is $\mathbb{Q}$-factorial by definition). Now let H' be an ample/Z divisor on X' and H its transform on X. There is a rational $t > 0$ such that $K_X + D + tH$ is antiample/Z and Klt. Now take a Klt $(0, \mathbb{Q})$-complement

$K_X + D + tH + A \sim_{\mathbb{Q}} 0/Z$. Thus, $K_{X'} + D' + tH' + A' \sim_{\mathbb{Q}} 0/Z$ on X'. From the assumptions $K_{X'} + D' + A'$ is antiample/Z and Klt. So X' is also a Klt pseudo-WLF.

If $X \to X'$ is a divisorial extremal contraction, then proceed as in the flip case by taking an ample/Z divisor H' on X'. $\square$

Definition 3.2.9 (Exceptional pairs) Let $(X/Z, B)$ be a pair of dim d. If $Z = \mathrm{pt.}$, then $(X/Z, B)$ is *exceptional* if there is at least a $(0, \mathbb{Q})$-complement/Z for $K_X + B$ and any $(0, \mathbb{Q})$-complement/Z $K_X + B^+$ is Klt. If $\dim Z > 0$ then $(X/Z, B)$ is *exceptional* if there is at least a $(0, \mathbb{Q})$-complement/Z for $K_X + B$ and any $(0, \mathbb{Q})$-complement/Z $K_X + B^+$ is plt on a log terminal resolution. Otherwise $(X/Z, B)$ is called *nonexceptional*.

Example 3.2.10 Let $(X/\mathrm{pt.}, B)$ be a Klt pair of dim d such that $K_X + B \sim_{\mathbb{Q}} 0$. Then, $(X/\mathrm{pt.}, B)$ is exceptional. Since if $K_X + B^+$ is a $(0, \mathbb{Q})$-complement for $K_X + B$, then $B^+ = B$.

Example 3.2.11 It is easy to see that $\left(\mathbb{P}^1/\mathrm{pt.}, B = \frac{1}{3}P_1 + \frac{1}{2}P_2 + \frac{1}{2}P_3 + \frac{1}{2}P_4\right)$ is exceptional where P_1, P_2, P_3 are distinct points on $\mathbb{P}^1$.

Example 3.2.12 Let $(P \in X, 0)$ be a Klt surface singularity. Then, $(P \in X, 0)$ is exceptional if and if the singularity is of type E_6, E_7 or E_8 [Pr, 6.1.2]. Moreover, $(P \in X, 0)$ is nonexceptional if and only if the singularity is of type A_r or D_r.

Example 3.2.13 Let $(X/\mathrm{pt.}, B)$ be a Klt WLF pair in dim 2. Then, $(X/\mathrm{pt.}, B)$ is nonexceptional if and only if there is a boundary $D \in \mathbb{Q}$ such that $K_X + D$ is antinef and strictly lc. Suppose there is a D with the mentioned properties. Then, $-(K_X + D)$ is semiample [sh2]. Moreover, there is a $(0, n)$-complement for $K_X + D$ for some $n \leq 57$ [Sh2, 2.3.1]. This important result is repeatedly used in this chapter. Conjecturally a similar statement holds in any dimension under $B \in \Phi_{\rm sm}$ [PSh, 1.12] or maybe under $B \in \Gamma_f$.

Example 3.2.14 Let $(P \in X, B)$ be a 2-dimensional Klt singularity where $B \in \Phi_{\rm sm}$. Suppose $\mathrm{mld}(P, X, B) < \frac{1}{6}$. Then, $(P \in X, B)$ is nonexceptional. By [PSh, 3.1] there is a $(0, n)$-complement/P $K_X + B^+$ where $n \in \{1, 2, 3, 4, 6\}$. Let $T \to X$ be a terminal model of $(P \in X, B)$ and E an exceptional/P divisor such that $a(E, X, B) < \frac{1}{6}$. Define $K_T + B_T := {}^*(K_X + B)$ and $K_T + {B_T}^+ := {}^*(K_X + B^+)$. Since $B \in \Phi_{\rm sm}$ then $B^+ \geq B$. Hence, $\mu_E({B_T}^+)$, the coefficient of E in ${B_T}^+$, is $> \frac{5}{6}$. On the other hand, $n(K_X + B)$ is Cartier so $\mu_E({B_T}^+) = \frac{t}{n}$ for some $t \in \mathbb{N}$. Thus $\mu_E({B_T}^+) = 1$ which in turn implies that $K_X + B^+$ is strictly lc.

Example 3.2.15 Let $X = \mathbb{P}^2/G$ for a finite subgroup $G \subset \mathrm{PGL}_3(\mathbb{C})$ and B a divisor such that $K_{\mathbb{P}^2} = {}^*(K_X + B)$ for the quotient morphism $\mathbb{P}^2 \to X$. Then, $(X/\mathrm{pt.}, B)$ is exceptional if and only if G has no semiinvariants of degree ≤ 3 [Pr, 10.3.3].

Remark 3.2.16 Boundedness of analytic (ϵ, n)-complements (that is, complements over an analytic neighbourhood of $P \in Z$) implies the boundedness of algebraic (ϵ, n)-complement because of the general GAGA principle [Sh1].

Lemma 3.2.17 *Let $Y/X/Z$ and $K_Y + B_Y$ be nef/X and $K_X + B := {}_*(K_Y + B_Y)$ be (ϵ, n)-complementary/Z. Moreover, assume that each* *nonexceptional/X component of B_Y that intersects an exceptional divisor/X has a standard coefficient then (Y, B_Y) will also be (ϵ, n)-complementary/Z.*

Proof See [PSh, 6.1].

Definition 3.2.18 (D-LMMP) Let D be an $\mathbb{R}$-Cartier divisor on a normal variety X. We say D-LMMP holds if the following hold:

D**-contraction** Any D-negative extremal ray R (in other words, $D \cdot R < 0$) on X can be contracted. And the same holds in the subsequent steps for the birational transform of D.

D**-flip** If $X \to Z$ is a small extremal D-negative contraction (flipping) as in the first step, then the corresponding D-flip exists, that is, there is a normal variety X^+ and a small extremal contraction $X^+ \to Z$ such that D^+, the birational transform of D, is $\mathbb{R}$-Cartier and ample/Z.

D**-termination** Any sequence of D-flips terminates.

By running *anti*-LMMP on a divisor D we mean $(-D)$-LMMP. If $D := K + B$ for a lc $\mathbb{R}$-Cartier divisor $K + B$, then D-LMMP holds in dim 3 by [Sh5].

Remark 3.2.19 Let D be an $\mathbb{R}$-Cartier divisor on a variety X of dim d and assume that LMMP_d holds. Moreover assume that $\beta D \sim_{\mathbb{R}} K + B$ for a lc $\mathbb{R}$-Cartier log divisor $K+B$ and $\beta > 0$. Then, D-LMMP holds since in this case D-LMMP and $(K + B)$-LMMP are equivalent. In particular, for any effective $\mathbb{R}$-Cartier divisor D on a Klt pair (X, B) the $K_X + B + D$-LMMP holds if $K_X + B + D$ is Klt.

Example 3.2.20 Let $(X/\mathrm{pt.}, B)$ be a d-dimensional Klt WLF where $B \in \mathbb{Q}$ and suppose LMMP_d holds. Then, $(-K)$-LMMP holds. By the assumptions and the base point free theorem [KMM] there is a Klt $(0, \mathbb{Q})$-complement $K+B^+ \sim_{\mathbb{Q}} 0$. There is a $t > 0$ such that $K+B^+ + tB^+ \sim_{\mathbb{Q}} tB^+$ is Klt. Since $-K \sim_{\mathbb{Q}} B^+$, $(-K)$-LMMP is equivalent to (B^+)-LMMP and in turn equivalent to (tB^+)-LMMP. Since $(K + B^+ + tB^+)$-LMMP holds so does $(-K)$-LMMP.

Example 3.2.21 Let $(X/\mathrm{pt.}, B)$ be a 2-dimensional Klt WLF. Then, for any divisor D on X the D-LMMP holds. Note that by definition X is $\mathbb{Q}$-factorial. Thus D is $\mathbb{R}$-Cartier. The WLF property guarantees D-contraction of any D-negative extremal ray on X [Sh3, Sh5]. The D-termination holds since the Picard number decreases after every contraction.

3.3 The case of curves

In this section we prove 3.1.7 for the case of curves. Note that 1-dimensional global weak log Fano pairs are just $(\mathbb{P}^1, B)$ for a boundary $B = \sum_i b_i B_i$ where $\sum_i b_i - 2 < 0$. The local case for curves is trivial.

Theorem 3.3.1 $\mathrm{WC}_{\delta,1,[0,1]}$ *holds; more precisely, suppose* $\frac{m-1}{m} \leq 1 - \delta < \frac{m}{m+1}$ *for* m *a natural number, then we have:*

- ◇ $\mathcal{N}_{\delta,1,[0,1]} = \bigcup_{0<k\leq m} \{k, k+1\}$.
- ◇ $(\mathbb{P}^1, B^+)$ *can be taken* $\frac{1}{m+1}$*-lc.*

Proof Let $B = \sum_i b_i B_i$, $b_h = \max\{b_i\}$ and $\frac{k-1}{k} \leq b < \frac{k}{k+1}$ for a natural number k. If $k = 1$, then $b < \frac{1}{2}$ and so we have a $(1,1)$-complement $K_{\mathbb{P}^1} + B^+$ where $B^+ = 0$. Since $\llcorner 2b_i \lrcorner = 0$.

Now assume that $k > 1$ and define $a_{i,t} = \llcorner (t+1)b_i \lrcorner$. Note that since $\sum_i b_i < 2$, then

$$\sum_i a_{i,k} = \sum_i \llcorner (k+1)b_i \lrcorner \leq \sum_i (k+1)b_i < 2k+2$$

If $K + B$ does not have a $(0,k)$-complement then $\sum_i a_{i,k} = 2k+1$. Since $\frac{k-1}{k} \leq b_h < \frac{k}{k+1}$ we have

$$\frac{(k+1)(k-1)}{k} = k + 1 - \frac{k+1}{k} = k - \frac{1}{k} \leq (k+1)b < \frac{(k+1)k}{k+1} = k$$

Thus $a_{h,k} = \llcorner (k+1)b_h \lrcorner = k-1$ and $1 - \frac{1}{k} \leq \langle (k+1)b_h \rangle < 1$ where $\langle \cdot \rangle$ stands for the fractional part.

Now $a_{i,k+1} = \llcorner (k+2)b_i \lrcorner = \llcorner (k+1)b_i + b_i \lrcorner$. So $a_{i,k+1}$ is equal to $a_{i,k}$ or $a_{i,k} + 1$. The latter happens iff $1 \leq b_i + \langle (k+1)b_i \rangle$. In particular, $b_h + \langle (k+1)b_h \rangle \geq \frac{k-1}{k} + 1 - \frac{1}{k} \geq 1$ so $a_{h,k+1} = a_{h,k} + 1$. On the other hand since

$$\sum_i a_{i,k} = \llcorner (k+1)b_i \lrcorner = 2k+1$$

and

$$\sum_i (k+1)b_i = \sum_i a_{i,k} + \sum_i \langle (k+1)b_i \rangle = 2k + 1 + \sum_i \langle (k+1)b_i \rangle < 2k+2$$

then $\sum_i \langle (k+1)b_i \rangle < 1$. Hence if $i \neq h$, then $\langle (k+1)b_i \rangle < \frac{1}{k}$ because $1 - \frac{1}{k} \leq \langle (k+1)b_h \rangle$. So if $i \neq h$ and $1 \leq \langle (k+1)b_i \rangle + b_i$, then $1 - \frac{1}{k} < b_i$.

If $K+B$ has no $(0,k+1)$-complement, then $\sum_i a_{i,k+1} = \llcorner(k+2)b_i\lrcorner = 2k+3$ therefore $1 \leq \langle(k+1)b_j\rangle + b_j$ must hold at least for some $j \neq h$. So $1-\frac{1}{k} < b_j \leq b_h$ which in turn implies $1-\frac{1}{k} \leq \langle(k+1)b_j\rangle$. Thus $1 \leq 2(1-\frac{1}{k}) \leq \langle(k+1)b_j\rangle + \langle(k+1)b_h\rangle$ and this is a contradiction. Hence $K+B$ has a $(0,k)$ or $(0,k+1)$-complement. If $K+B$ has a $(0,k)$-complement, then $\frac{\llcorner(k+1)b_h\lrcorner}{k} = 1-\frac{1}{k}$ and $\sum_i a_{i,k} \leq 2k$. If it has a $(0,k+1)$-complement, then $\frac{\llcorner(k+1)b_h\lrcorner}{k+1} \leq \frac{k}{k+1}$ and $\sum_i a_{i,k+1} \leq 2k+2$. Therefore, we can construct a $(\frac{1}{k},k)$-complement or a $(\frac{1}{k+1},k+1)$-complement $K_{\mathbb{P}^1}+B^+$ for $K_{\mathbb{P}^1}+B$. Since $0<k\leq m$, $\mathcal{N}_{\delta,1,[0,1]} = \bigcup_{0<k\leq m}\{k,k+1\}$. $\square$

3.4 The case of surfaces

We divide the surface case of Conjecture 3.1.7 into the following cases:

Local isomorphic $X \to Z$ is the identity.

Local birational $X \to Z$ is birational.

Local over curve Z is a curve.

Global $Z = \mathrm{pt}$.

3.5 Local isomorphic case

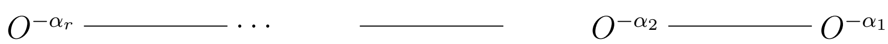
The main theorem in this section is Theorem 3.5.1. We use classification of surface singularities.

Theorem 3.5.1 *Conjecture* $\mathrm{WC}_{\delta,2,\{0\}}$ *(3.1.7) holds in the local isomorphic case, that is, when $X \to Z$ is the identity and $\Gamma = \{0\}$.*

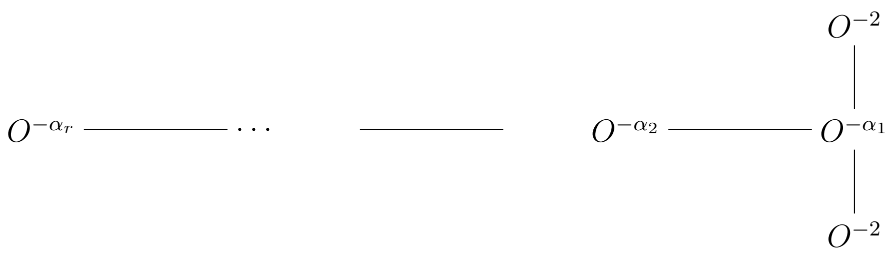
Proof Note that $(X, 0)$ is Klt$/P \in Z$ by assumptions of Conjecture 3.1.7 ($\delta > 0$). If $\delta > 1$, then X is smooth at P so we are already done. If $\delta = 1$ then X is canonical at P so K_X is Cartier. In this case K_X is a $(1,1)$-complement$/P$ for K_X. From now on we assume that $\delta < 1$.

If the singularity at P is of type E_6, E_7 or E_8, then there are only a finite number of possibilities for such singularities up to analytic isomorphism because of the δ-lc assumption [Pr, 6.1.2].

If the singularity at P is of type A_r, then the graph of the resolution is as



where $\alpha_i \geq 2$. If the singularity at P is of type D_r, then the graph is as



where $\alpha_i \geq 2$.

$\mathbf{A}_r$ case: Let $K_W - \sum_i e_i E_i = {}^*K_Z$ where e_i are the discrepancies for a log resolution $W \to Z$ near P. The following lemma is well known and a proof can be found in [AM, 1.2].

Lemma 3.5.2 *The numbers $(-E_i^2)$ are bounded from above in terms of δ.*

□

By computing the intersection numbers $(K_W - \sum_i e_i E_i) \cdot E_j$ we get the following system:

$$\begin{cases} a_1(-E_1^2) - a_2 - 1 = 0 \\ a_2(-E_2^2) - a_1 - a_3 = 0 \\ a_3(-E_3^2) - a_2 - a_4 = 0 \\ \vdots \\ a_{r-1}(-E_{r-1}^2) - a_{r-2} - a_r = 0 \\ a_r(-E_r^2) - a_{r-1} - 1 = 0 \end{cases}$$

where a_i is the log discrepancy of E_i with respect to K_Z.

From the equation $a_i(-E_i^2) - a_{i-1} - a_{i+1} \leq 0$ we get the inequality $a_i(-E_i^2 - 2) + a_i - a_{i-1} \leq a_{i+1} - a_i$ which shows that if $a_{i-1} \leq a_i$, then $a_i \leq a_{i+1}$ and moreover if $a_{i-1} < a_i$ then $a_i < a_{i+1}$. So the solution for the system above must satisfy the following:

$$a_1 \geq \cdots \geq a_i \leq \cdots \leq a_r \tag{3.5.2.1}$$

for some $i \geq 1$. If $r \leq 2$ (or any fixed number), then the theorem is trivial. So we may assume that $r > 3$ and also can assume $i \neq r$ unless $a_1 = a_2 = \cdots = a_r$. Now, for any $i \leq j < r$, if $-E_j^2 > 2$, then $a_{j+1} - a_j \geq a_j(-E_j^2 - 2) \geq \delta$. Hence if $l := \#\{j \mid -E_j^2 > 2 \text{ and } i \leq j < r\}$, then $a_r \geq l\delta$. Thus $a_r(-E_r^2 - 1) + a_r - a_{r-1} \geq l\delta$, which contradicts the last equation in the system for l large enough. In any case, $l\delta \leq 1$ and $l \leq \frac{1}{\delta}$, so l is bounded .

Similarly, we deduce that $l' := \#\{j \mid -E_j^2 > 2 \text{ and } 1 \leq j \leq i\}$ is bounded. Then, $l + l' \leq \frac{2}{\delta}$.

