

ALGEBRAIC GEOMETRY PART III EXAMPLES AND EXERCISES 2

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Example 1. *Product of quasi-projective algebraic sets: we embed the set theoretic product $\mathbb{P}_k^n \times \mathbb{P}_k^m$ into some projective space so that we can consider it as an algebraic set. Define the map $\phi: \mathbb{P}_k^n \times \mathbb{P}_k^m \rightarrow \mathbb{P}_k^N$ where $N = (n+1)(m+1) - 1$ by*

$$\phi((x_0 : \cdots : x_n), (y_0 : \cdots : y_m)) = (x_0 y_0 : x_0 y_1 : \cdots : x_i y_j : \cdots : x_n y_m)$$

Lets take s_{ij} to be a variable which corresponds to the coordinate of \mathbb{P}_k^N where $x_i y_j$ appears, and let Z be the image of ϕ . Then, Z is a closed subset of \mathbb{P}_k^N given by all the equations $s_{ij} s_{uv} - s_{iv} s_{uj} = 0$. ϕ is called the segre embedding.

Now if $X \subseteq \mathbb{P}_k^n$ and $Y \subseteq \mathbb{P}_k^m$ are quasi-projective algebraic sets, then, $\phi(X \times Y)$ is a quasi-projective algebraic subset of \mathbb{P}_k^N .

Exercise 2. *Let $\phi: X \rightarrow Y$ and $\psi: Y \rightarrow Z$ be regular maps of quasi-projective algebraic sets and $W \subseteq X$ be a closed or open subset of X . Prove that*

- (i) the composition $\psi\phi: X \rightarrow Z$ is a regular map,*
- (ii) the natural embedding $i: W \rightarrow X$ is a regular map,*
- (iii) the restriction map $\phi|_W: W \rightarrow Y$ is a regular map.*

Example 3. *Dimension of a single point is zero and dimension of \mathbb{A}_k^1 is 1. Dimension of \emptyset by convention is -1 .*

Exercise 4. *Let X be a quasi-projective algebraic set. Prove that if $Y \subseteq X$ is a closed subset of X , then $\dim Y \leq \dim X$. Prove that if $X = X_1 \cup \cdots \cup X_l$ where X_i are closed subsets, then $\dim X = \max\{\dim X_i\}$.*

Exercise 5. *Let X be a quasi-projective variety and U a dense open subset. Prove that $\dim U = \dim X$. In particular, $\dim \mathbb{P}_k^n = n$.*

Exercise 6. *Let $g, h \in k[t_1, \dots, t_n]$ and $x \in \mathbb{A}_k^n$. Prove that $d_x(g+h) = d_x g + d_x h$ and $d_x(gh) = g(x)d_x h + h(x)d_x g$.*

Example 7. *Let $X = \mathbb{A}_k^n$. Then, $T_{X,x} = \mathbb{A}_k^n$ for every x .*

Example 8. Let $X \subseteq \mathbb{A}_k^n$ with $I_X = \langle f \rangle$. Then, $T_{X,x}$ is defined by the single equation $d_x f = 0$. Thus, $T_{X,x} = \mathbb{A}_k^n$ iff $\frac{\partial f}{\partial t_j}(x) = 0$ for all j . If some $\frac{\partial f}{\partial t_j}(x) \neq 0$, then $\dim_k T_{X,x} = n - 1$.

Exercise 9. Define a differential map $d_x: \mathcal{O}_{X,x} \rightarrow T_{X,x}^*$ by

$$d_x(f/g) = \frac{g(x)d_x f - f(x)d_x g}{g(x)^2}$$

Prove that this is well-defined and it gives an isomorphism $m_x/m_x^2 \rightarrow T_{X,x}^*$ as k -vector spaces where m_x is the maximal ideal. (Hint: use the fact that the differential map $d_x: I_x/I_x^2 \rightarrow T_{X,x}^*$ defined before factors through $I_x/I_x^2 \rightarrow m_x/m_x^2 \rightarrow T_{X,x}^*$)

Exercise 10. Let X and Y be affine algebraic sets, $x \in X$ and $y \in Y$. Prove that if x and y are smooth points, then (x, y) is a smooth point of $X \times Y$.