Now suppose that $a_{i_2} = \cdots = a_i = \cdots = a_{i_1}$, $a_{i_1-1} \neq a_{i_1}$ (unless $i_1 = 1$) and $a_{i_2} \neq a_{i_2+1}$ (unless $i_2 = r$) where $i_2 \leq i \leq i_1$. Assume that $i_1 \neq i$ or $i_2 \neq i$. If $i_1 \neq i$ and all a_j are not equal (to 1), then we have

$$
\begin{aligned}
1 = (-E_r^2 - 1)a_r + a_r - a_{r-1} \\
&\geq (r - i_1)(a_{i_1+1} - a_{i_1}) \\
&= (r - i_1)[(-E_{i_1}^2 - 2)a_{i_1} + a_{i_1} - a_{i_1-1}] \\
&= (r - i_1)(-E_{i_1}^2 - 2)a_{i_1} \geq (r - i_1)\delta
\end{aligned}
$$

because $-E_{i_1}^2$ cannot be equal to 2.

So $(r - i_1)\delta \leq 1$ which in turn implies that $r - i_1 \leq \frac{1}{\delta}$ is bounded. Similarly, we deduce that i_2 is bounded.

These observations show that, given that all $-E_k^2$ are bounded, the denominators of a_k are bounded. Therefore, the index of K_Z at P is bounded and so we are done in this case.

But if $i_1 = i = i_2$, then the situation is different. Note that in this case $\delta \leq (-E_i^2 - 2)a_i = a_{i-1} - a_i + a_{i+1} - a_i$. Hence $\frac{\delta}{2} \leq a_{i-1} - a_i$ or $\frac{\delta}{2} \leq a_{i+1} - a_i$. If $\frac{\delta}{2} \leq a_{i+1} - a_i$, then similar to the calculations we just carried out above, $r - i$ is bounded. But it can happen that $a_{i-1} - a_i$ is very small so we will not be able to bound i. The same argument applies to the case $\frac{\delta}{2} \leq a_{i-1} - a_i$.

Actually, we try to find a solution with bounded denominators for the following system:

$$
\begin{cases}
u_1(-E_1^2) - u_2 - 1 \leq 0 \\
u_2(-E_2^2) - u_1 - u_3 \leq 0 \\
u_3(-E_3^2) - u_2 - u_4 \leq 0 \\
\vdots \\
u_{r-1}(-E_{r-1}^2) - u_{r-2} - u_r \leq 0 \\
u_r(-E_r^2) - u_{r-1} - 1 \leq 0
\end{cases}
$$

To find such a solution, note that if $-E_{i-1}^2 > 2$, then $\delta \leq (-E_{i-1}^2 - 2)a_{i-1} = a_{i-2} - a_{i-1} + a_i - a_{i-1} \leq a_{i-2} - a_{i-1}$. Hence similar computations to the above show that i is bounded. Now let j be the smallest number such that $-E_j^2 = \cdots = -E_{i-1}^2 = 2$ (remember that we have assumed $\frac{\delta}{2} \leq a_{i+1} - a_i$). Hence j is bounded. Now take $u_j = \cdots = u_i = \frac{1}{I}$ for a natural number I. Then, the following equations are satisfied if $i - j > 2$:

$$
\begin{cases}
u_{j+1}(-E_{j+1}^2) - u_j - u_{j+2} = 2u_j - u_j - u_j = 0 \\
\vdots \\
u_{i-1}(-E_{i-1}^2) - u_{i-2} - u_i = 2u_{i-1} - u_{i-2} - u_i = 0
\end{cases}
$$

Since $r - i$ and j are bounded the number of remaining equations is bounded. Therefore, there is a bounded I such that there is a solution $(u_1, \ldots, u_r)$ where $u_j = \cdots = u_i = \frac{1}{I}$. This completes the proof of A_r case.

Form the solution $(u_1, \ldots, u_r)$ we construct a Klt log divisor $K_W + D$ with bounded index such that $-(K_W + D)$ is nef and big$/P \in Z$. Now we may use Remark 3.2.16. This completes the proof of A_r case.

Remark 3.5.3 In Shokurov's case [Sh1, §5], where $\delta = \epsilon = 0$, we just take $u_1 = \cdots = u_r = 0$.

$\mathbf{D}_r$ case: We have a chain $E_1, \ldots, E_r$ of exceptional divisors together with E and E', where E and E' intersect only E_1. In this case we have the following system:

$$\begin{cases} a(-E^2) - a_1 - 1 = 0 \\ a'(-E'^2) - a_1 - 1 = 0 \\ a_1(-E_1^2) - a - a' - a_2 + 1 = 0 \\ a_2(-E_2^2) - a_1 - a_3 = 0 \\ a_3(-E_3^2) - a_2 - a_4 = 0 \\ \vdots \\ a_{r-1}(-E_{r-1}^2) - a_{r-2} - a_r = 0 \\ a_r(-E_r^2) - a_{r-1} - 1 = 0 \end{cases}$$

Note that $-E^2 = -E'^2 = 2$, so $2a - a_1 - 1 = 0$ and $2a' - a_1 - 1 = 0$. Hence $a + a' = a_1 + 1$ and the third equation becomes $a_1(-E_1^2 - 1) - a_2 = 0$. We now consider the system obtained from the last system after ignoring the first two equations:

$$\begin{cases} a_1(-E_1^2 - 1) - a_2 = 0 \\ a_2(-E_2^2) - a_1 - a_3 = 0 \\ a_3(-E_3^2) - a_2 - a_4 = 0 \\ \vdots \\ a_{r-1}(-E_{r-1}^2) - a_{r-2} - a_r = 0 \\ a_r(-E_r^2) - a_{r-1} - 1 = 0 \end{cases}$$

Any solution of this system satisfies the following:

$$a_1 = \cdots = a_i < a_{i+1} < \cdots < a_r$$

If $i = r$, then $a = a' = a_1 = \cdots = a_r = 1$. So we may assume $i < r$. We show that $r - i$ is bounded. In this case, if $i > 1$, then $-E_1^2 = \cdots = -E_{i-1}^2 = 2$ but $-E_i^2 > 2$. Now $\delta(-E_i^2 - 2) \leq a_i(-E_i^2 - 2) + a_i - a_{i-1} = a_{i+1} - a_i$ (if $i = 1$ then $\delta(-E_1^2 - 2) \leq a_1(-E_1^2 - 2) = a_2 - a_1$). We also have $a_{k+1} - a_k \leq a_{k+2} - a_{k+1}$ for $i \leq k < r - 1$. On the other hand, $\sum_{i \leq k < r} a_{k+1} - a_k \leq a_r < a_r + a_r - a_{r-1} < 1$. So we conclude that $r - i$ is bounded.

Moreover since $-E_k^2$ is bounded, this proves that the denominators of all a_k in the D_r case are bounded hence the index of K_Z at P is bounded. In this case $B^+ = 0$ and we complete the proof of Theorem 3.5.1.

□

Remark 3.5.4 All the bounds occurring in the proof are effective and can be calculated in terms of δ.

Remark 3.5.5 Essentially, the boundedness properties that we proved and used in the proof of Theorem 3.5.1 have been more or less discovered by other mathematicians independently. In particular, Shokurov has used these ideas in an unpublished preprint on mlds [Sh8].

Remark 3.5.6 Here we recall the diagrams for the E_6, E_7 and E_8 types of singularities [Pr, 6.1.2]. The following is a general case of such singularities:

$$\begin{array}{ccccc} \mathbb{C}^2/\mathbb{Z}_{m_1} & \text{——} & O^{-p} & \text{——} & \mathbb{C}^2/\mathbb{Z}_{m_2} \\ & & | & & \\ & & O^{-2} & & \end{array}$$

where the only possibilities for (m_1, m_2) are $(3, 3)$, $(3, 4)$ and $(3, 5)$. So the only possible diagrams are as follows: For $(m_1, m_2) = (3, 3)$ we have

1

$$\begin{array}{ccccc} O^{-3} & \text{——} & O^{-p} & \text{——} & O^{-3} \\ & & | & & \\ & & O^{-2} & & \end{array}$$

2

$$\begin{array}{ccccccc} O^{-2} & \text{——} & O^{-2} & \text{——} & O^{-p} & \text{——} & O^{-3} \\ & & & & | & & \\ & & & & O^{-2} & & \end{array}$$

3

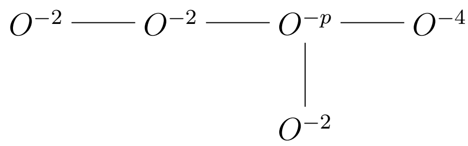
$$\begin{array}{ccccccccc} O^{-2} & \text{——} & O^{-2} & \text{——} & O^{-p} & \text{——} & O^{-2} & \text{——} & O^{-2} \\ & & & & | & & & & \\ & & & & O^{-2} & & & & \end{array}$$

For $(m_1, m_2) = (3, 4)$ we have

4

$$\begin{array}{ccccc} O^{-3} & \text{——} & O^{-p} & \text{——} & O^{-4} \\ & & | & & \\ & & O^{-2} & & \end{array}$$

5



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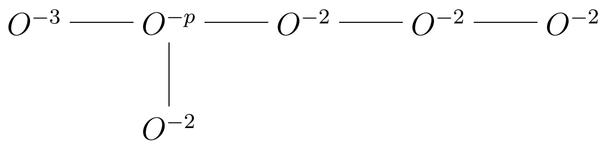

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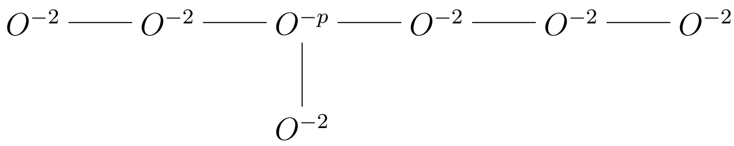
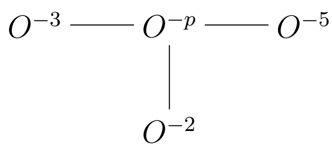


Finally for $(m_1, m_2) = (3, 5)$ we have

8



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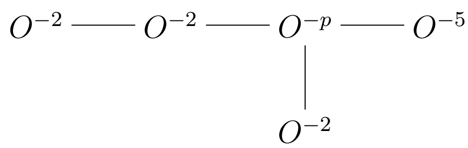

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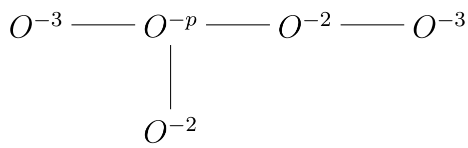

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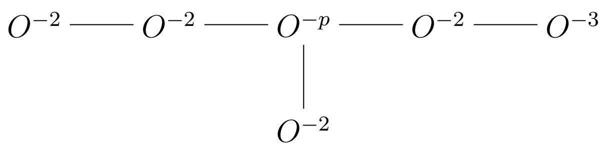
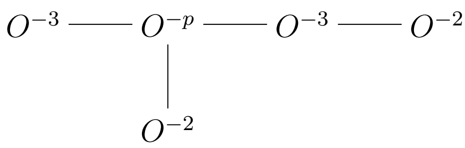


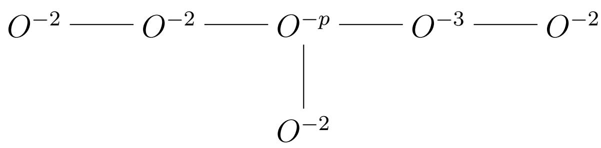
O−2
O−2
O−p
O−2
O−3
O−2

12



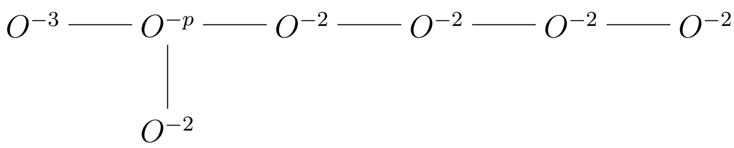
O−3
O−p
O−3
O−2
O−2

13



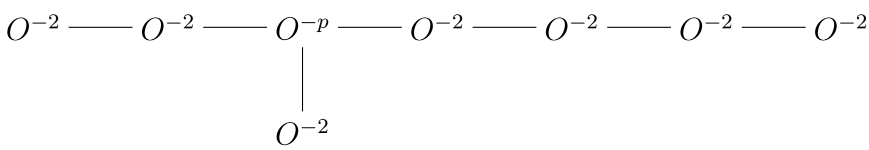
O−2
O−2
O−p
O−3
O−2
O−2

14



O−3
O−p
O−2
O−2
O−2
O−2
O−2

15



O−2
O−2
O−p
O−2
O−2
O−2
O−2
O−2

3.6 Local birational case

In this section whenever we write $/Z$ we mean $/P \in Z$ for a fixed point P on Z.

Theorem 3.6.1 *Conjecture* $\mathrm{WC}_{\delta,2,\{0\}}$ *(3.1.7) holds in the birational case, that is, when* $X \to Z$ *is birational and* $\Gamma = \{0\}$.

Strategy of the proof: Let W be a minimal resolution of X and let $\{E_i\}$, $\{F_j\}$ be the exceptional divisors $/Z$ on W where the E_i are exceptional$/X$ but F_j are not. We use the notation E for a typical E_i and similarly F for F_j or its birational transform). We construct an antinef$/Z$ and Klt log divisor $K_W + \Omega = K_W + \sum_i u_i E_i + \sum_j u_j F_j$ where $u_i, u_j < 1$ are rational numbers with bounded denominators. Then, we use Remark 3.2.16.

Proof By contracting those curves where $-K_X$ is numerically zero, we can assume that $-K_X$ is ample$/Z$ (we can pull back the complement). Let W be the minimal resolution of X. Then, since K_W is nef$/X$ by the negativity lemma we have $K_W - \sum_i e_i E_i = K_W + \sum_i (1 - a_i) E_i \equiv {}^*K_X$ where $e_i \leq 0$.

Definition 3.6.2 For any smooth model Y where $W/Y/Z$ we define $\overline{\mathrm{exc}}(Y/Z)$ to be the graph of the exceptional curves ignoring the birational transform of exceptional divisors of type F. For an exceptional$/Z$ divisor G on Y not of type F, $\overline{\mathrm{exc}}(Y/Z)_G$ means the connected component of $\overline{\mathrm{exc}}(Y/Z)$ where G belongs to.

Lemma 3.6.3 *We have the following on* W:

◇ *The exceptional divisors/*Z *on* W *are with simple normal crossings.*

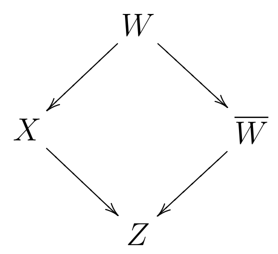
◇ *Each F (that is, each exceptional divisor of type F) is a -1-curve.*

◇ *The model $\overline{W}$ obtained by blowing down -1-curves$/Z$ is the minimal resolution of Z.*

◇ *Each F cuts at most two exceptional divisors of type E.*

Proof Let F be an exceptional divisor$/Z$ on W which is not exceptional$/X$. Then, $(K_W - \sum_i e_i E_i) \cdot F = K_W \cdot F + \sum_i (-e_i) E_i \cdot F = 2p_a(F) - 2 - F^2 + \sum_i (-e_i) E_i \cdot F < 0$ where $p_a(F)$ stands for the arithmetic genus of the curve F. Then, $2p_a(F) - 2 - F^2 < 0$ and so $p_a(F) = 0$ and $-F^2 = 1$. In other words F is a -1-curve.
On the other hand by contracting -1-curves$/Z$ (i.e. running the classical minimal model theory for smooth surfaces on W/Z) we get a model $\overline{W}/Z$ where $K_{\overline{W}}$ is nef$/Z$. Actually $\overline{W}$ is the minimal resolution of $P \in Z$.

The exceptional divisors$/Z$ on $\overline{W}$ are with simple normal crossings and since W is obtained from $\overline{W}$ by a sequence of blow ups, then the exceptional divisors$/Z$ on W are also with simple normal crossings. Further, since all the F, exceptional$/Z$ but not$/X$, are contracted$/\overline{W}$ then they can intersect at most two of E_i because exc$(\overline{W}/Z)$ is with simple normal crossings and F is exceptional$/\overline{W}$.



$\square$

Moreover no two exceptional divisors of type F can intersect on W because they are both -1-curves. This means that the intersection points of any two exceptional divisor$/Z$ on X is a singular point of X. Also any exceptional divisor$/Z$ on X contains at most two singular points of X.

Let $\{Q_k\}_k$ be the singular points of X. If none of the points $\{Q_k\}$ is of type A_r, then the proof of Theorem 3.5.1 shows that the index of K_X is bounded so we are done. But if there is one point of type A_r, then the proof is more complicated. Surprisingly, the A_r type is the most simple case in the sense of Shokurov, that is, when $\delta = 0$ (see Remark 3.5.3). Similar to the proof of Theorem 3.5.1 we try to understand the structure of $\mathrm{exc}(W/Z)$ and the blow ups $W \to \overline{W}$.

Definition 3.6.4 A smooth model $\ddot{W}$ where $W/\ddot{W}$ and $\ddot{W}/\overline{W}$ are series of smooth blow ups, is called a *blow up model* of $\overline{W}$. Such a model is *perfect* if there is X' such that $K_{\ddot{W}}$ is nef$/X'$ and $X/X'/Z$. In other words, it is the minimal resolution of X'. The connected components of $\overline{\mathrm{exc}}(\ddot{W}/Z)$ are either of type A_r, D_r, E_6, E_7 or E_8 for a perfect blow up model.

Definition 3.6.5 We call the divisor $K_W + \omega = K_W + \sum_i (1 - a_i)E_i = {}^*K_X$ the *primary log divisor*. The pair (X, B) has a $(0, n)$-complement $K_X + B^+$ over Z by Shokurov [Sh2] ($n \in \{1, 2, 3, 4, 6\}$). From now on we call it a *Shokurov complement*. So $K_W + \omega_{Sh} + C = K_W + \sum_i (1 - a_i^{Sh})E_i + \sum_j (1 - a_j^{Sh})F_j + C = {}^*(K_X + B^+)$ where C is the birational transform of the nonexceptional part of B^+. We call $K_W + \omega_{Sh}$ a *Shokurov log divisor* and the numbers a_i^{Sh} and a_j^{Sh} *Shokurov log discrepancies.*

Definition 3.6.6 Consider the graph $\text{exc}(W/Z)$. If we ignore those F that appear with zero coefficient in ω_{Sh} (that is, $a^{Sh} = 1$), then we get a graph $\text{exc}(W/Z)_{>0}$ with some connected components. The connected graph $\mathcal{C}$ consisting of exceptional/Z curves with $a^{Sh} = 0$, belong to one of the components of the graph $\text{exc}(W/Z)_{>0}$ which we show by $\mathcal{G}$ ($\mathcal{C}$ is connected because of the connectedness of the locus of log canonical centres/$P \in Z$). Now contracting all -1-curves/Z in $\mathcal{G}$ and continuing the contractions of subsequent -1-curves/Z which appear in $\mathcal{G}$, we finally get a model which we denote by $W_{\mathcal{G}}$. The transform of $\mathcal{G}$ on $W_{\mathcal{G}}$ is denoted by $\mathcal{G}_1$ and similarly the transform of $\mathcal{C}$ is $\mathcal{C}_1$.

Definition 3.6.7 A chain of exceptional curves consisting of $G_{\beta_1}, \dots, G_{\beta_r}$ is called *strictly monotonic* if $r = 1$ or if $a_{\beta_1} < a_{\beta_2} < \cdots < a_{\beta_r}$ (these are log discrepancies with respect to K_X). G_{β_1} is called the *base curve.*

Definition 3.6.8 Let $G \in \text{exc}(\ddot{W}/Z)$ for a smooth blow up model $\ddot{W}$. Then, we define the *negativity* of G on this model as $N_{\ddot{W}}(G) = (K_{\ddot{W}} + {}_*\omega) \cdot G \leq 0$. We also define the *total negativity* by $N_{\ddot{W}} = \sum_\alpha N_{\ddot{W}}(G_\alpha)$ where G_α runs over all exceptional divisors/Z on $\ddot{W}$. For $G \in \overline{\text{exc}}(\ddot{W}/Z)$ we define $N_{\ddot{W},G} = \sum_\alpha N_{\ddot{W}}(G_\alpha)$ where the sum runs over all members of $\overline{\text{exc}}(\ddot{W}/Z)_G$. Similarly, we define the negativity functions N^{Sh} and N^+ replacing ω with ω_{Sh} and $\omega_{Sh} + C$ respectively. Note that the latter is always zero, because $K_W + \omega_{Sh} + C \equiv 0/Z$.

Definition 3.6.9 Let $\ddot{W}/Z$ be a smooth blow up model and $\xi \in \ddot{W}$. If ξ belongs to two exceptional divisors/Z, then the blow up at ξ is a *double blow up.* If ξ belongs to just one exceptional divisor/Z, then the blow up at

ξ is a *single blow up*. If ξ belongs to two components of $_*(\omega_{Sh}+C)$, then the blow up at ξ is a *double*$^+$ *blow up*. If ξ belongs to just one component of $_*(\omega_{Sh}+C)$, then the blow up at ξ is a *single*$^+$ *blow up*.

Lemma 3.6.10 *For any exceptional $G_\beta \in \overline{\text{exc}}(\ddot{W}/Z)$ on a blow up model $\ddot{W}$ we have:*

- ◇ $-1+\delta \leq N_{\ddot{W},G_\beta}$ *if* $\overline{\text{exc}}(\ddot{W}/Z)_{G_\beta}$ *is of type* D_r, E_6, E_7 *or* E_8. *In particular, in these cases* $-1+\delta \leq N_{\ddot{W}}(G_\beta)$ *holds.*

- ◇ $2(-1+\delta) \leq N_{\ddot{W},G_\beta}$ *and* $-1+\delta \leq N_{\ddot{W}}(G_\beta)$ *if* $\overline{\text{exc}}(\ddot{W}/Z)_{G_\beta}$ *is of type* A_r *unless it is strictly monotonic.*

Proof D_r case: Similar to the notation in the proof of Theorem 3.5.1 let $G_\beta, G_{\beta'}, G_{\beta_1}, \ldots, G_{\beta_r}$ be the exceptional divisors in $\overline{\text{exc}}(\ddot{W}/Z)_{G_\beta}$. Then, from the equations in the proof of Theorem 3.5.1 for the D_r case we get the following system for the log discrepancies:

$$\begin{cases} 2a_\beta - a_{\beta_1} - 1 \leq 0 \\ 2a_{\beta'} - a_{\beta_1} - 1 \leq 0 \\ 2a_{\beta_1} - a_\beta - a_{\beta'} - a_{\beta_2} + 1 \leq 0 \\ 2a_{\beta_2} - a_{\beta_1} - a_{\beta_3} \leq 0 \\ \vdots \\ 2a_{\beta_{r-1}} - a_{\beta_{r-2}} - a_{\beta_r} \leq 0 \\ 2a_{\beta_r} - a_{\beta_{r-1}} - 1 \leq 0 \end{cases}$$

Adding the first and the second inequalities gives $2a_\beta+2a_{\beta'}-2a_{\beta_1}-2\leq 0$. Accordingly, the third inequality becomes $a_{\beta_1}\leq a_{\beta_2}$ and so $a_{\beta_1}\leq a_{\beta_2}\leq \cdots \leq$

a_{β_r}. Therefore

$$\begin{aligned} N_{\ddot{W},G_\beta} & \\ &\geq a_\beta + a_{\beta'} + a_{\beta_r} - a_{\beta_1} - 2 \\ &\geq a_\beta + a_{\beta'} + a_{\beta_2} - a_{\beta_1} - 2 \\ &\geq 2a_{\beta_1} + 1 - a_{\beta_1} - 2 \\ &\geq a_{\beta_1} - 1 \geq \delta - 1 \end{aligned}$$

because $2a_{\beta_1} + 1 \leq a_\beta + a_{\beta'} + a_{\beta_2}$ and X is δ-lc.

A_r case (nonstrictly monotonic): In this case assume that the exceptional divisors in $\overline{\mathrm{exc}}(\ddot{W}/Z)_{G_\beta}$ are $G_{\beta_1}, \dots, G_{\beta_r}$. We get the system:

$$\begin{cases} 2a_{\beta_1} - a_{\beta_2} - 1 \leq 0 \\ 2a_{\beta_2} - a_{\beta_1} - a_{\beta_3} \leq 0 \\ \vdots \\ 2a_{\beta_{r-1}} - a_{\beta_{r-2}} - a_{\beta_r} \leq 0 \\ 2a_{\beta_r} - a_{\beta_{r-1}} - 1 \leq 0 \end{cases}$$

So there will be k such that $a_{\beta_1} \geq a_{\beta_2} \geq \cdots \geq a_{\beta_k} \leq a_{\beta_r}$. Thus $N_{\ddot{W}}(G_{\beta_1}) \geq a_{\beta_1} + a_{\beta_1} - a_{\beta_2} - 1 \geq a_{\beta_1} - 1 \geq \delta - 1$. In this way we get the similar inequalities for all other inequalities except for $N_{\ddot{W}}(G_{\beta_k})$. Suppose $N_{\ddot{W}}(G_{\beta_k}) < \delta - 1$. So we get $2a_{\beta_k} - a_{\beta_{k-1}} - a_{\beta_{k+1}} < \delta - 1$ and so $1 - \delta < a_{\beta_{k-1}} + a_{\beta_{k+1}} - 2a_{\beta_k} \leq a_{\beta_1} + a_{\beta_r} - 2a_{\beta_k}$. On the other hand by adding all the inequalities in the system we get $N_{\ddot{W},G_\beta} \geq a_{\beta_1} + a_{\beta_r} - 2 > 1 - \delta + 2a_{\beta_k} - 2 \geq \delta - 1$. This contradicts the fact that $N_{\ddot{W}}(G_{\beta_k}) \geq N_{\ddot{W},G_{\beta_k}}$.

To get the inequality for $N_{\ddot{W},G_{\beta_k}}$ add all the equations in the system above. Note that if $r = 2$, then $a_{\beta_1} = a_{\beta_2}$ and lemma is immediate.

E_6, E_7, E_8 cases:[3.6.10.1] In these cases the graph $\overline{\text{exc}}(\ddot{W}/Z)_{G_\beta}$ is as in Remark 3.5.6. It is enough to substitute 2 for all the self-intersection numbers because the negativity becomes smaller. We start from the smallest possible graph, that is, case 1 in 3.5.6.

$$\begin{cases} 2a_\beta - a_{\beta_2} - 1 \leq 0 \\ 2a_{\beta_2} - a_\beta - a_{\beta_1} - a_{\beta_3} + 1 \leq 0 \\ 2a_{\beta_1} - a_{\beta_2} - 1 \leq 0 \\ 2a_{\beta_3} - a_{\beta_2} - 1 \leq 0 \end{cases}$$

Adding all inequalities we get $N_{\ddot{W},G_\beta} = a_\beta + a_{\beta_1} + a_{\beta_3} - a_{\beta_2} - 2$. By the second inequality we have $a_\beta + a_{\beta_1} + a_{\beta_3} - a_{\beta_2} \geq a_{\beta_2} + 1$, so $N_{\ddot{W},G_\beta} \geq a_{\beta_2} + 1 - 2 \geq \delta - 1$. In fact, this was a special case of the D_r type inequalities (the similarity of the system not necessarily the graph $\overline{\text{exc}}(\ddot{W}/Z)_{G_\beta}$). Note that the inequality for the total negativity implies the inequality for the negativity of each exceptional curve.

Now we prove the other cases by induction on the number of the exceptional curves. The minimum is four exceptional curves and we have just proved this case. Suppose we have proved the lemma for graphs with $\leq k-1$ exceptional curves and that our graph has k members. Let the exceptional curves be $G_\beta, G_{\beta_1}, \ldots, G_{\beta_{k-1}}$ and such that G_{β_l} cuts $G_\beta, G_{\beta_{l-1}}$ and $G_{\beta_{l+1}}$. If $l = 2$ or $l = k-2$, then we obtain again a system of type D_r. Otherwise,

[3.6.10.1] I only prove that $-1 + \delta \leq N_{\ddot{W}}(G)$ for any exceptional G. We will not need the inequality for total negativity.

since $-a_\beta + 1 \geq 0$ we get a system as follows

$$\begin{cases} 2a_{\beta_1} - a_{\beta_2} - 1 \leq 0 \\ 2a_{\beta_2} - a_{\beta_3} - a_{\beta_1} \leq 0 \\ \vdots \\ 2a_{\beta_{k-1}} - a_{\beta_{k-2}} - 1 \leq 0 \end{cases}$$

This is a system of type A_{k-1}, so we have either $a_{\beta_1} \geq a_{\beta_2}$ or $a_{\beta_{k-1}} \geq a_{\beta_{k-2}}$. We study the first case (the other case being similar). Now note that $N_{\ddot{W}}(G_{\beta_1}) \geq 2a_{\beta_1} - a_{\beta_2} - 1 = a_{\beta_1} - a_{\beta_2} + a_{\beta_1} - 1 \geq \delta - 1$. By ignoring G_{β_1} we get a system for a graph with a smaller number of elements:

$$\begin{cases} 2a_{\beta_2} - a_{\beta_3} - 1 \leq 2a_{\beta_2} - a_{\beta_3} - a_{\beta_1} \leq 0 \\ \vdots \\ 2a_{\beta_l} - a_{\beta_{l-1}} - a_{\beta_{l+1}} - a_\beta + 1 \leq 0 \\ \vdots \\ 2a_{\beta_{k-1}} - a_{\beta_{k-2}} - 1 \leq 0 \end{cases}$$

and the lemma is proved by induction. □

Lemma 3.6.11 *Suppose $\xi \in \ddot{W}/\overline{W}$ ($\ddot{W}$ is a blow up model). Let $\tilde{W}$ be the blow up of $\ddot{W}$ at ξ and G_α the exceptional divisor of the blow up. Then,*

If G_α is the double blow up of G_β and G_γ (that is, $\xi \in G_\beta \cap G_\gamma$), then:

- ◇ $N_{\tilde{W}}(G_\alpha) = a_\alpha - a_\beta - a_\gamma$ *where a_α is the log discrepancy of G_α for K_X and similarly a_β and a_γ.*
- ◇ $N_{\tilde{W}}(G_\beta) = N_{\ddot{W}}(G_\beta) - N_{\tilde{W}}(G_\alpha)$ *and* $N_{\tilde{W}}(G_\gamma) = N_{\ddot{W}}(G_\gamma) - N_{\tilde{W}}(G_\alpha)$.
- ◇ $N_{\tilde{W}} = N_{\ddot{W}} - N_{\tilde{W}}(G_\alpha)$.

If G_α is the single blow up of G_β, then:

◇ $N_{\tilde{W}}(G_\beta) = N_{\ddot{W}}(G_\beta) - N_{\tilde{W}}(G_\alpha)$, $N_{\tilde{W}}(G_\alpha) = a_\alpha - a_\beta - 1 \leq -\delta$ *and* $N_{\ddot{W}}(G_\beta) + \delta \leq 0$.

◇ $N_{\tilde{W}} = N_{\ddot{W}}$.

Proof Standard computations. □

Corollary 3.6.12 *Let $\ddot{W}$ be a blow up model/$\overline{W}$. If G_α is a single blow up of G_β on $\ddot{W}$ and $N_{\ddot{W}}(G_\beta) \geq \delta - 1$, then $a_\alpha \geq a_\beta + \delta$.*

Proof Since G_α is a single blow up of G_β, $1 + a_\beta - a_\alpha + N_{\ddot{W}}(G_\beta) \leq 0$ and so $1 + a_\beta - a_\alpha + \delta - 1 \leq 0$. Therefore, $a_\beta + \delta \leq a_\alpha$.

Definition 3.6.13 Let ξ be a point on a blow up model $\ddot{W}$. Define the *multiplicity of double blow ups* as

$$\mu_{db}(\xi) = \max\left\{\#\{\text{double blow ups}/\xi \text{ before having a single blow up}/\xi\}\right\}$$

where the maximum is taken over all sequences of blow ups from $\ddot{W}$ to W. The next lemma shows the boundedness of this number.

Lemma 3.6.14 *$\mu_{db}(\xi)$ is bounded.*

Proof By Lemma 3.6.11 each double blow up adds a non-negative number to the total negativity of the system. Moreover, the total negativity is bounded because the total negativity on $\overline{W}$ is bounded.[3.6.14.1] Therefore, except for a bounded number of double blow ups, we have

$$\frac{-\delta}{2} \leq N_{\tilde{W}}(G_\alpha) = a_\alpha - a_\beta - a_\gamma \leq 0$$

where G_α is the double blow up of some G_β and G_γ and $G_\beta \cap G_\gamma/\xi$. The inequality shows that $a_\beta + \frac{\delta}{2} \leq a_\beta + a_\gamma - \frac{\delta}{2} \leq a_\alpha$ and similarly $a_\gamma + \frac{\delta}{2} \leq a_\alpha$. In other words the log discrepancy is increasing at least by $\frac{\delta}{2}$. Since log discrepancies are in $[\delta, 1]$, the number of these double blow ups has to be bounded. □

Definition 3.6.15 Let $\xi \in \ddot{W}/W$ a blow up model. Define the *single blow up multiplicity* of ξ as:

$$\mu_{sb}(\xi) = \max\left\{\#\{G\colon G \text{ is the exceptional divisor of a single blow up}/\xi\}\right\}$$

The maximum is taken over all sequences of blow ups from $\ddot{W}$ to W. Moreover, define $\mu_{sb}(G_\beta) = \sum_{\xi \in G_\beta} \mu_{sb}(\xi)$ and $\mu_{sb}(\ddot{W}) = \sum_{\xi \in \ddot{W}} \mu_{sb}(\xi)$.

So, if ξ_2/ξ_1 (these points may be on different models), then $\mu_{sb}(\xi_1) \geq \mu_{sb}(\xi_2)$.

Remark 3.6.16 Usually there is not a unique sequence of blow ups from $\ddot{W}$ to W. In fact, if ξ_1, ξ_2 are distinct points on $\ddot{W}$ and they are centres of some exceptional divisors on W, then it does not matter which one we blow up first in order to get to W.

[3.6.14.1] This boundedness for the A_r and D_r cases is shown in Lemma 3.6.10 and in the other cases it is obvious.

Definition 3.6.17 Let $\xi \in \overline{\text{exc}}(\ddot{W}/Z)$ be a point on a blow up model $\ddot{W}$. We call such a point a *generating point* if there is an exceptional divisor G_α/ξ on a blow up model $\tilde{W}$ such that $N_{\tilde{W}}(G_\alpha) < \delta - 1$.

Remark 3.6.18 By Lemma 3.6.10 and Lemma 3.6.11 if $\xi \in \overline{\text{exc}}(\ddot{W}/Z)_{G_\beta}$ and $\overline{\text{exc}}(\ddot{W}/Z)_{G_\beta}$ is of type A_r (non-strictly monotonic) , D_r, E_6, E_7 or E_8, then ξ can not be a generating point. Moreover, again by Lemma 3.6.10, if $\overline{\text{exc}}(\ddot{W}/Z)_{G_\beta}$ is strictly monotonic, then there can be at most one generating point in $\overline{\text{exc}}(\ddot{W}/Z)_{G_\beta}$ and it can only belong to the base curve.

Lemma 3.6.19 *$\mu_{sb}(\xi)$ is bounded if $\xi \in \ddot{W}$ is not a generating point.*

Proof If G_α/ξ is a single blown up exceptional divisor, then $\delta - 1 \leq N_{\tilde{W}}(G_\alpha)$ since ξ is not a generating point. So if G_α is a single blow up/ξ of G_β, then $a_\alpha \geq a_\beta + \delta$, that is, it increases the log discrepancy at least by δ. Moreover, as in the proof of Lemma 3.6.14, except for a bounded number of double blow ups, any double blow up/ξ increases the log discrepancy at least by $\frac{\delta}{2}$. Hence, there can be only a bounded number of blow ups/ξ from $\ddot{W}$ to W. □

Corollary 3.6.20 *The number of exceptional curves/ξ on W is bounded for any nongenerating point $\xi \in \ddot{W}$.*

We now continue the proof of Theorem 3.6.1. If no divisor in ω_{Sh} has coefficient 1, then this is what we are looking for. Since in this case $K_W + \omega_{Sh}$ will be a $\frac{1}{6}$-lc log divisor. If the opposite happens,that is, some divisors appear with coefficient 1 in ω_{Sh}, then these divisors will form a connected chain $\mathcal{C}$

which does not intersect with any other exceptional divisor/Z with positive coefficient in ω_{Sh}, except the edges of this chain. Some of the exceptional divisors of type F may appear with positive coefficients and some with zero coefficients in ω_{Sh}.