Example 11. Let $X \subset \mathbb{A}_k^2$ be defined by $f = t_2^2 - t_1^3$. Then, $0 = (0, 0)$ is a singular point because $T_{X,0} = \mathbb{A}_k^2$. Earlier we proved that X is birational to the affine line \mathbb{A}_k^1 but not isomorphic. The presence of the singular point 0 prevents an isomorphism.

Example 12. Suppose that the characteristic of k is not 2 and let $X \subset \mathbb{P}_k^n$ be defined by $F = s_0^2 + \cdots + s_m^2 = 0$ where $m \leq n$. If $m = n$, then X has no singular points. But if $m < n$, then all the points $x = (x_0 : \cdots : x_n) \in X$ with $x_0 = \cdots = x_m = 0$ are singular.

Example 13. Let $X \subset \mathbb{A}_k^2$ be defined by $f = t_2^2 - t_1^3 - t_1^2$. Then, $0 = (0, 0)$ is a singular point because $T_{X,0} = \mathbb{A}_k^2$. At the origin, X has two branches and the part of f with smallest degree, that is, $t_2^2 - t_1^2$ gives the equations of the tangent lines to these branches.

Example 14. An elliptic curve $X \subset \mathbb{P}_k^2$ is a smooth curve given by F of degree 3. For example, $F = t_2^2 t_0 - t_1^3 + t_1 t_0^2$ defines an elliptic curve.

Exercise 15. Find the singular points of

$$X = V(t_1^2 t_2^2 + t_0^2 t_2^2 + t_0^2 t_1^2 - t_0 t_1 t_2 t_3) \subset \mathbb{P}_k^3$$

Exercise 16. Let $X = \mathbb{A}_k^1$, $x \in X$ and $f \in k[t]$. Prove that f is a local parameter at x iff x is a simple root of f , i.e. $t - x$ divides f but $(t - x)^2$ does not.

Exercise 17. Let $x \in X$ be a singular point. Prove that any $f \in \mathcal{O}_{X,x}$ has infinitely many different associated formal power series.

Proof. We may replace X by an affine neighborhood such that every component passes through x and $k[X] \subset \mathcal{O}_{X,x}$.

First assume that X is irreducible. Let $u_1, \dots, u_d \in k[X]$ such that they form a basis of m_x/m_x^2 . Since x is a singular point, $d > \dim X$. Thus, u_1, \dots, u_d are algebraically dependent over k and $h(u_1, \dots, u_d) = 0$ for some nonzero pol $h \in k[s_1, \dots, s_d]$. Now $\Lambda = h$ is a power series associated to 0. This proves that we can associate infinitely many power series to 0 and so to any f .

Now assume that X is not irreducible and $X = X_1 \cup \dots \cup X_n$ is the decomposition of X into irreducible components. As above, we can find $h_i \in k[s_1, \dots, s_d]$ such that $h_i(u_1, \dots, u_d) = 0$ on X_i . Put $h = h_1 \cdots h_n$. The rest is similar. \square

Exercise 18. Let $\phi: \mathbb{A}_k^2 \rightarrow \mathbb{A}_k^1$ be the regular map which sends $(x_1, x_2) \in \mathbb{A}_k^2$ to $x_1 \in \mathbb{A}_k^1$. Prove that ϕ is not a finite map.

Exercise 19. Let $\phi: \mathbb{P}_k^1 \rightarrow \mathbb{P}_k^1$ be the regular map which sends $(x_0 : x_1)$ to $(x_0^l : x_1^l)$ for a fixed $l \in \mathbb{N}$. Prove that ϕ is a finite map.

Exercise 20. Let X be a smooth quasi-projective algebraic set. Prove that it is normal.

Exercise 21. Let X be a quasi-projective algebraic variety of dimension one. Prove that if X is normal, then it is smooth.

Exercise 22. Prove that the normalisation of a quasi-proj algebraic variety is unique if it exists.

Exercise 23. Find an example of a variety which is normal but not smooth (look at $V(t_1^2 + t_2^2 + t_3^2)$).