The image of the graph $\mathcal{G}$ on $W_{\mathcal{G}}$, that is $\mathcal{G}_1$ (see Definition 3.6.6), is either of type A_r, D_r, E_6, E_7 or E_8 because similar to what we proved above for $\overline{W}$ the model $W_{\mathcal{G}}$ is the minimal resolution of some surface, namely, the minimal resolution of the surface $X_{\mathcal{G}}$ obtained from X by contracting the exceptional/Z curves on X whose birational transform belong to $\mathcal{G}$. In fact, there is no -1-curve/$X_{\mathcal{G}}$ on $W_{\mathcal{G}}$.

Now, suppose $\mathcal{G}_1$ is of type A_r and not strictly monotonic. Let the push-down on $W_{\mathcal{G}}$ of the chain $\mathcal{C}$ be $\mathcal{C}_1$. Let the exceptional divisors of $\mathcal{G}_1$ be $G_{\beta_1}, \dots, G_{\beta_r}$ and assume that the chain $\mathcal{C}_1$ consists of $G_{\beta_k}, \dots, G_{\beta_l}$. Hence $N^{Sh}_{W_{\mathcal{G}}}(G_{\beta_k}) \leq -\frac{1}{6}$, $N^{Sh}_{W_{\mathcal{G}}}(G_{\beta_{k+1}}) = \cdots = N^{Sh}_{W_{\mathcal{G}}}(G_{\beta_{l-1}}) = 0$ and $N^{Sh}_{W_{\mathcal{G}}}(G_{\beta_l}) \leq -\frac{1}{6}$. Here the superscript Sh means that we compute the negativity according to the Shokurov log divisor not the primary log divisor. Note that if $a^{Sh}_{\beta} > 0$ for some β, then $a^{Sh}_{\beta} \geq \frac{1}{6}$ because the denominator of a^{Sh}_{β} is in $\{1, 2, 3, 4, 6\}$. The chain $\mathcal{C}_1$ is of type A_{l-k+1}. From the constructions in the local isomorphic section we can replace the Shokurov log numbers $a^{Sh}_{\beta_k} = 0, \dots, a^{Sh}_{\beta_l} = 0$ with new log numbers with bounded denominators and preserve all other Shokurov log numbers in the graph $\mathrm{exc}(W_{\mathcal{G}}/Z)$ so that we obtain a new log divisor $K_{W_{\mathcal{G}}} + \Omega_1$ on $W_{\mathcal{G}}$ which is antinef/Z and Klt. Now put $K_W + \Omega =^* (K_{W_{\mathcal{G}}} + \Omega_1)$. The only problem with Ω is that it may have negative coefficients (it is a subboundary). Remark 3.6.18 and Corollary 3.6.20 guarantee that the negativity of these coefficients is bounded from below. Moreover if an exceptional divisor has negative coefficient in Ω, then it must belong to the graph $\mathcal{G}$. But any exceptional divisor in $\mathcal{G}$ appears with positive coefficient in ω_{Sh}. Since $\omega_{Sh} \geq \omega$ and by the definition of $\mathcal{G}$, any exceptional divisor of type F in $\mathcal{G}$ has positive coefficient at least $\frac{1}{6}$. If E is not of type F but belongs to $\mathcal{G}$, then since B^+ is not zero $P \in Z$ we get positive coefficients in ω_{Sh} for all exceptional/Z curves which are not of type F. Thus all members of $\mathcal{G} = \text{exc}(W/Q)$ appear with positive coefficient in ω_{Sh}.

Now, consider the sum

$$K_W + \Omega + I[K_W + \omega_{Sh}] = (1+I)K_W + [\Omega + I\omega_{Sh}]$$

where I is an integer. Given that the negative coefficients appearing in Ω are bounded from below, this implies that there is a large bounded I such that the sum $\Omega + I\omega_{Sh}$ is an effective divisor. So by construction the log divisor $K_W + \frac{[\Omega+I\omega_{Sh}]}{1+I}$ is ϵ-lc and antinef/Z for some fixed rational number $0 < \epsilon$ and the denominators of the coefficients in the log divisor are bounded.

Now assume that $\mathcal{G}_1$ is strictly monotonic and the generating curve is G_{β_1}. By Corollary 3.6.20 and Remark 3.6.18 the only place where we may have difficulties is a generating point ξ on the generating curve if there is any such point.

We blow up ξ and get the exceptional divisor G_{α_1}. The chain $G_{\alpha_1}, G_{\beta_1}, \dots, G_{\beta_r}$ is not exactly of type A_{r+1} because G_{α_1} is a -1-curve. But still we can claim that there is at most a base on this chain and it can only be on G_{α_1}. Obviously a generating point cannot be on $G_{\beta_2}, \dots, G_{\beta_r}$. Now suppose that the intersection point of G_{α_1} and G_{β_r} is a generating point. Then, the sum of negativities of all $G_{\alpha_1}, G_{\beta_1}, \dots, G_{\beta_r}$ must be less than $2\delta - 2$. This is impossible because the sum of negativities of all $G_{\beta_1}, \dots, G_{\beta_r}$ on $W_{\mathcal{G}}$ is at least $2\delta - 2$ (remember that blowing up reduces negativity).

Now if on G_{α_1} there is a generating point ξ_1, then again we blow up this point to get G_{α_2} and so on. This process has to stop after finitely many steps (not after bounded steps!). Let the final model be W_ξ and let $G_{\alpha_1}, \dots, G_{\alpha_s}$ be the new exceptional divisors. In fact, we have constructed a chain (because there was at most one generating point on each curve) and by adding the new exceptional divisors to $\mathcal{G}_1$ we get a new graph $\mathcal{G}_2$. Now there is no base point on $\mathcal{G}_2$. All the divisors G_{α_i} have self-intersection equal to -2 except G_{α_s} which is a -1-curve.

Next let $\mathcal{C}_2$ be the pushdown of $\mathcal{C}$, that is, the connected chain of curves with coefficient one in ω_{Sh} on W_ξ. If G_{α_s} is not in $\mathcal{C}_2$, then we proceed exactly as in the non-monotonic case above; that is we assign appropriate coefficients to the members of $\mathcal{C}_2$ and keep all other coefficients in ω_{Sh} on W_ξ. If G_{α_s} is in $\mathcal{C}_2$, then let $\mathcal{C}'$ be the chain $\mathcal{C}_2$ except the member G_{α_s}. This new chain (i.e. $\mathcal{C}'$) if of type A_x and so we can assign appropriate coefficients to its members and put the coefficient of G_{α_s} simply equal to zero and retain all other coefficients in ω_{Sh} on W_ξ. In any case, we construct a Klt log divisor $K + \Omega$ on W_ξ which is antinef$/Z$ and the boundary coefficients are with bounded denominators. The rest is as in the non-monotonic case above.

Suppose the graph $\mathcal{G}_1$ is of type D_r and $\mathcal{C}_\infty \neq \emptyset$ (if it is empty, then we already have Ω_1). Assume that the members of $\mathcal{G}_1$ are $G_\beta, G_{\beta'}, G_{\beta_1}, \dots, G_{\beta_r}$ and the members of $\mathcal{C}_1$ are $G_{\beta_k}, \dots, G_{\beta_l}$. As in the proof of Lemma 3.6.10 for the D_r case, we have $a^{Sh}_{\beta_1} \leq a^{Sh}_{\beta_2} \leq \dots$. So $k = 1$, therefore $2a^{Sh}_\beta - 0 - 1 \leq 0$

and so $a_{\beta}^{Sh} \leq \frac{1}{2}$. Similarly $a_{\beta'}^{Sh} \leq \frac{1}{2}$. The chain $\mathcal{C}_1$ is of type A_l and so we can change the coefficients of its members in ω_{Sh} on $W_{\mathcal{G}}$. The rest of the argument is very similar to the above cases. Just note that there is no base point in this case.

The cases E_6, E_7 and E_8 are settled by Remark 3.6.18 and Corollary 3.6.20. In these cases, the graph $\mathcal{G}$ is bounded, so assigning the primary log numbers to the members of $\mathcal{G}_1$ and Shokurov log numbers to the rest of the graph $\mathrm{exc}(W_{\mathcal{G}}/Z)$ gives a log divisor which can be used as $K_{W_{\mathcal{G}}} + \Omega_1$. This completes the proof of Theorem 3.6.1.

□

3.7 Global case

The main theorem of this section is the following theorem. A generalised version of this and the BAB_2 follow as corollaries.

Theorem 3.7.1 *Conjecture* $\mathrm{WC}_{\delta,2,\{0\}}$ *(3.1.7) holds in the global case, that is, when* $Z = \mathrm{pt.}$ *and* $\Gamma = \{0\}$.

Proof We divide the problem into two main cases: exceptional and nonexceptional. $(X, 0)$ is nonexceptional if there is a strictly lc $(0, \mathbb{Q})$-complement $K_X + M$. By [Sh2, 2.3.1], under our assumptions on X, nonexceptionallity is equivalent to the fact that K_X has a strictly lc $(0, n)$-complement for some $n < 58$. We prove that the exceptional cases are bounded. But in the nonexceptional case we only prove the existence of an (ϵ, n)-complement for a bounded n. Later we show that this implies the boundedness of X.

First assume that $(X,0)$ is **nonexceptional**.

1. Lets denote the set of accumulation points of the mlds in dim 2 for lc pairs (T,B) where $B \in \Phi_{\rm sm}$, by $\mathrm{Accum}_{2,\Phi_{\rm sm}}$. Then, $\mathrm{Accum}_{2,\Phi_{\rm sm}} \cap [0,1] = \{1-z\}_{z\in\Phi_{\rm sm}} = \{\frac{1}{k}\}_{k\in\mathbb{N}} \cup \{0\}$ [Sh8]. Now if there is a $\tau > 0$ such that $\mathrm{mld}(P,T,B) \notin [\frac{1}{k}, \frac{1}{k}+\tau]$ for any natural number k and any point $P \in T$, then there will be only a finite number of possibilities for the index of $K_T + B$ at P if (T,B) is $\frac{1}{m}$-lc for some $m \in \mathbb{N}$. Now Borisov-M$^{\rm c}$Kernan [Mc, 1.2] implies the boundedness of all such T if $-(K_T+B)$ is nef and big and τ and m are fixed. In the following steps we try to reduce our problem to this situation in some cases.

2. **Definition 3.7.2** Let $B = \sum b_iB_i$ be a boundary on a variety T and $\tau > 0$ a real number. Define

$$D_\tau := \sum_{b_i \notin [\frac{k-1}{k}-\tau, \frac{k-1}{k}]} b_iB_i + \sum_{b_i \in [\frac{k-1}{k}-\tau, \frac{k-1}{k}]} \frac{k-1}{k}B_i$$

where in the first sum $b_i \notin [\frac{k-1}{k} - \tau, \frac{k-1}{k}]$ for any natural number k but in the second sum k is the smallest natural number satisfying $b_i \in [\frac{k-1}{k} - \tau, \frac{k-1}{k}]$.

Lemma 3.7.3 *For any natural number m, there is a real number $\tau > 0$ (depending only on m) such that if (T,B) is a surface log pair, $P \in T$, $K_T + B$ is $\frac{1}{m}$-lc at P and $D_\tau \in \Phi_{\rm sm}$, then $K_T + D_\tau$ is also $\frac{1}{m}$-lc at P.*

Proof By applying the ACC to all surface pairs with standard boundary, we get a fixed rational number $v > 0$ such that, if any $K_T + D_\tau$ is not $\frac{1}{m}$-lc at P, then $\mathrm{mld}(P, T, D_\tau) < \frac{1}{m} - v$.

Now assume that the lemma is not true. Then, there is a sequence $\tau_1 > \tau_2 > \dots$ and a sequence of pairs $\{(T_i, B_i)\}$ such that the lemma does not hold for τ_i and (T_i, B_i) and $P_i \in T_i$. In other words $\mathrm{mld}(P_i, T_i, D_{\tau_i}) < \frac{1}{m} - v$.

Write $B_i = F_i + C_i$ where $F_i = \sum f_{i,x} F_{i,x}$ and $C_i = \sum c_{i,y} C_{i,y}$ have no common components and the coefficient of any component of C_i is equal to the coefficient of the same component in D_{τ_i} but the coefficient of any component of F_i is less than the coefficient of the same component in D_{τ_i}.

Now there is a set $\{s_{1,x}\} \subseteq [\frac{m-1}{m} - \tau_1, \frac{m-1}{m}]$ of rational numbers such that $\mathrm{mld}(P_1, T_1, \sum s_{1,x} F_{1,x} + C_1) = \frac{1}{m} - v$. There is an i_2 such that $\max\{s_{1,x}\} < \frac{m-1}{m} - \tau_{i_2}$. So there is also a set $\{s_{2,x}\} \subseteq [\frac{m-1}{m} - \tau_{i_2}, \frac{m-1}{m}]$ such that $\mathrm{mld}(P_{i_2}, T_{i_2}, \sum s_{2,x} F_{i_2,x} + C_{i_2}) = \frac{1}{m} - \frac{v}{2}$. By continuing this process we find $\{s_{j,x}\} \subseteq [\frac{m-1}{m} - \tau_{i_j}, \frac{m-1}{m}]$ such that $\max\{s_{i_{j-1},x}\} < \frac{m-1}{m} - \tau_{i_j}$. Hence we can find a set $\{s_{j,x}\} \subseteq [\frac{m-1}{m} - \tau_{i_j}, \frac{m-1}{m}]$ such that $\mathrm{mld}(P_{i_j}, T_{i_j}, \sum s_{j,x} F_{i_j,x} + C_{i_j}) = \frac{1}{m} - \frac{v}{j}$.

We have thus constructed a set $\bigcup\{s_{j,x}\}$ of rational numbers which satisfies the DCC condition but such that there is an increasing set of mlds corresponding to boundaries with coefficients in $\bigcup\{s_{j,x}\}$. This is a contradiction with the ACC for mlds. $\square$

3. Let m be the smallest number such that $\frac{1}{m} \leq \delta$. Let $h = \min\{\frac{k-1}{k} - \frac{u}{r!} > 0\}_{1 \leq k \leq m}$ where u, k are natural numbers and $r = \max\{m, 57\}$. Now choose a τ for m as in Lemma 3.7.3 such that $\tau < h$.

Blow up one exceptional divisor E via $f\colon Y \to X$ such that the log discrepancy satisfies $\frac{1}{k} \leq a(E, X, 0) \leq \frac{1}{k} + \tau$ for some $k > 1$ (if such E does not exist, then go to step 1). The crepant log divisor $K_Y + B_Y$ is $\frac{1}{m}$-lc and so by Lemma 3.7.3 $K_Y + D_\tau$ is also $\frac{1}{m}$-lc (D_τ is constructed for B_Y). Let $K_X + B^+$ be a $(0, n)$-complement for some $n < 58$ and let $K_Y + B_Y^+$ be the crepant blow up. Then, by the way we chose τ we have $D_\tau \leq B^+$. Now run the anti-LMMP on $K_Y + D_\tau$ (Definition 3.2.18 and Example 3.2.21), i.e, contract any birational type extremal ray R such that $(K_Y + D_\tau) \cdot R > 0$. At the end of this process we get a model X_1 and the corresponding map $g\colon Y \to X_1$. After contracting those birational extremal rays where $K_{X_1} + D_\tau$ is numerically zero, we get a model S_1 with one of the following properties:

- ◇ $\rho(S_1) = 1$ and $K_{S_1} + D_\tau \equiv K_{S_1} + B_{S_1}^+ \equiv 0$ and $\frac{1}{m}$-lc.
- ◇ $\rho(S_1) = 2$ and $(K_{S_1} + D_\tau) \cdot R = 0$ for a nonbirational type extremal ray R on S_1 and $K_{S_1} + D_\tau$ is $\frac{1}{m}$-lc.
- ◇ $-(K_{S_1} + D_\tau)$ is nef and big and $K_{S_1} + D_\tau$ is $\frac{1}{m}$-lc.

where $K_{S_1} + D_\tau$ is the birational transform of $K_Y + D_\tau$.

In any case $-(K_{S_1} + D_\tau)$ is nef because $D_\tau \leq B_{S_1}^+$ and so D_τ can not be positive on a non-birational extremal ray. $K_{S_1} + D_\tau$ is $\frac{1}{m}$-lc by the way we have chosen τ.

4. If the first case occurs in the division in step 3, then we are done.

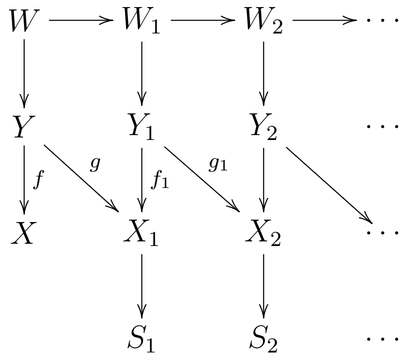
5. If the second case occurs in the division in step 3, then R defines a fibration $\phi\colon S_1 \to Z$. In this case, $B_{S_1}^+ = D_\tau + N$ where each component

of N is a fibre of ϕ and there are only a finite number of possibilities for the coefficients of N. Now if the index of $K_{S_1} + D_\tau$ is bounded, then we can replace N by $N' \sim_{\mathbb{Q}} N$ such that each component of N' is a general fibre of ϕ, there are only a finite number of possibilities for the coefficients of N', $K_{S_1} + D_\tau + N'$ is $\frac{1}{m}$-lc and has a bounded index. Note that the components of N' are smooth curves and intersect the components of D_τ transversally in smooth points of S_1.

6. Now assume that the third case or the second case occurs in the division in step 3. Let C be a curve contracted by $g\colon Y \to X_1$ constructed in step 3. If C is not a component of B_Y, then the log discrepancy of C with respect to $K_{X_1}+B_{X_1}$ is at least 1 where $K_{X_1}+B_{X_1}$ is the birational transform of $K_Y + B_Y$. Moreover $g(C) \in \operatorname{Supp} B_{X_1} \neq \emptyset$. So the log discrepancy of C with respect to K_{X_1} is more than 1. This means that C is not a divisor on a minimal resolution $W_1 \to X_1$. Let $W \to X$ be a minimal resolution. Then, there is a morphism $W \to W_1$. Hence $\operatorname{exc}(W_1/X_1) \subseteq \operatorname{exc}(W/X)$. Now if $C \in \operatorname{exc}(W/X)$ is exceptional$/X_1$, then $a(C, X_1, D_\tau) < a(C, X, 0)$.

7. Let $(X_1, B_1) := (X_1, D_\tau)$ and repeat the process. In other words we blow up again one exceptional divisor E via $f_1\colon Y_1 \to X_1$ such that the log discrepancy satisfies $\frac{1}{k} \leq a(E, X_1, B_1) \leq \frac{1}{k} + \tau$ for some natural number $k > 1$. The crepant log divisor $K_{Y_1} + B_{1,Y_1}$ is $\frac{1}{m}$-lc and so by Lemma 3.7.3 $K_{Y_1} + D_{1,\tau}$ is $\frac{1}{m}$-lc. Note that the point which is blown up on X_1 can not be smooth since $\tau < h$ as defined in step 3. So according to step 6 the blown up divisor E is a member of $\operatorname{exc}(W/X)$. Now we

again run the anti-LMMP on $K_{Y_1} + D_{1,\tau}$ and proceed as in step 3.



8. Steps 6,7 show that each time we blow up a member of $\mathrm{exc}(W/X)$ say E. And if we blow that divisor down in some step, then the log discrepancy $a(E, X_j, B_j)$ will decrease. That divisor will not be blown up again unless the log discrepancy drops by at least $\frac{1}{2(m-1)} - \frac{1}{2m}$. So after finitely many steps either case one occurs in the division in step 3 or we get a model X_i with a standard boundary B_i such that there is no E where $\frac{1}{k} \le a(E, X_i, B_i) \le \frac{1}{k} + \tau$ for any $1 < k \le m$. The latter implies the boundedness of the index of $K_{X_i} + B_i = K_{X_i} + D_{i-1,\tau}$. If $-(K_{X_i} + B_i)$ is nef and big (case one), then (X_i, B_i) will be bounded by step 1. Otherwise we have the second case in the division above and so by step 5 we are done (the index of $K_{X_i} + D_{i-1,\tau} + N'$ is bounded).

Now we treat the **exceptional** case: From now on we assume that $(X, 0)$ is exceptional.

9. Let $W \to X$ be a minimal resolution. Let $\tau \in (0, \frac{1}{2})$ be a rational number. If $(X, 0)$ is $\frac{1}{2}+\tau$-lc, then we know that X belongs to a bounded family according to step 1 above. So we assume that $(X, 0)$ is not $\frac{1}{2}+\tau$-

lc. Then, blow up an exceptional curve E_1 with log discrepancy $a_{E_1} = a(E_1, X, 0) \leq \frac{1}{2} + \tau$ to get $Y \to X$. Put $K_Y + B_Y = {}^*K_X$. Let $t \geq 0$ be such that there is an extremal ray R such that $(K_Y + B_Y + tE_1) \cdot R = 0$ and $E_1 \cdot R > 0$ (and s.t. $K_Y + B_Y + tE_1$ Klt and antinef). Such R exists otherwise there is a $t > 0$ such that $K_Y + B_Y + tE_1$ is lc (and not Klt) and antinef. This is a contradiction by [Sh2, 2.3.1]. Now if R is of birational type, then contract it via $Y \to Y_1$.

Again by increasing t we obtain an extremal ray R_1 on Y_1 such that $(K_{Y_1} + B_{Y_1} + tE_1) \cdot R_1 = 0$ and $E_1 \cdot R_1 > 0$ (preserving the nefness of $-(K_{Y_1} + B_{Y_1} + tE_1)$). If it is birational, then contract it and so on. After finitely many steps we get a model $(V_1, B_{V_1} + t_1E_1)$ and a number $t_1 > 0$ with the following possible outcomes:

(3.7.3.1)

- ◇ $(V_1, B_{V_1} + t_1E_1)$ is Klt, $\rho(V_1) = 1$ and $K_{V_1} + B_{V_1} + t_1E_1$ is antinef.
- ◇ $(V_1, B_{V_1} + t_1E_1)$ is Klt and $\rho(V_1) = 2$ and there is a non-birational extremal ray R on V_1. Moreover $K_{V_1} + B_{V_1} + t_1E_1$ and K_{V_1} are antinef.
- ◇ $(V_1, B_{V_1} + t_1E_1)$ is Klt and $\rho(V_1) = 2$ and there is a non-birational extremal ray R on V_1. Moreover $K_{V_1} + B_{V_1} + t_1E_1$ is antinef but K_{V_1} is not antinef.

Define $K_{V_1} + D_1 := K_{V_1} + B_{V_1} + t_1E_1$. Note that in all the cases above E_1 is a divisor on V_1 and the coefficients of B_{V_1} and D_1 are $\geq \frac{1}{2} - \tau$.

Lemma 3.7.4 *Let $P \in U$ be a δ-lc surface singularity. Moreover suppose that there is at most one exceptional/U divisor such that $a(E, U, 0) < \frac{1}{2} + \tau$. Then, the index of K_U is bounded at P and the bound only depends on δ and τ.*

Proof We only need to prove this when the singularity is of type A_r (otherwise the index is bounded). If there is no E/P such that $a(E, U, 0)) < \frac{1}{2} + \frac{\tau}{2}$, then step 1 shows that the index is bounded. But if there is one E/P such that $a(E, U, 0) < \frac{1}{2} + \frac{\tau}{2}$, then using the notation of 3.5.2.1, we have $a_{i+1} - a_i \geq \frac{\tau}{2}$ and $a_{i-1} - a_i \geq \frac{\tau}{2}$. This implies the boundedness of r and hence the boundedness of the index of K_U at P.
$\square$

10. Let U/pt. be a surface with the following properties:

- ◇ $\rho(U) = 1$.
- ◇ $K_U + G_U$ antinef, Klt and exceptional.
- ◇ K_U antiample.

Now blow up two divisors E and E' as $f : Y_U \to U$ such that $a(E, U, 0) < \frac{1}{2} + \tau$ and $a(E', U, 0) < \frac{1}{2} + \tau$ (suppose there are such divisors). Choose $t, t' \geq 0$ such that $(f^*(K_U + G_U) + tE + t'E') \cdot R = 0$ for an extremal ray R s.t. $R \cdot E \geq 0$ and $R \cdot E' \geq 0$ and $f^*(K_U + G_U) + tE + t'E'$ is antinef and Klt. We contract R to get $g : Y_U \to U'$. We call such operation a **hat of first type**. Note that E and E' are divisors on U' and $\rho(U') = 2$. Define $K_{U'} + G_{U'}$ to be the pushdown of $f^*(K_U + G_U) + tE + t'E'$.

If K_U is δ-lc and such E, E' do not exist as above, then the index of K_U will be bounded by Lemma 3.7.4. So U will be bounded.

11. Let $U/\mathrm{pt}.$ be a surface with the following properties:

- ◇ $\rho(U) = 2$.
- ◇ $K_U + G_U$ antinef, Klt and exceptional.
- ◇ $-K_U$ nef and big.

Now blow up a divisor E to get $f\colon Y_U \to U$ such that $a(E, U, 0) < \frac{1}{2} + \tau$ (suppose there is such E). Let $t \geq 0$ be such that $(f^*(K_U + G_U) + tE) \cdot R = 0$ for an extremal ray R s.t. $R \cdot E \geq 0$ and $f^*(K_U + G_U) + tE$ is antinef and Klt. We contract R to get $g\colon Y_U \to U'$. We call such operation a **hat of second type**. Note that E is a divisor on U' and $\rho(U') = 2$. Define $K_{U'} + G_{U'}$ to be the pushdown of $f^*(K_U + G_U) + tE$.

If K_U is δ-lc and such E does not exist as above, then the index of K_U and so U will be bounded by Lemma 3.7.4.

12. Let $U/\mathrm{pt}.$ be a surface with the following properties:

- ◇ $\rho(U) = 2$ and U is pseudo-WLF/$\mathrm{pt}.$.
- ◇ There is a birational type extremal ray R_{bir} and the other extremal ray of U is of fibration type.
- ◇ $K_U + G_U$ is antinef, Klt and exceptional.
- ◇ $K_U \cdot R_{bir} > 0$.

Then, we say that U is of **2-bir** type. Let C be the divisor that defines R_{bir} on U. There is a $c \in (0,1)$ such that $(K_U + cC) \cdot C = 0$. Now blow up E as $Y_U \to U$ such that $a(E, U, cC) < \frac{1}{2} + \tau$ (suppose there is such E). Now let $t \geq 0$ such that $f^*(K_U + G_U + tC) \cdot R = 0$ for an extremal ray R s.t. $R \cdot E \geq 0$, $R \cdot C \geq 0$ and $f^*(K_U + G_U + tC)$ is antinef and Klt. We contract R to get $g \colon Y_U \to U'$. We call such operation a **hat of third type**. Define $K_{U'} + G_{U'}$ to be the pushdown of $f^*(K_U + G_U + tC)$. Note that in this case E and C are both divisors on U' and $\rho(U') = 2$.

If $K_U + cC$ is δ-lc and such E does not exist as above, then contract $C \colon U \to U_1$. Thus the index of K_{U_1} will be bounded at each point by Lemma 3.7.4 and so U_1 and consequently U will be bounded.

$$\begin{array}{ccc} Y_U & & \\ \downarrow f & \searrow g & \\ U & & U' \end{array}$$

13. Let $U/\mathrm{pt}.$ be a surface such that $\rho(U) = 2$. Moreover suppose that $K_U + G_U$ is antinef, Klt and exceptional where $G_U \neq 0$. Moreover suppose there are two exceptional curves H_1 and H_2 on U. In this case let C be a component of G_U and let $t \geq 0$ such that $(K_U + G_U + tC) \cdot H_i = 0$ for $i = 1$ or 2 and $K_U + G_U + tC$ Klt and antinef (assume $i = 1$). We contract H_1 as $U \to U_1$ and define $K_{U_1} + G_{U_1}$ to be the pushdown of $K_U + G_U + tC$.

Definition 3.7.5 Define $K_U + \Delta_U$ as follows: $K_U + \Delta_U := K_U$

in step 10 and step 11. $K_U + \Delta_U := K_U + cC$ in step 12. Finally $K_{U_1} + \Delta_{U_1} := K_{U_1}$ in step 13.

14. The following lemmas are crucial to our proof.

Lemma 3.7.6 *Let $\mathcal{U}$ be a bounded family of surfaces with Picard number one or two and let $0 < x < 1$ be a rational number. Moreover assume the following for each member U:*

- ◇ *$-(K_U + B)$ is nef and big for a boundary B with coefficients $\geq x$.*
- ◇ *$K_U + B$ is Klt.*

Then, $(U, \operatorname{Supp} B)$ is bounded.

Proof We prove that there is a finite set Λ_f such that for each U there is a boundary $M \in \Lambda_f$ s.t. $-(K_U + M)$ is nef and big and $M \leq B$.

If $\rho(U) = 1$, then simply take $M = x \sum_\alpha B_\alpha$ where $B = \sum_\alpha b_\alpha B_\alpha$. Obviously $-(K_U + M)$ is nef and big and since U belongs to a bounded family hence $(U, \operatorname{Supp} M)$ is bounded.

Now suppose $\rho(U) = 2$. Let $N = x \sum_\alpha B_\alpha$. If $-(K_U + N)$ is not nef, then there is an exceptional curve E on U with $(K_U + N) \cdot E > 0$. Let $\theta \colon U \to U'$ be the contraction of E. By our assumptions, $K_{U'} + B'$, the pushdown of $K_U + B$, is antiample. So $K_{U'} + N'$, the pushdown of $K_U + N$, is also antiample. The boundedness of U implies the boundedness of U' (since we have a bound for the Picard number of a minimal resolution of U'). Thus $-(K_U + M) := -\theta^*(K_{U'} + N') = -(K_U + N + yE)$ is nef and big and there is only a finite number of possibilities for $y > 0$. This proves the boundedness of $(U, \operatorname{Supp}(N + yE))$. Note that in the arguments above $\operatorname{Supp} B = \operatorname{Supp} M$. $\square$

Main Lemma 3.7.7 *Suppose that* $\mathcal{U} = \{(U, \operatorname{Supp} D)\}$ *is a bounded family of log pairs of dim* d *where* $K_U + D$ *is antinef and* ϵ*-lc for a fixed* $\epsilon > 0$. *Then, the set of partial resolutions of all* $(U, D) \in \mathcal{U}$ *is a bounded family.*

Note that here we do not assume (U, D) to be bounded, that is, the coefficients of D may not necessarily be in a finite set.

Proof Let (U, D) be a member of the family. By our assumptions the number of components of D is bounded (independent of (U, D)) and so we can consider any divisor supported in D as a point in a real finite dimensional space. Let $D = \sum_{1 \leq i \leq q} d_i D_i$ and define

$$\mathcal{H}_U := \{(h_1, \dots, h_q) \in \mathbb{R}^q \mid K_U + \sum_{1 \leq i \leq q} h_i D_i \text{ is antinef and } \epsilon\text{-lc}\}$$

So $\mathcal{H}_U$ is a subset of the cube $[0, 1]^q$ and since being ϵ-lc and antinef are closed conditions, then $\mathcal{H}_U$ is a closed and hence compact subset of $[0, 1]^q$. For each $H \in \mathcal{H}_U$ the corresponding pair (U, H) is ϵ-lc. Let $Y_H \to U$ be a terminal blow up of (U, H) and denote by R_H the set of exceptional/U divisors on Y_H. For different H we may have different R_H, but the union of all R_H is a finite set when H runs through $\mathcal{H}_U$.

Suppose otherwise, so there is a sequence $\{H_1, \dots, H_m, \dots\} \subseteq \mathcal{H}_U$ such that the union of all R_{H_i} is not finite. Since $\mathcal{H}_U$ is compact, there is at least an accumulation point for the sequence in $\mathcal{H}_U$, say $\overline{H}$ (we can assume that this is the only accumulation point). So $(U, \overline{H})$ is ϵ-lc. Let $v = (1, \dots, 1) \in \mathbb{R}^q$. Then, there are $\alpha, \beta > 0$ such that $K_U + H_\alpha$ is $\epsilon - \beta$-lc where $\epsilon - \beta > 0$ and H_α is the corresponding divisor of $\overline{H} + \alpha v$. In particular this implies that there is a d-dimensional disc $\mathbb{B} \subseteq [0, 1]^q$ with positive radius $\overline{H}$ as its centre such that $K_U + H$ is $\epsilon - \beta$-lc and $R_H \subseteq R_{H_\alpha}$ for any $H \in \mathbb{B}$. This contradicts the way we chose the sequence $\{H_1, \dots, H_m, \dots\}$. The function $R : \mathcal{H}_U \to \mathbb{N}$ gives a finite decomposition of the set $\mathcal{H}_U$. This means that there is only a finite number of partial resolutions for all (U, H) where $H \in \mathcal{H}_U$ for a fixed U. Using Noetherian induction we complete the proof of the lemma. $\square$

Now we prove a statement similar to [Sh2, 4.2].

Lemma 3.7.8 *Let $\mathcal{U} = \{(U, \operatorname{Supp} D)\}$ be a bounded family where we assume that each (U, D) is Klt and exceptional and $K + D$ is antinef. Then, there is a constant $\gamma > 0$ such that each (U, D) is γ-lc.*

Proof For $(U, \operatorname{Supp} D)$ a member of the family let

$$\mathcal{H}_U = \{H = \sum h_k D_k |\ K + H \text{ is log canonical and } -(K + H) \text{ is nef}\}$$

where $D = \sum d_k D_k$.

$\mathcal{H}_U$ is a closed subset of a multi-dimensional cube (with bounded dimension) and so it is compact. There is a biggest number $e_U > 0$ such that (U, H) is e_U-lc for any $H \in \mathcal{H}_U$. Since the family is bounded, $\{e_U\}_{U \in \mathcal{U}}$ is bounded from below away from zero. $\square$

Now we return to the division in 3.7.3.1 and deal with each case as follows:

15. (First case in 3.7.3.1) Perform a hat of the first type for $U := V_1$ and $K_U + G_U := K_{V_1} + D_1$ (so we blow up E, E'). Then, we get $V_2 := U'$ and $K_{V_2} + D_2 := K_{U'} + G_{U'}$ as defined above and $Y_1 := Y_U$. Now V_2 would be as in step 11, 12 or 13 so we can perform the appropriate operation as explained in each case. If V_2 is as in step 11, then $a(E, V_2, \Delta_{V_2}) = 1$ and $a(E', V_2, \Delta_{V_2}) = 1$. If V_2 is as in step 12, then E or E' is not exceptional so we have $a(E, V_2, \Delta_{V_2}) = 1$ or $a(E', V_2, \Delta_{V_2}) = 1$. But if V_2 is as in step 13, then we get U_1 as defined in step 13 and so $a(E, U_1, \Delta_{U_1}) = 1$ or $a(E', U_1, \Delta_{U_1}) = 1$. In the latter case we define (replace) $(V_2, D_2) := (U_1, G_{U_1})$.

    So, whatever case we have for V_2 we have $a(A, V_2, \Delta_{V_2}) = 1$ for at least one $A \in \mathrm{exc}(Y/X)$.

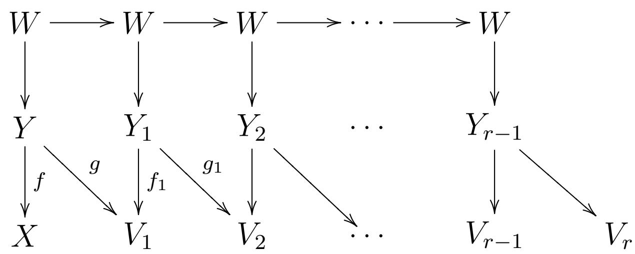
16. (Second case in 3.7.3.1) Here we perform a hat of second type for $U := V_1$ and $K_U + G_U := K_{V_1} + D_1$ to get $V_2 := U'$ and $K_{V_2} + D_2 := K_{U'} + G_{U'}$. If V_2 is as in step 11, then $a(E, V_2, \Delta_{V_2}) = 1$. If V_2 is as in step 12, then go to step 17. But if V_2 is as in step 13, then we get U_1 as defined in step 13 where $K_U + G_U := K_{V_2} + D_2$ and then continue the process for U_1 as in step 15.