Example 24. Let $X = \mathbb{A}_k^n$. Let D be a prime divisor on X . The ideal I_D is a prime ideal in $k[t_1, \dots, t_n]$. Suppose that $D \subseteq V(g)$. Then, $g \in I_D$ because I_D is a prime ideal. This means that I_D is a principal ideal which in turn implies that $D \sim 0$. Therefore, $\text{Cl}(X) = 0$.

Example 25. Let $X = \mathbb{P}_k^n$. Let D be a prime divisor on X . Then, $I_D = \langle F \rangle$ for some homogenous polynomial F of degree l . Let L be the prime divisor defined by the linear pol $s_0 = 0$. Since F/s_0^l is a rational function on X , $D - lL \sim 0$. In other words, every divisor is a multiple of L , therefore $\text{Cl}(X) = \mathbb{Z}$.

Exercise 26. Let X be a normal quasi-projective algebraic variety and f a rational function on X . Prove that $\text{Div}(f) \geq 0$ iff f is regular on X .

Exercise 27. Let X be a normal quasi-projective algebraic variety and $D \sim D'$ linearly equivalent divisors on X . Prove that $H^0(X, D) \simeq H^0(X, D')$.

Example 28. Let X be a normal quasi-projective algebraic variety and D a divisor on X . Prove that the abelian groups $H^0(U, D|_U)$ define a sheaf on X . This is called the sheaf associated to D and is denoted by $\mathcal{O}_X(D)$.

Exercise 29. Let X be a normal quasi-projective algebraic variety and D a divisor on X . Prove that the rings $k[U]$ define a sheaf on X . This is called the structure sheaf of X and is denoted by \mathcal{O}_X . Note that this is the same as $\mathcal{O}_X(0)$.

Exercise 30. Let X be a smooth projective curve. Prove that there is a finite regular map $\phi: X \rightarrow \mathbb{P}_k^1$.

Example 31. Let X be a smooth curve and assume that there is a rational function f on X such that $\text{Div}(f) = D_1 - D_2$. We prove that $X \simeq \mathbb{P}_k^1$. Let $\phi: X \rightarrow \mathbb{P}_k^1$ be the rational map defined by $\phi = (1 : f)$ which is regular because X is smooth. On the other hand, $\phi^*(1 : 0) = D_1$. Thus, $\deg(\phi) = 1$ which means that $k(X) = k(\mathbb{P}_k^1)$. So, X and \mathbb{P}_k^1 are birational and therefore, isomorphic.

Exercise 32. Let X be a quasi-projective algebraic set. Prove that the abelian groups $\Omega[U]$ define a sheaf on X which is denoted by Ω . This sheaf is called the sheaf of regular differential forms.

Example 33. Let $X = \mathbb{A}_k^n$. For $f \in k[X]$, we have $df = \sum_{j=1}^n \frac{\partial f}{\partial t_j} dt_j$. Then, $\Omega[X]$ is the free $k[X]$ -module with the basis dt_1, \dots, dt_n .

Example 34. Let $X = \mathbb{P}_k^1$. A regular differential form ω on X is determined by its restriction to the two affine open subsets $U = \mathbb{P}_k^1 - \{(0 : 1)\}$ and $V = \mathbb{P}_k^1 - \{(1 : 0)\}$. On U we take the variable t and on V the variable s . On $U \cap V$, we have $ts = 1$. Let $\omega|_U = f dt$ and $\omega|_V = g ds$. On $U \cap V$, $ds = -\frac{1}{t^2} dt$. So, on this intersection, $f dt = -\frac{g}{t^2} dt$. Therefore, $f(t) = -\frac{g(1/t)}{t^2}$ which implies that $f = g = 0$. So, $\Omega[X] = 0$.

Example 35. Let $X = \mathbb{P}_k^1$ and let $U = \mathbb{P}_k^1 - \{(0 : 1)\}$ and $V = \mathbb{P}_k^1 - \{(1 : 0)\}$. On U we take the variable t and on V the variable s . On $U \cap V$, we have $ts = 1$. Let ω be the rational differential form defined by ds on V . So, $\omega = ds = -\frac{1}{t^2} dt$. Therefore, $\text{Div}(\omega) = -2(1 : 0)$ which means that $K_X = -2(1 : 0)$.