Here, in some cases we may not be able to make the singularities better for $K+\Delta$ immediately on V_2 but the algorithm ensures us that we will be able to do that in latter steps.

17. (Third case in 3.7.3.1) In this case V_1 is 2-bir. We perform a hat of the third type where $U := V_1$ and $K_U + G_U := K_{V_1} + D_1$ so we get $V_2 := U'$ and $Y_1 := Y_U$ and $K_{V_2} + D_2 := K_{U'} + G_{U'}$. If V_2 is as in step 11, then $a(E, V_2, \Delta_{V_2}) = 1$ and $a(C, V_2, \Delta_{V_2}) = 1$ (E is the blown divisor and C is on V_1, as in step 12 for $U := V_1$). If V_2 is as in step 12, then $a(E, V_2, \Delta_{V_2}) = 1$ or $a(C, V_2, \Delta_{V_2}) = 1$. Now if V_2 is as in step 13, then we get U_1 as defined in step 13 and so $a(E, U_1, \Delta_{U_1}) = 1$ or $a(C, U_1, \Delta_{U_1}) = 1$. Then, we define (replace) $(V_2, D_2) := (U_1, G_{U_1})$.

    So whatever case we have for V_2 we have $a(A, V_2, \Delta_2) = 1$, after the appropriate operations, for at least one $A \in \operatorname{exc}(Y/X)$.

18. After finitely many steps we get V_r where W/V_r such that $K_W + D := {}^*(K_{V_r} + D_r)$ with effective D where V_r is bounded. Since all the coefficients of B_{V_r} are $\geq \frac{1}{2} - \tau$ (B_{V_r} is the birational transform of B_W where $K_W + B_W = {}^*K_X$), (V_r, B_{V_r}) is also bounded by Lemma 3.7.6. By construction $\operatorname{Supp} D_r = \operatorname{Supp} B_{V_r}$ and so (V_r, D_r) is bounded. Lemma 3.7.7 implies the boundedness of W and so of X.



□

Corollary 3.7.9 *The* $\mathrm{BAB}_{\delta,2,[0,1]}$ *(Conjecture 3.1.11) holds.*

Proof **Reduction to the case** $B = 0$**:** We run the anti-LMMP on the divisor K_X (Definition 3.2.18 and Example 3.2.21); if there is an extremal ray R such that $K_X \cdot R > 0$, then contract R to get $X \to X_1$. Note that $B \cdot R < 0$ (because $(K_X + B) \cdot R \leq 0$) so the bigness of $-K_X$ will be preserved (So R has to be of birational type). Repeat the same process for X_1, that is, if there is an extremal ray R_1 such that $K_{X_1} \cdot R_1 > 0$, then contract it and so on. Since in each step we get a pseudo-WLF then the canonical class cannot become nef. Let $\overline{X}$ be the last model in our process, then $-K_{\overline{X}}$ is nef ad big. Now the boundedness of $\overline{X}$ implies the boundedness of X. So we replace X by $\overline{X}$, that is, from now on we can assume $B = 0$.

By Theorem 3.7.1 $(X, 0)$ has an (ϵ, n)-complement $K_X + B^+$ for some $n \in \mathcal{N}_{\delta,2,\{0\}}$. Now let $W \to X$ be a minimal resolution and $\phi \colon W \to S$ be the map obtained by running the classical MMP on W, that is, contracting -1-curves to get a minimal S. As it is well known S is $\mathbb{P}^2$ or a smooth ruled surface with no -1-curves.

Let $B_S^+ = \sum b_{i,S}^+ B_{i,S}^+$ be the pushdown of B_W^+ on S where $K_W + B_W^+$ is the crepant pullback of $K_X + B^+$. Then, define

$$A_S := \frac{b_{1,S}^+}{2} B_{1,S}^+ + \sum_{i \neq 1} b_{i,S}^+ B_{i,S}^+$$

If $S = \mathbb{P}^2$, then $-(K_S + A_S)$ is ample and $\operatorname{Supp} A_S = \operatorname{Supp} B_S^+$. By Lemma 3.7.6 $(S, \operatorname{Supp} A_S = \operatorname{Supp} B_S^+)$ is bounded. Then, Lemma 3.7.7 implies the boundedness of W and so of X.

Now assume that S is a ruled surface. If there is no exceptional curve (with negative self-intersection) on S, then $-(K_S + A_S)$ is nef and big if we take $B_{1,S}^+$ a non-fibre component of B_S^+. Since S is smooth, S is bounded and so $(S, \operatorname{Supp} A_S = \operatorname{Supp} B_S^+)$ is bounded.

But if there is an exceptional divisor C on S, then contract C as $S \to S'$. So S is a minimal resolution of S'. Since $\rho(S) = 2$ and $(S', 0)$ is δ-lc, the index of each integral divisor on S' is bounded. So, S' is bounded and hence $(S', \operatorname{Supp} B_{S'}^+)$ is also bounded. This implies the boundedness of S, W and so of X. Note that $B_{S'}^+ \neq 0$ as S' is WLF.

□

Corollary 3.7.10 *Conjecture* $\mathrm{WC}_{\delta,2,\Gamma_f}$ *(3.1.7) holds for any finite set* $\Gamma_f \subseteq [0,1]$ *of rational numbers.*

Proof It follows from Corollary 3.7.9 □.

3.8 Second proof of the global case

Remember that all the varieties are algebraic surfaces unless otherwise stated. We first prove the boundedness of varieties and then prove the boundedness of complements. This is somehow the opposite of what we did in the last section. However our proof was inspired by the theory of complements. The following proof makes heavy use of properties of algebraic surfaces. That means that it is not expected to generalise to higher dimension. The method also has some similarity with the proof of Alexeev and Mori [AM] in the sense that both analyse a series of blow ups, but in different ways.

Theorem 3.8.1 *The* $\mathrm{BAB}_{\delta,2,[0,1]}$ *(3.1.11) holds.*

Proof Now we reduce to the case $B = 0$. Run the anti LMMP on the pair $(X, 0)$ i.e. if $-K_X$ is not nef, then contract an extremal ray R where $K_X \cdot R > 0$. This obviously contracts a curve in B. Repeating this process gives us a model $(X', 0)$ where $-K_{X'}$ is nef and big. Otherwise X' must be with Picard number one and $K_{X'}$ nef. But this is impossible by our assumptions. We prove the boundedness of $\{X'\}$ which in turn implies the boundedness of $\{X\}$. Now we replace (X, B) with $(X', 0)$ but we denote it by $(X, 0)$. We also assume that $\delta < 1$ otherwise X will be smooth and so with bounded index.

Let $W \to X$ be a minimal resolution. The main idea is to prove that there is only a bounded number of possibilities for the coefficients in B_W where $K_W + B_W = {}^*K_X$, that is, the index of K_X is bounded.

Strategy: We apply the familiar division into nonexceptional and exceptional cases.

First assume that $(X, 0)$ is nonexceptional. So there will be a $(0, n)$-complement $K_X + B^+$ for $n < 58$. If we run the classical MMP on the pair $(W, 0)$, then we end up with S which is either $\mathbb{P}^2$ or a ruled surface. Since $-(K_S + B_S) = -_*(K_W + B_W)$ is nef and big, K_S cannot be nef. Let $K_W + B_W^+ = {}^*(K_X + B_X^+)$

Lemma 3.8.2 *Let G be a component of the boundary B_S^+ where $K_S + B_S^+ = {}_*(K_W + B_W^+)$. Then, G^2 is bounded from below and above. Moreover there is only a bounded number of components in B_S^+.*

Proof The boundedness of G^2 follows from the next lemma and the fact that X is δ-lc. The boundedness of number of components in B_S^+ is left to the reader. □

The more general lemma below will also be needed later.

Lemma 3.8.3 *Let $(T/Z, B_T)$ be an δ-lc WLF pair where T is either $\mathbb{P}^2/\mathrm{pt.}$ or a smooth ruled surface (with no -1-curves) over a curve and suppose $K_T + \overline{B}$ is antinef and lc for a boundary $\overline{B}$. Let M, B'_T be effective divisors with no common component such that $\overline{B} = B'_T + M$. Then, M^2 is bounded from above.*

Proof First assume that $T = \mathbb{P}^2$. In this case the lemma is obvious because if M^2 is too big, then so is $\deg M$ and so it contradicts the fact that $\deg M \leq 3$.

Now assume that T is a ruled surface where F is a general fibre other than those curves in the boundary and let C be a section. The Mori cone of T is generated by its two edges. F generates one of the edges. If all the components of M are fibres, then $M^2 = 0$ and we are done. So, assume otherwise and let $M \equiv aC + bF$, then $0 < M \cdot F = (aC + bF) \cdot F = a$ so a is positive. Let $C^2 = -e$ and consider the following two cases:

1. $e \geq 0$: We know that $K_T \equiv -2C + (2g - 2 - e)F$ where g is a non-negative number [H, V, 2.11]. So we have

$$0 \geq (K_T + M + tC) \cdot F = -2 + a + t$$

for some $t \geq 0$ where $B'_T \equiv tC + uF$ ($u \geq 0$ since $e \geq 0$). Hence $a + t \leq 2$. Calculations give $M^2 = a(2b - ae)$. Since a and e are both nonnegative, M^2 big implies that b is big. But on the other hand we have:

$$0 \geq (K_T + M + tC) \cdot C = (-2 + a + t)(-e) + 2g - 2 - e + b$$

This gives a contradiction if b is too big because e is also bounded. The boundedness of e follows from the fact that T is δ-lc. In the local isomorphic section, we proved that exceptional divisors have bounded selfintersection numbers.

2. $e < 0$: in this case, by [H, V, 2.12] we have $e + 2g \geq 0$ and so:

$$\begin{aligned} 0 \geq (K_T + M) \cdot C &= (-2 + a)(-e) + 2g - 2 - e + b \\ &= 2g + e - 2 - (ae/2) + (2b - ae)/2 \end{aligned}$$

Now since $2g + e - (ae/2) \geq 0$, $(2b - ae)/2 \leq 2$. So, M^2 is bounded because a is also bounded. □

Let $P \in X$ be a singular point. If P is not in the support of B^+, then the index of K_X at P is at most 57 and so bounded. Now suppose that P is in the support of B^+. If the singularity of P is of type E_6, E_7, E_8 or D_r, then again the index of K_X at P is bounded. So assume that the singularity at P is of type A_r. The goal is to prove that the number of curves in $\mathrm{exc}(W/P)$ is bounded. We must prove that the number of -2-curves is bounded because the number of other curves is bounded by the proof of local isomorphic case. Note that the coefficient of any $E \in \mathrm{exc}(W/P)$ in B_W^+ is positive and there is only a bounded number of possibilities for these coefficients. Let $\mathcal{C}$ be the longest connected subchain of -2-curves in $\mathrm{exc}(W/P)$. Run the classical MMP on W to get a model W' such that there is a -1-curve F on W' s.t. it is the first -1-curve that intersects the chain $\mathcal{C}$ (if there is no such W' and F, then $\mathcal{C}$ must consist of a single curve). We have two cases:

1. F intersects, transversally and in one point, only one curve in $\mathcal{C}$, say E. First suppose that E is a middle curve, that is, there are E' and E'' in the chain which both intersect E. Now contract F so E becomes a -1-curve. Then, contract E and then E' and then all those which are on the side of E'. In this case by contracting each curve we increase E''^2 by one. Hence E'' will be a divisor on S in B_S^+ with high self-intersection. By Lemma 3.8.3 there can be only a bounded number of curves in $\mathcal{C}$ on the side of E'. Similarly there is only a bounded number of curves on the side of E''. So we are done in this case.

Now suppose that E is on the edge of the chain and intersects E'. Let t_E and t_F be the coefficients of E and F in B_W^+ and similarly for other curves. Let h be the intersection number of F with the curves in $B_{W'}^+$ except those in $\mathcal{C}$ and F itself. Now we have

$$0 = (K_{W'} + B_{W'}^+) \cdot F = t_E + h - 1 - t_F$$

and hence $h = 1 + t_F - t_E$. If $h \neq 0$, then F intersects some other curve not in the chain $\mathcal{C}$. By contracting F then E and then other curves in the chain we get a contradiction again. Now suppose $h = 0$, that is, $t_E = 1$ and $t_F = 0$. In this case let x be the intersection of E with the curves in $B^+_{W'}$ except those in $\mathcal{C}$. So we have

$$0 = (K_{W'} + B^+_{W'}) \cdot E = -2t_E + t_{E'} + x$$

therefore $x = 2t_E - t_{E'} > 0$ and similarly we again get a contradiction.

2. Now assume that F intersects the chain in more than one curve or intersects a curve with intersection number more than one. Suppose the chain $\mathcal{C}$ consists of $E_1, \dots, E_s$ and F intersects $E_{j_1}, \dots, E_{j_l}$. Note that l is bounded. If $F \cdot E_{j_k} > 1$ for all $0 \leq k \leq l$, then contract F. So $E^2_{j_k} \geq 0$ after contraction of F. In addition, they will not be contracted later and so they appear in the boundary B^+_S. Now replace $\mathcal{C}$ with longest connected subchain when we disregard all E_{j_k}. Now go to step one again. If it does not hold return to step two and so on.

Now suppose $F \cdot E_{j_k} = 1$ for some k. So F must intersect at least another E_{j_t} where $t = k+1$ or $t = k-1$. Now contract F so E_{j_k} becomes a -1-curve and it will intersect E_{j_t}. Contracting E_{j_k} and possible subsequent -1-curves will prove that there is a bounded number of curves between E_{j_t} and E_{j_k}. Now after contracting E_{j_k} and all other curves between E_{j_t} and E_{j_k} we have $E^2_{j_m} \geq 0$ for each $m \neq k$. So we again take the longest connected subchain excluding all E_{j_t}. Repeat the procedure. It must stop after a bounded number of steps because the number of curves in B^+_S is bounded. This boundedness implies that there is only a bounded number of possibilities for the coefficients in B_W where $K_W + B_W = {}^*K_X$. By Borisov-M$^{\text{c}}$Kernan W belongs to a bounded family and so complements will be bounded.

Here the proof of the nonexceptional case finishes and from now on we assume that $(X, 0)$ is exceptional.

Let $W \to X$ be a minimal resolution. Let $\tau \in (0, \frac{1}{2})$ be a rational number. If $(X, 0)$ is $\frac{1}{2} + \tau$-lc, then we know that X belongs to a bounded family according to step 1 in the proof of Theorem 3.7.1. So we may assume that $(X, 0)$ is not $\frac{1}{2} + \tau$-lc. Blow up all exceptional curves E with log discrepancy $a_E = a(E, X, 0) \leq \frac{1}{2} + \tau$ to get $Y \to X$ and put $K_Y + B_Y = {}^*K_X$. Fix E_1, one of these exceptional divisors. Let $t \geq 0$ be such that there is an extremal ray R such that $(K_Y + B_Y + tE_1) \cdot R = 0$ and $E_1 \cdot R > 0$ (and s.t. $K_Y + B_Y + tE_1$ is Klt and antinef). Such R exists otherwise there is a $t > 0$ such that $K_Y + B_Y + tE_1$ is lc (and not Klt) and antinef. This is a contradiction by [Sh2, 2.3.1]. Now contract $R \colon Y \to Y_1$ if it is of birational type.

Again by increasing t there will be an extremal ray R_1 on Y_1 such that $(K_{Y_1} + B_{Y_1} + tE_1) \cdot R_1 = 0$ and $E_1 \cdot R_1 > 0$ (preserving the nefness of $-(K_{Y_1} + B_{Y_1} + tE_1)$). If it is of birational type, then contract it and so on. After finitely many steps we get a model $(V_1, B_{V_1} + t_1E_1)$ and a number $t_1 > 0$ with the following possible outcomes:

(3.8.3.1)

- ◇ $(V_1, B_{V_1} + t_1E_1)$ is Klt, $\rho(V_1) = 1$ and $K_{V_1} + B_{V_1} + t_1E_1 \equiv 0$.
- ◇ $(V_1, B_{V_1} + t_1E_1)$ is Klt and $\rho(V_1) = 2$ and there is a non-birational extremal ray R on V_1 such that $(K_{V_1} + B_{V_1} + t_1E_1) \cdot R = 0$. Moreover $K_{V_1} + B_{V_1} + t_1E_1$ is antinef.

Note that for each element $E \in \mathrm{exc}(Y/X)$, either E is a divisor on V_1 or it is contracted to a point in the support of E_1.

Lemma 3.8.4 *For any $h > 0$ there is a $\mu > 0$ such that if (T, B) is a δ-lc pair (δ is already fixed) with a component C of B passing through $P \in T$, with a coefficient $t \geq h$, then either K_T is $\delta+\mu$-lc at P or $1-a_E > \mu$ for each exceptional divisor E/P on a minimal resolution of T (a_E = log discrepancy of (T, B) at E).*

Proof If P is smooth or has E_6, E_7, E_8 or D_r type of singularity, then the lemma is clear since the index of K_T at P is bounded in all these cases (see the local isomorphic section). In all these cases there will be an $\mu > 0$ such that K_T is $\delta + \mu$-lc at P.

Now suppose that the singularity at P is of type A_r. Take a minimal resolution $W_T \to T$ with $\mathrm{exc}(W_T/P) = \{E_1, \dots, E_r\}$ (notation as in the local isomorphic section) and suppose that j is the maximal number such that $\mathrm{mld}(P, T, 0) = a'_j$ (a'_* is the log discrepancy of $(T, 0)$ at E_*) for an exceptional divisor E_j/P. Actually we may assume that $r - j$ is bounded. By the local isomorphic section , the distance of E_j from one of the edges of $\mathrm{exc}(W_T/P)$ is bounded. We denote the birational transform of C on W_T again by C. Suppose C intersects E_k in $\mathrm{exc}(W_T/P)$. If $k \neq 1$ or r, then $(-E_k^2)a_k - a_{k-1} - a_{k+1} + x = 0$ where a_* shows the log discrepancy of the pair (T, B) at E_* and $x \geq h$. So either $a_{k-1} - a_k \geq \frac{h}{2}$ or $a_{k+1} - a_k \geq \frac{h}{2}$. In either case the distance of E_k is bounded from one of the edges of $\mathrm{exc}(W_T/P)$. If this edge is the same edge as for Ej, then again the lemma is clear since the coefficients of E_k and E_j in ${}^*C$ (now C is on T and ${}^*C$ on W_T) are bounded from below (in other words they are not too small).