Since $\deg K_X = -2$, $H^0(X, K_X) = 0$ hence $g(X) = 0$.

Exercise 36. Let X and X' be two isomorphic smooth projective curves. Prove that $g(X) = g(X')$.

Example 37. Let X be a smooth projective curve. Applying the Riemann-Roch theorem to $D = K_X$, we get

$$\dim_k H^0(X, K_X) - \dim_k H^0(X, 0) = \deg K_X + 1 - g(X)$$

which implies that $\deg K_X = 2g(X) - 2$.

Example 38. Earlier, we proved that $g(\mathbb{P}_k^1) = 0$. Let $D \geq 0$ be a divisor on X . The Riemann-Roch theorem implies that

$$\dim_k H^0(X, D) = \deg D + 1$$

Example 39. We prove that \mathbb{P}_k^1 is the only smooth projective curve with genus zero. Let X be a smooth projective curve with $g(X) = 0$. Let D be a prime divisor on X , that is, just a point. By the Riemann-Roch theorem,

$$\dim_k H^0(X, D) = 2$$

which implies that there is a non-constant $f \in k(X)$ such that $D' := \text{Div}(f) + D \geq 0$. Thus, $\text{Div}(f) = D' - D$ and since $\deg \text{Div}(f) = 0$, D' is also a prime divisor. So, $X \simeq \mathbb{P}_k^1$.

Example 40. An elliptic curve is a smooth projective curve X of $g(X) = 1$. So, $\deg K_X = 0$. Let $D \geq 0$ be an effective divisor on X . By the Riemann-Roch theorem

$$\dim_k H^0(X, D) - \dim_k H^0(X, K_X - D) = \deg D$$

which says that if $D \neq 0$, then $\dim_k H^0(X, D) = \deg D$ and if $D = 0$, then obviously $\dim_k H^0(X, D) = 1$.

We can also calculate the canonical divisor K_X . By definition, $g(X) = \dim_k H^0(X, K_X) = 1$ and as mentioned $\deg K_X = 0$. So, there is an effective divisor D such that $D \sim K_X$ and $\deg D = 0$. Therefore, $D = 0$ and $K_X \sim 0$.

Example 41. Let X be an elliptic curve and let $x \in X$. Let $\text{Cl}^0(X)$ be the subgroup of $\text{Cl}(X)$ consisting of divisors with degree zero. Now define a map $\theta: X \rightarrow \text{Cl}^0(X)$ by $\theta(y) = y - x$ for any $y \in X$. We prove that this is one-to-one.

The map θ is injective, otherwise $y - y'$ is the divisor of some rational function for some $y \neq y'$. But this is not possible by Example ???. Now let D be a divisor on X such that $\deg D = 0$. By the Riemann-Roch theorem

$$\dim_k H^0(X, D + x) = \deg(D + x) = 1$$

which implies that there is $f \in k(X)$ such that $D' := \text{Div}(f) + D + x \geq 0$. So, $\deg D' = 1$. This means that D' is a single point hence $D \sim D' - x$, that is, $D = \theta(D')$. Thus, θ is bijective.

On the other hand, $\text{Cl}^0(X)$ is an abelian group, so this makes X into an abelian group. This is the famous group law on elliptic curves.

Example 42. For any smooth projective curve $X \subseteq \mathbb{P}_k^2$ of degree n , that is, defined by a homogeneous polynomial of degree n , it is possible to calculate its genus directly. In this case, $g(X) = (n-1)(n-2)/2$. This follows from the adjunction formula. In particular, we see that curves of degree 1 and 2 are all of genus zero and so they are isomorphic to the projective line by Example 39.

If $n = 3$, then $g(X) = 1$ and so X is an elliptic curve according to Example 40. Another thing we can get from this is that two curves of different degree cannot be isomorphic except when one is of degree 1 and the other of degree 2.