Now assume that E_k and E_j are close to different edges. In this case we claim that the coefficients of the members of $\mathrm{exc}(W_T/P)$ in $\overline{B}_{W_T}$ are bounded from below where $K_W + \overline{B}_{W_T} = {}^*(K_T + tC)$. Suppose that the smallest coefficient occurs at E_m. A simple calculation shows that we can assume that E_m is one of the edges of $\mathrm{exc}(W_T/P)$. Hence E_m is within a bounded distance from E_j or from E_k.

Suppose that E_m is within a bounded distance from E_j. If $a'_j \geq \frac{1+\delta}{2}$, then K_T is $\frac{1+\delta}{2}$-lc at P. So we can assume that $a'_j < \frac{1+\delta}{2}$. We prove that all the numbers $1 - a'_j, \dots, 1 - a'_r$ are bounded from below. In fact, if $1 < j < r$, then $(-E_j^2)a'_j - a'_{j-1} - a'_{j+1} = 0$ (note that $-E_j^2 > 2$ in this case). Now if $a'_{j-1} - a'_j \geq \frac{\delta}{2}$, then the chain will be bounded and thus the index of K_T at P. But if $a'_{j+1} - a'_j \geq \frac{\delta}{2}$ then $a'_r - a'_{r-1} \geq \frac{\delta}{2}$ and so $(-E_r^2 - 1)a'_r = 1 - (a'_r - a'_{r-1}) \leq 1 - \frac{\delta}{2}$. Hence if $m = r$, then we are done. But if $m = 1$, then again the whole chain is bounded and so the index of K_T at P. Now if $j = r$, then again the chain is bounded if $m = 1$ and $a'_m = a'_j = a'_r < \frac{1+\delta}{2}$ if $m = r$.

In the second case, that is, if E_m is within a bounded distance from E_k, then the coefficient of E_m in ${}^*C$ on W is bounded from below. □

Lemma 3.8.5 *For any $h > 0$ there is a $\gamma > 0$ such that if $(T/\mathrm{pt.}, B)$ is a δ-lc WLF pair (δ is already fixed) with a component C of B passing through $P \in T$ and $t \geq h$ where t is the coefficient of C in B, then K_T is $\delta + \gamma$-lc at P.*

Proof As discussed in Lemma 3.8.4 we may assume that the singularity at P is of type A_r. Moreover, we assume that $1 - a_k > \mu$ for some fixed number $\mu > 0$ where a_k is the log discrepancy of the pair (T, B) at an exceptional divisor E_k/P on W_T. Here $W_T \to T$ is a minimal resolution and $\text{exc}(W_T/P) = \{E_1, \ldots, E_r\}$. Let $\mathcal{C}$ be the longest connected sub-chain of -2-curves in $\text{exc}(W_T/P)$ and W_1 a model where $\mathcal{C}$ is intersected by a -1-curve F for the first time, that is, we blow down -1-curves on W_T until we get a model W_1 and a morphism $W_T \to W_1$ such that W_1 is the first model where there is a -1-curve F intersecting $\mathcal{C}$ (on W_1). Let $K_{W_T} + B^+ \equiv 0$ be a (lc) $(0, \mathbb{Q})$-complement of $K_{W_T} + B_{W_T}$. Assume that F intersects E_j in $\mathcal{C}$ and let t_{E_j} and t_F be the coefficients of E_j and F in B^+ on W_T (similar notation for the coefficients of other exceptional divisors). Then, an argument as in the proof of the nonexceptional case gives a contradiction:

1. Suppose F intersects, transversally and in one point, only one curve in $\mathcal{C}$, say E_j. First suppose that E_j is a middle curve, that is, there are E_{j-1} and E_{j+1} in $\mathcal{C}$ which both intersect E_j. Now contract F so E_j becomes a -1-curve. Then, contract E_j and then E_{j-1} and then all those which are on the of E_{j-1}. By contracting each curve we increase E_{j+1}^2 by one. If we continue contracting -1-curves we get S ($S = \mathbb{P}^2$ or a ruled surface with no -1-curve) where E_{j+1} is a component of B_S. By Lemma 3.8.3 there can be only a bounded number of curves in $\mathcal{C}$ on the side of E_{j-1}. Similarly there is only a bounded number of curves in $\mathcal{C}$ on the side of E_{j+1}. So we are done in this case.

Now suppose that E_j is on the edge of the chain $\mathcal{C}$ and that it intersects E_{j-1}. Let ${B^+}_{W_1} = \dot{B^+} + M$ (M and $\dot{B^+}$ with no common component) where each component of $\dot{B^+}$ is either F or an element of $\mathcal{C}$. Now we have

$$0 = (K_{W_1} + {B^+}_{W_1}) \cdot F = t_{E_j} - 1 - t_F + (M \cdot F)$$

and thus $M \cdot F = 1 + t_F - t_{E_j}$. Similarly let $B^+{}_{W_1} = \ddot{B}^+ + N$ (N and $\ddot{B}^+$ with no common component) where each component of $\ddot{B}^+$ is either F or an element of $\mathcal{C}$. Then, we have

$$0 = (K_{W_1} + B^+{}_{W_1}) \cdot E_j = -2t_{E_j} + t_{E_{j-1}} + t_F + (N \cdot E_j)$$

and so $t_{E_j} = t_{E_{j-1}} - t_{E_j} + t_F + (N \cdot E_j) > \mu$. Hence $t_{E_{j-1}} - t_{E_j} > \frac{\mu}{3}$ or $t_F > \frac{\mu}{3}$ or $(N \cdot E_j) > \frac{\mu}{3}$.

If $t_F > \frac{\mu}{3}$, then by contracting F we increase M^2 by at least $(M \cdot F)^2 \geq t_F^2 > (\frac{\mu}{3})^2$. We have the same increase when we contract E_j and then E_{j-1} and so on. So Lemma 3.8.3 shows the boundedness of $\mathcal{C}$.

If $(N \cdot E_j) > \frac{\mu}{3}$, then proceed similar to the last paragraph.

If $t_{E_{j-1}} - t_{E_j} > \frac{\mu}{3}$, then $t_{E_{j-1}} > t_{E_j} + \frac{\mu}{3}$. This implies that $t_{E_j} \leq 1 - \frac{\mu}{3}$, hence $M \cdot F \geq \frac{\mu}{3}$ and so we continue as above.

2. Now assume that F intersects $\mathcal{C}$ in more than one curve or intersects a curve in $\mathcal{C}$ with intersection number more than one. Suppose the chain $\mathcal{C}$ consists of $E_s, \dots, E_u$ and F intersects $E_{j_1}, \dots, E_{j_l}$. Note that l is bounded.

If $F \cdot E_{j_k} > 1$ for all $1 \leq k \leq l$, then contract F. So $E_{j_k}^2 \geq 0$ after contraction of F hence E_{j_k} can not be contracted. Therefore, it appears in the boundary on a "minimal" model S (namely, S is the projective plane or a smooth ruled surface with no -1-curve). Replace $\mathcal{C}$ with its longest connected subchain when we disregard all E_{j_k}. From here we can return to step one and repeat the argument.

Now suppose $F \cdot E_{j_k} = 1$ for some k. So F must intersect at least another E_{j_q} where $q = k+1$ or $q = k-1$. Now contract F so E_{j_k} becomes a -1-curve and would intersect E_{j_q}. Contracting E_{j_k} and possible subsequent -1-curves will prove that there are only a bounded number of curves between E_{j_q} and

E_{j_k} in $\mathcal{C}$. Now after contracting E_{j_k} and all other curves between E_{j_q} and E_{j_k} we will have $E_{j_m}^2 \geq 0$ for each $m \neq k$. So again we take the longest connected subchain excluding $E_{j_1}, \ldots, E_{j_l}$ and return to step one.

This process must stop after a bounded number of steps because the number of curves in B_S^+ with coefficient $> \mu$ is bounded (S is again a "minimal" model). To prove this latter boundedness note that $(K_S + B^+{}_S) \cdot F = 0$, where we assume that S is a ruled surface and F a fibre. This implies that there is only a bounded number of non-fibre components in B_S^+ with coefficient $> \mu$. Let L be a section and t_L be its coefficient in B_S^+ and F_i fibre components of B_S^+ with $t_{F_i} > \mu$. Then,

$$\begin{aligned} 0 \geq \; & (K_S + t_L L + \textstyle\sum_i t_{F_i} F_i) \cdot L \\ = \; & (-2L + (2g-2-e)F + t_L L + \textstyle\sum_i t_{F_i} F_i) \cdot L \\ = \; & -t_L e + e + 2g - 2 + \textstyle\sum_i t_{F_i} \end{aligned}$$

which proves that there is a bounded number of F_i ($L^2 = -e$ and $e + 2g \geq 0$ if $e < 0$). So the chain $\mathcal{C}$ must have a bounded length. This implies that if we throw C away in the boundary B, then the mld at P will increase by at least a fix number $\gamma > 0$ (γ does not depend on P or T). This proves the lemma. $\square$

Lemma 3.8.5 settles the first case in 3.8.3.1 by deleting the boundary B_{V_1}.

Now assume the second case in 3.8.3.1. Let F be a general fibre of the contraction defined by the extremal ray R. If the other extremal ray of V_1 defines a birational map $V_1 \to Z$, then let H be the exceptional divisor of this contraction (otherwise delete the boundary and use 3.8.5).

If K_{V_1} is antinef, then use again 3.8.5. If K_{V_1} is not antinef and if $E_1 \neq H$ then apply Lemma 3.8.5 to (Z, B_Z). Boundedness of Z implies the boundedness of V_1 and so we can apply Lemma 3.7.6. But if K_{V_1} is not antinef and $E_1 = H$, then perform a hat of the third type as defined in the proof of Theorem 3.7.1 with $(U, G_U) := (V_1, B_{V_1} + t_1 E_1)$ and $V_2 := U'$. We can use Lemma 3.8.5 on V_2 or after contracting a curve on V_2 to get the boundedness of V_2. Boundedness of V_2 implies the boundedness of V_1. $\square$

Corollary 3.8.6 *Conjecture* $\mathrm{WC}_{\delta,2,\Gamma_f}$ *(3.1.7) holds in the global case where* Γ_f *is a finite subset of rational numbers in* $[0,1]$.

Proof Obvious by Theorem 3.8.1.

3.9 An example

Example 3.9.1 Let m be a positive natural number. For any $\mu \in (0,1)$ and any $\tau > 0$ there is a model $(X, 0)$ satisfying the following:

1. X is $\frac{1}{m}$-lc.

2. There is a partial resolution $Y \to X$ such that $K_Y + B_Y := {}^*K_X$ is $\frac{1}{m} + \mu$-lc in codim 2 and $b_i > \frac{m-1}{m} - \mu$. Set $D := \sum \frac{m-1}{m} B_i$.

3. $K_Y + D$ is not $\frac{1}{m} + \tau$-lc in codim 2.

Proof Let $P \in X$ with X smooth outside P. Suppose that the minimal resolution of P has the following diagram:

$$O^{-3} \text{ —— } O^{-2} \text{ —— } \cdots \qquad \text{ —— } O^{-2} \text{ —— } O^{-2} \text{ —— } O^{-4}$$

where the numbers show the self-intersections.

This diagram has the following corresponding system on a minimal resolution where a_i stand for the log discrepancies:

$$\begin{cases} 3a_1 - a_2 - 1 = 0 \\ 2a_2 - a_1 - a_3 = 0 \\ \vdots \\ 2a_{r-1} - a_{r-2} - a_r = 0 \\ 4a_r - a_{r-1} - 1 = 0 \end{cases}$$

Now let $t = a_{r-1} - a_r$. Then, $a_{r-2} - a_{r-1} = t, \ldots, a_1 - a_2 = t$ and $a_r = \frac{1+t}{3}$ and $a_1 = \frac{1-t}{2}$. The longer the chain the smaller the t is and the discrepancies vary from $-\frac{1+t}{2}$ to $\frac{t-2}{3}$. Other a_i can be calculated as $a_i = a_1 - (i-1)t = \frac{1-t}{2} - (i-1)t = \frac{1-(2i-1)t}{2}$.

Suppose that j is such that $a_j < \frac{1}{m} + \mu$ but $a_{j-1} \geq \frac{1}{m} + \mu$. So the exceptional divisors corresponding to $a_r, a_{r-1}, \ldots, a_j$ will appear on Y but the others will not. Now we try to compute the log discrepancies of the pair (Y, D). The minimal resolution for $P \in X$ is also the minimal resolution for Y. But only $E_1, \ldots, E_{j-1}$ are exceptional/Y. The system for the new log discrepancies (for (Y, D)) is as follows:

$$\begin{cases} 3a'_1 - a'_2 - 1 = 0 \\ 2a'_2 - a'_1 - a'_3 = 0 \\ \vdots \\ 2a'_{j-2} - a'_{j-3} - a'_{j-1} = 0 \\ 2a'_{j-1} - a'_{j-2} - \frac{1}{m} = 0 \end{cases}$$

Let $s = a'_{j-2} - a'_{j-1}$ so as before, we have $a'_{j-1} = \frac{1}{m} + s$ and $a'_1 = \frac{1-s}{2}$. If j is big (i.e. if t is small enough), then s will be small and so $a'_{j-1} = \frac{1}{m} + s < \frac{1}{m} + \tau$. Hence (Y, D) is not $\frac{1}{m} + \tau$-lc.

□

3.10 Local cases revisited

Using the methods in the proof of the global case, we give a new proof of the local cases. Here again by $/Z$ we mean $/P \in Z$ for a fixed P. The following is the main theorem in this section.

Theorem 3.10.1 *Conjecture* $\mathrm{WC}_{\delta,2,\Phi_{\mathrm{sm}}}$ *(3.1.7) holds in the local case, that is, when we have* $\dim Z \geq 1$ *and* $\Gamma = \Phi_{\mathrm{sm}}$.

Proof Our proof is similar to the nonexceptional global case. Here the pair $(X/Z, B)$ is a WLF surface log pair where (X, B) is δ-lc and $B \in \Phi_{\mathrm{sm}}$. Fix $P \in Z$. Then, there exists a regular $(0, n)$-complement$/P \in Z$, $K + B^+$ for some $n \in \{1, 2, 3, 4, 6\}$ by [Sh2].

1. Recall the first step in the proof of Theorem 3.7.1.

2. Recall Definition 3.7.2 and Lemma 3.7.3. Let m be the smallest number such that $\frac{1}{m} \leq \delta$. Let $h = \min\{\frac{k-1}{k} - \frac{u}{r!} > 0\}_{1 \leq k \leq m}$ where u, k are natural numbers and $r = \max\{m, 6\}$. Now choose a τ for m as in Lemma 3.7.3 such that $\tau < h$.

   Blow up one exceptional divisor E/P via $f\colon Y \to X$ such that the log discrepancy satisfies $\frac{1}{k} \leq a(E, X, B) \leq \frac{1}{k} + \tau$ for some k (if such E does not exist, then return to step 1). The crepant log divisor $K_Y + B_Y$ is $\frac{1}{m}$-lc and so by the choice of τ, $K_Y + D_\tau$ is also $\frac{1}{m}$-lc (D_τ is constructed for B_Y). Let $K_Y + B_Y^+$ be the crepant blow up of $K_X + B^+$. Then, again by the way we chose τ we have $D_\tau \leq B_Y^+$. Now run the anti-LMMP$/P \in Z$ (reflmmp) over $K_Y + D_\tau$ i.e. contract any birational type extremal ray $R/P \in Z$ such that $(K_Y + D_\tau) \cdot R > 0$. At the end we get a model X_1 with one of the following properties:

   - $(K_{X_1} + D_\tau) \equiv 0/P \in Z$ and $K_{X_1} + D_\tau$ is $\frac{1}{m}$-lc.
   - $-(K_{X_1} + D_\tau)$ is nef and big$/P \in Z$ and $K_{X_1} + D_\tau$ is $\frac{1}{m}$-lc.

   where $K_{X_1} + D_\tau$ is the birational transform of $K_Y + D_\tau$ and let $g\colon Y \to X_1$ be the corresponding morphism.

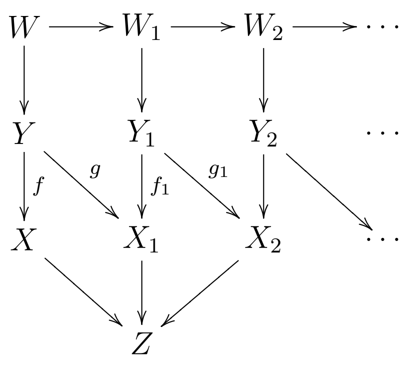
   The nefness of $-(K_{X_1} + D_\tau)$ comes from the fact that $D_\tau \leq B_1^+$. We see that $K_{X_1} + D_\tau$ is $\frac{1}{m}$-lc by applying Lemma 3.7.3.

3. Whichever case occurs above, to construct a complement, it is enough to bound the index of $K_{X_1} + D_\tau/P$.

4. Let C be a curve contracted by $g\colon Y \to X_1$. If C is not a component of B_Y, then the log discrepancy of C with respect to $K_{X_1} + B_{X_1}$ is at least

1 where $K_{X_1} + B_{X_1}$ is the birational transform of $K_Y + B_Y$. Moreover $g(C) \in \operatorname{Supp} B_{X_1} \neq \emptyset$. So the log discrepancy of C with respect to K_{X_1} is more than 1. This means that C is not a divisor on a minimal resolution $W_1 \to X_1$. Let $W \to X$ be a minimal resolution. Then, there is a morphism $W \to W_1$. Hence $\operatorname{exc}(W_1/X_1) \subseteq \operatorname{exc}(W/X) \cup \operatorname{Supp}(B = B_X)$. Now if $C \in \operatorname{exc}(W/X) \cup \operatorname{Supp} B$ is contracted by g, then $a(C, X_1, D_\tau) < a(C, X, B)$.

5. Let $(X_1, B_1) := (X_1, D_\tau)$ and repeat the process. In other words again we blow up one exceptional divisor E via $f_1 \colon Y_1 \to X_1$ such that the log discrepancy satisfies $\frac{1}{k} \leq a(E, X_1, B_1) \leq \frac{1}{k} + \tau$ for some natural number $k > 1$. The crepant log divisor $K_{Y_1} + B_{1,Y_1}$ is $\frac{1}{m}$-lc and so by Lemma 3.7.3 $K_{Y_1} + D_{1,\tau}$ is $\frac{1}{m}$-lc. Note that the point which is blown up on X_1 cannot be smooth since $\tau < h$ as defined above. So according to the last step the blown up divisor E is a member of $\operatorname{exc}(W/X) \cup \operatorname{Supp} B$. Now we run again the anti-LMMP on $K_{Y_1} + D_{1,\tau}$ and proceed as in step 2.



6. Steps 4 and 5 show that each time we blow up a member of $\operatorname{exc}(W/X) \cup$

Supp B say E. If we blow that divisor down in some step, then the log discrepancy $a(E, X_j, B_j)$ will decrease. That divisor will not be blown up again unless the log discrepancy drops by at least $\frac{1}{2(m-1)} - \frac{1}{2m}$ (this is not a sharp bound). So after finitely many steps we get a model X_i with a standard boundary B_i for which there is no E/P where $\frac{1}{k} \leq a(E, X_i, B_i) \leq \frac{1}{k} + \tau$ for any $1 < k \leq m$. Hence the index of $-(K_{X_i} + B_i)/P$ is bounded and so we can construct an appropriate complement for $(X_i, B_i)/Z$. This implies the existence of the desired complement for $(X, B)/Z$.

□

4 Epsilon-log canonical complements in higher dimensions

In this chapter we consider the (ϵ, n)-lc complements in higher dimensions, that is, in dimensions more than two. This is a joint work in progress with V.V. Shokurov. In subsection 4.1 we try to work out the proof of Theorem 3.7.1 in dimension 3 and we point out the problems we have to solve in order to finish the proof of Conjecture 3.1.7 in dimension 3. In subsection 4.2 we outline Shokurov's plan on the same problem.

Let $X \to Z$ be an extremal K_X-negative contraction where X is a 2-dimensional pseudo-WLF and Z is a curve. We know that $Z \simeq \mathbb{P}^1$ since Z must be rationally connected as X is. Moreover $\rho(X) = 2$. Similar Mori fibre spaces in higher dimensions are not that simple. This makes the boundedness problem of (ϵ, n)-lc complements more difficult in higher dimensions. We also don't know yet whether the index of $K_X + B$ will be bounded if we fix the mld at a point.

In chapter 3 we first proved the boundedness of ϵ-lc complements (Theorem 3.7.1 and Theorem 3.10.1) and then the BAB (Corollary 3.7.9). But in higher dimensions we expect to solve both problems together. In other words in some cases where it is difficult to prove the boundedness of varieties, it seems easier to prove the boundedness of complements; specially when we deal with a fibre space. Conversely when it is difficult to prove the boundedness of ϵ-lc complements, it is better to prove the boundedness of pairs; this is usually the case when the pairs are exceptional.

Lemma 4.0.2 *Let $X \dashrightarrow X'$ be a flip/Z and assume that (X, B) is (ϵ, n)-*

complementary/Z. Then, (X', B') is (ϵ, n)-complementary/Z where B' is the birational transform of B.

Proof Obvious from the definition of (ϵ, n)-complements. $\square$

Note that in the previous Lemma it doesnot matter with respect to which log divisor the flipping is taken to be.

Lemma 4.0.3 *Let (Y, B) be a pair and $Y \dashrightarrow Y'/Z$ be a composition of divisorial contractions and flips/Z such that in each step we contract an extremal ray R where $(K+B).R \geq 0$. Suppose $B' = \sum b'_i B'_i$ is the birational transform of B, the pair (Y', B') is (ϵ, n)-complementary/Z and $(n+1)b'_i \geq nb'_i$ for each coefficient b'_i. Then, (Y, B) is also (ϵ, n)-complementary/Z.*

Proof Clear by Lemmas 3.2.17 and 4.0.2. $\square$

4.1 Epsilon-lc complements in dimension 3

In this section we propose a plan toward the resolution of Conjecture 3.1.7 in dimension 3.

We repeat the proof of 3.7.1, in dimension 3, step by step:

1. Under the assumptions of Conjecture 3.1.7 for $d = 3$ and $\Gamma = \{0\}$, first assume that $(X, 0)$ is nonexceptional.

2. We do not have much information about the accumulation points of mlds in dimension 3. Actually we still have not proved ACC in dimension 3 (Conjecture 3.1.15). As pointed out in the introduction of

chapter 3, only one case of ACC in dimension 3 is remained to be proved. Remember that Shokurov's program tries to use complements in dimension $d-1$ in order to prove the ACC in dimension d. So it is reasonable to assume ACC in dimension $d-1$.

Lets denote by $\mathrm{Accum}_{d,\Gamma}$ the set of accumulation points of mlds of d-dimensional lc pairs (T, B), where $B \in \Gamma$.

3. We may be able to use inductive complements; since $(X, 0)$ is not exceptional, it is expected that there is an inductive $(0, n)$-complement $K_X + B^+$ where $n \in \mathcal{N}_2$. Inductive complements are those which are extended from lower dimensional complements [PSh, 1.12].

4. Remember definition 3.7.2. We can similarly define $D_{\tau,A}$ for a boundary B, with respect to a real number $\tau \geq 0$ and a set $A \subseteq [0, 1]$:

$$D_{\tau,A} := \sum_{b_i \notin [a-\tau,a]} b_i B_i + \sum_{b_i \in [a-\tau,a]} a B_i$$

where in the first term $b_i \notin [a-\tau, a]$ for any $a \in A$ but in the second term $a \in A$ is the biggest number satisfying $b_i \in [a-\tau, a]$.

Definition 4.1.1 Let $A \subseteq [0, 1]$ and let (T, B) be a log pair. We say that (T, B) is A-lc if (T, B) is x-lc where $x := 1 - \sup\{A\}$.

Assuming the ACC in dimension 3 a statement similar to Lemma 3.7.3 may hold: For any $\gamma > 0$ and finite set $A \subseteq [0, 1]$ containing $1-\gamma$ there is a real number $\tau > 0$ such that if (T, B_T) is a 3-fold log pair, $P \in T$, $K_T + B_T$ is γ-lc in codim 2 at P and $D_{\tau,A} \in A$, then $K_T + D_{\tau,A}$ is also γ-lc in codim 2 at P.

Moreover we expect that there is a $\tau > 0$ such that the following conditions hold as well:

- If $B_T \in A$ and E is the exceptional divisor of a smooth blow up of T, then $a(E, T, B_T) \notin [1 - a, 1 - a + \tau]$ for any $a \in A$.
- If $B_T \in A$ and the pair (T, B_T) is nonexceptional, then we can refine $\mathcal{N}_2$ so that there is a $(0, n)$-complement $K_T + B_T^+$ for some $n \in \mathcal{N}_2$ with $B_T \leq B_T^+$.

5. Let $A_1 := \{a_1\}$ where $1 - a_1 = \max\left(\text{Accum}_{3,\{0\}} \cap [0, \delta]\right)$. Now blow up all exceptional divisor E such that $a(E, T, B_T) \in [1 - a, 1 - a + \tau]$ for some $a \in A_1$ to get $f : Y \to X$. Construct D_{τ,A_1} for B_Y where $K_Y + B_Y$ is the crepant pull back. Hence (Y, D_{τ,A_1}) is A_1-lc. Run the D-LMMP where $D := -(K_Y + D_{\tau,A_1})$. At the end we get $Y \dashrightarrow X_1$ and $X_1 \dashrightarrow S_1$ such that $-(K_{X_1} + D_{\tau,A_1})$ is nef and $\equiv 0/S_1$ and $-(K_{S_1} + D_{\tau,A_1}).R > 0$ for any birational type extremal ray R.

6. There are the following possibilities for the model S_1:

   ◇ $\rho(S_1) = 1$, $-(K_{S_1} + D_{\tau,A_1}) = -(K_{S_1} + B^+) \equiv 0$ and $K_{S_1} + D_{\tau,A_1}$ is A_1-lc.

   ◇ There is a fibration type extremal ray R such that , $-(K_{S_1} + D_{\tau,A_1}).R = 0$ and $K_{S_1} + D_{\tau,A_1}$ is A_1-lc.

   ◇ $-(K_{S_1} + D_{\tau,A_1})$ is nef and big and $K_{S_1} + D_{\tau,A_1}$ is A_1-lc.

7. In the first case of the above division we are done. In the second and third case we replace $(X, 0)$ by $(X_1, B_1) := (X_1, D_{\tau,A_1})$ and return to

step one and repeat. At each repetition of the process, we get new coefficients. In other words, we need to replace A_i with A_{i+1} such that $A_i \subseteq A_{i+1}$. We need to prove that $\cup_{i\to\infty} A_i$ is finite.

8. At the end, we get a model (X_r, B_r) which is terminal in codim 2. Then, we hope to prove the boundedness of the index of $K_{X_r} + B_r$ possibly after some more blow ups and blow downs. This will settle the problem if $-(K_{X_r} + B_r)$ is nef and big. Otherwise we may have a fibration and $K_{X_r} + B_r^+ = K_{X_r} + B_r + N$ where N is vertical. Then, we may replace N by N' and construct a desirable complement $K_{X_r} + B_r + N'$. At the end we need to prove that the boundedness of the complement implies the boundedness of the pairs.

9. Next let $(X, 0)$ be exceptional. Since $\mathrm{BAB}_{1,3,\{0\}}$ (3.1.11) holds by [KMMT], by assuming ACC in dimension 3, we can see that there is a $\tau > 0$ such that $\mathrm{BAB}_{1-\tau,3,\{0\}}$ also holds. Blow up an exceptional/X divisor E_1 with log discrepancy $a_{E_1} = a(E_1, X, 0) \leq 1 - \tau$ to get $Y \to X$ and put $K_Y + B_Y = {}^*K_X$. Let $t \geq 0$ be a number such that there is an extremal ray R with the properties $(K_Y + B_Y + tE_1).R = 0$ and $E_1.R > 0$ (and $K_Y + B_Y + tE_1$ Klt and antinef). Such R exists otherwise there is a $t > 0$ such that $K_Y + B_Y + tE_1$ is lc (and not Klt) and antiample. This contradicts the fact that $(X, 0)$ is exceptional. Now contract $R\colon Y \to Y_1$ if it is of birational type (and perform the flip if it is a flipping).

   By increasing t again, we find that there is an extremal ray R_1 on Y_1 such that $(K_{Y_1} + B_{Y_1} + tE_1).R_1 = 0$ and $E_1.R_1 > 0$ (preserving the nefness of $-(K_{Y_1}+B_{Y_1}+tE_1)$). If it is of birational type, then contract it and so on. After finitely many steps we get a model $(V_1, B_{V_1}+t_1E_1)$ and a number $t_1 > 0$ with the following possible outcomes:

- ◇ $(V_1, B_{V_1}+t_1E_1)$ is Klt, $\rho(V_1) = 1$ and $K_{V_1}+B_{V_1}+t_1E_1 \equiv 0$.
- ◇ $(V_1, B_{V_1}+t_1E_1)$ is Klt and there is a fibre type extremal ray R on V_1 such that $(K_{V_1}+B_{V_1}+t_1E_1).R = 0$ and $K_{V_1}+B_{V_1}+t_1E_1$ is antinef.

If the second case occurs, then we do not know $\rho(V_1)$ unlike the surface case where $\rho(V_1) = 2$.

10. In the proof of Theorem 3.7.1 we introduced three types of hat. Here, also we can similarly define hats but it is not clear yet how to proceed.

4.2 Epsilon-lc complements in dimension 3: Shokurov's approach

Here we explain Shokurov's approach to the problem discussed in 4.1.

1. We know that $\mathrm{BAB}_{1,3,\{0\}}$ (3.1.11) holds by [KMMT]. Let a be the smallest positive real number with the following property: $\mathrm{BAB}_{a',3,\{0\}}$ holds for any $a' > a$. The idea is to prove that $\mathrm{BAB}_{a,3,\{0\}}$ holds and thus, assuming the ACC in dimension 3, to prove $a = 0$. Now assume that $\mathrm{BAB}_{\epsilon',3,\{0\}}$ holds for any $\epsilon' > \epsilon$ where $1 > \epsilon > 0$.

2. Prove $\mathrm{SC}_{\epsilon,3}$ (Conjecture 3.1.9) in the local case. Moreover prove that the local ϵ-lc complement indices can be chosen in a way that there is

a $\tau > 0$ such that if $1-\epsilon-\tau \le b \le 1-\epsilon$, then $\llcorner (n+1)b \lrcorner \ge n(1-\epsilon)$ for any local ϵ-lc complement index n.

3. Blow up all exceptional divisor E such that $\epsilon \le a(E,X,0) \le \epsilon+\tau$ to get $f\colon Y \to X$. Then, $D_{\tau,\{1-\epsilon\}} := \sum_i (1-\epsilon)B_i$ where $B_Y = \sum_i b_i B_i$ is the crepant pull back boundary. Then, run the D-LMMP for $D := -(K_Y + D_{\tau,\{1-\epsilon\}})$. At the end we get $g\colon Y \dashrightarrow X_1$ and $X_1 \dashrightarrow S_1$ such that $-(K_{X_1} + D_{\tau,\{1-\epsilon\}})$ is nef and $\equiv 0/S_1$ and $-(K_{S_1} + D_{\tau,\{1-\epsilon\}}).R > 0$ for any birational type extremal ray R.

4. There are the following possibilities for the model S_1:

    - ◇ $\rho(S_1) = 1$, $K_{S_1} + D_{\tau,\{1-\epsilon\}}$ is ample and $K_{S_1} + D_{\tau,\{1-\epsilon\}}$ is ϵ-lc.
    - ◇ $-(K_{S_1} + D_{\tau,\{1-\epsilon\}}).R = 0$ for a fibre type extremal ray R and the log divisor $K_{S_1} + D_{\tau,\{1-\epsilon\}}$ is ϵ-lc.
    - ◇ $-(K_{S_1} + D_{\tau,\{1-\epsilon\}})$ is nef and big and $K_{S_1} + D_{\tau,\{1-\epsilon\}}$ is ϵ-lc.

5. If the first case happens in the above division, then delete the boundary, so $(S_1, 0)$ is $\epsilon+\tau$-lc and so the pair is bounded by the assumptions.

6. **Definition 4.2.1** Let $f\colon T \to Z$ be a contraction and $K_T + B \sim_{\mathbb{R}} 0/Z$. Put $D_Z := \sum_i d_i D_i$ where d_i is defined as follows:

$$1 - d_i = \sup\{c \mid K_T + B + cf^*D_i \text{ is lc over the generic point of } D_i\}$$

7. If the second case of step 4 occurs, then we need the following general Conjecture, due to Shokurov [PSh1] and Kawamata [K3], which is useful in many situations:

Conjecture 4.2.2 (Adjunction) *Let $(T/Z, B)$ be a lc pair of dimension d such that $K_T + B \sim_{\mathbb{R}} 0/Z$. Define the unique class M_Z up to $\mathbb{R}$-linear equivalence as $K_T + B \sim_{\mathbb{R}} {}^*(K_Z + D_Z + M_Z)$. Then, the followings hold:*

Adjunction *We can choose an $M_Z \geq 0$ in its $\mathbb{R}$-linear equivalence class so that $(Z, D_Z + M_Z)$ is lc.*

Effective adjunction *Fix Γ_f. Then, there is a constant $I \in \mathbb{N}$ depending only on d and Γ_f such that $|IM_Z|$ is a free linear system for an appropriate choice of M_Z. In addition*

$$I(K_T + B) \sim {}^*I(K_Z + D_Z + M_Z).$$

It is expected that the effective adjunction implies the boundedness of S_1 under our assumptions.

8. If the third case of step 4 occurs, then we need to repeat the process with a bigger ϵ. We have new coefficients in the boundary. Moreover we need to prove that this process stops after a bounded number of steps.

9. If every time the third case occurs, then at the end we get a pair (X_r, B_r) which is terminal in codim 2 and $-(K_{X_r} + B_r)$ is nef and big. After some more blow ups and blow downs we may prove that the index of $K_{X_r} + B_r$ is bounded.

4.3 List of notation and terminology for chapter three-four

$\mathbb{N}$	*The set of natural numbers* $\{1, 2, \dots\}$.
$\mathbb{R}^+$	*The set of positive real numbers. Similar notation for* $\mathbb{Q}$.
dim	*Dimension or dimensional.*
WLF	*Weak log Fano.* $(X/Z, B)$ *is WLF if* X/Z *is a projective contraction and* $-(K_X + B)$ *is nef and big/*Z *and* X *is* $\mathbb{Q}$*-factorial.*
pseudo-WLF	*Pseudo weak log Fano/*Z*, that is, there is a* B *where* $(X/Z, B)$ *is WLF.*
Φ_{sm}	*The set of standard boundary multiplicities, that is,* $\{\frac{k-1}{k}\}_{k\in\mathbb{N}} \cup \{1\}$.
Γ_f	*A finite subset of* $[0, 1]$.
$\mathrm{mld}(\mu, X, B)$	*The log minimal discrepancy of* (X, B) *at the centre* μ.
$P(D)$	*The smallest positive natural number* r *such that* rD *is a Cartier divisor at* P.
$\mathrm{WC}_{\delta,d,\Gamma}$	*The weak Conjecture on the boundedness of* ϵ*-lc complements in dimension d. See 3.1.7*
$\mathrm{SC}_{\delta,d}$	*The strong Conjecture on the boundedness of* ϵ*-lc complements in dimension d. See 3.1.9*
$\mathrm{BAB}_{\delta,d,\Gamma}$	*The Alexeev-Borisovs Conjecture on the boundedness of d-dimensional* δ*-lc WLF varieties. See 3.1.11*
LT_d	*The log termination Conjecture in dimension d. See 3.1.16*
$\mathrm{ACC}_{d,\Gamma}$	*The ACC Conjecture on mlds in dimension d. See 3.1.15*

4.4 References for chapter three-four:

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