ALGEBRAIC GEOMETRY

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Contents

1. Introduction

These notes are for a first graduate course on algebraic geometry. It is assumed that the students are not familiar with algebraic geometry so we have started from scratch. I have taken a moderate approach emphasising both geometrical and algebraic thinking. We have borrowed few main theorems of commutative algebra but rigorous proofs are given for the rest, except the sections on sheaves and cohomology.

Algebraic geometry is about studying a system of polynomial equations. To be more precise, we first specify a field k and consider the polynomial ring $k[t_1, \dots, t_n]$ consisting of polynomials with coefficients in k. Next, we pick $f_1, \dots, f_r$ in $k[t_1, \dots, t_n]$ and consider the set of solutions

$$\{(x_1, \dots, x_n) \mid f_1(x_1, \dots, x_n) = 0, \dots, f_r(x_1, \dots, x_n) = 0\}$$

where $x_i \in k$. There are two main questions:

- Is the set of solutions non-empty?

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- If not empty, how does this set look like?

The most primitive form of algebraic geometry is linear algebra, that is, if we choose the f_i with $\deg f_i \leq 1$. Here we can use matrices to investigate the solutions of the system. In particular, the set of solutions always "looks like" a collection of vector spaces over k.

Now assume that $k = \mathbb{Q}$. Then, we enter the field of Diophantine geometry which is generally considered to be in the realm of number theory. By taking $f_1 = t_1^2 + t_2^2 + 1$ we realise that the set of solutions can be empty and by taking $f_1 = t_1^m + t_2^m - t_3^m$ you realise that we are talking about serious stuff (Fermat's last theorem). In fact, roughly speaking, for every statement in mathematics there is a Diophantine equation so that the statement is provable if and only if the equation has a solution. Therefore, there cannot be any easy way of determining whether a system has solutions or not.

In this course we always assume that k is an algebraically closed field (unless stated otherwise) so the first question is not very interesting. We will mainly deal with the second question. Evidently there are two sides to this story. On the one hand we have geometry (the set of solutions) which has to do with intuition, and on the other hand we have algebra (the polynomial ring) which has to do with precise arguments and technicalities.

Remember that topology deals with continuous functions, analytic geometry deals with analytic (holomorphic) functions, and differential geometry deals with differentiable functions. In contrast, algebraic geometry deals with polynomial maps. Since a polynomial is differentiable, holomorphic, and continuous (at least when $k = \mathbb{C}$), algebraic geometers often use tools and ideas from topology, analysis, and differential geometry. However, since polynomials have rather special properties, algebraic geometry has its own unique techniques and methods, and these often help other subjects when their problems are formulated in terms of polynomials.

We assume good knowledge of commutative algebra, that is, general properties of rings, ideals, modules, fields, etc.

References. The main reference for this course is the book of Shafarevich [5]. We will also make use of the books of Harris [2] and Hartshorne [3]. For the commutative algebra background the main source is Atiyah-Macdonald [1] but Matsumura [4] and Zariski-Samuel [6] are also occasionally used.

2. Affine varieties

Throughout this document, we assume k to be an algebraically closed field unless stated otherwise. We denote $k[t_1, \ldots, t_n]$ for the ring of polynomials over k with the variables $t_1, \ldots, t_n$. It is well-known that this is a unique factorisation domain (UFD).

Corresponding to the algebraic object $k[t_1, \cdots, t_n]$ we have the following geometric object.

Definition 2.1 (Affine space) The n-dimensional affine space $\mathbb{A}_k^n$ is defined as

$$\mathbb{A}_k^n = \{x = (x_1, \ldots, x_n) \mid x_i \in k\}$$

We call $\mathbb{A}_k^1$ the affine line and $\mathbb{A}_k^2$ the affine plane. For the moment the word n-dimensional is just a name but later on we will see that $\mathbb{A}_k^n$ really has dimension $= n$.

We are interested in some special subsets of the n-dimensional affine space $\mathbb{A}_k^n$.

Definition 2.2 (Affine algebraic set) An affine algebraic set X is a set

$$X = V(f_1, \ldots, f_m) = \{x \in \mathbb{A}_k^n \mid f_i(x) = 0, \forall i\}$$

where $f_i \in k[t_1, \ldots, t_n]$. That is, an affine algebraic set is the set of common solutions of finitely many polynomials.

One can also define algebraic sets using ideals rather than polynomials.

Definition 2.3 For an ideal I of $k[t_1, \ldots, t_n]$ define its affine algebraic set as

$$V(I) = \{x \in \mathbb{A}_k^n \mid f(x) = 0, \forall f \in I\}$$

The algebraic set of an ideal is not a new object by the following theorem.

Theorem 2.4 (Hilbert basis theorem). *Every ideal I of $k[t_1, \ldots, t_n]$ is finitely generated, that is, there are $f_1, \ldots, f_m \in I$ such that $I = \langle f_1, \ldots, f_m \rangle$. In other words, the ring $k[t_1, \ldots, t_n]$ is noetherian.*

It is easy to see that if $I = \langle f_1, \ldots, f_m \rangle$, then $V(I) = V(f_1, \ldots, f_m)$.

Example 2.5 The affine algebraic subsets of $\mathbb{A}_k^1$ are $\emptyset$, $\mathbb{A}_k^1$ and its finite subsets.

Example 2.6 Let $f \in k[t_1, t_2]$ be of degree ≤ 1, that is, $f = at_1 + bt_2 + c$ where $a, b, c \in k$. If $b \neq 0$, then we have

$$X = \{(x_1, \frac{ax_1 + c}{-b}) \in \mathbb{A}_k^2 \mid x_1 \in k\}$$

We get a similar description of X if $a \neq 0$. On the other hand, if $a = b = 0$, then $X = \mathbb{A}_k^2$ if $c = 0$ but $X = \emptyset$ if $c \neq 0$.

Example 2.7 When $n = 2$, $V(t_1 - t_2)$ is a line, $V(t_1 t_2)$ is the union of two lines that is the two axis, and $V(t_1, t_2)$ is the origin $(0, 0)$.

The reason we use ideals rather than polynomials is that it is more convenient to relate algebraic sets and ideals.

Theorem 2.8. *Let I, J be ideals of $k[t_1, \dots, t_n]$. Then,* $V(IJ) = V(I \cap J) = V(I) \cup V(J)$ *and* $V(I + J) = V(I) \cap V(J)$.

Proof. Do as exercise. □

If we are given an affine algebraic set we can try to reconstruct its defining ideal. It is usually not possible to recover the ideal but we can recover the biggest possible ideal which defines the given algebraic set.

Definition 2.9 Let $X \subseteq \mathbb{A}_k^n$ be an affine algebraic set. Define its ideal I_X as

$$I_X = \{f \in k[t_1, \dots, t_n] \mid f(x) = 0, \forall x \in X\}$$

Example 2.10 Let $X = V(I) \subset \mathbb{A}_k^1$ where $I = \langle t^2 \rangle$. Then, $I_X = \langle t \rangle$. In particular, this shows that some times $I \neq I_X$.

Definition 2.11 (Zariski topology) We define a topology on $\mathbb{A}_k^n$ by taking the affine algebraic sets as closed subsets. An open subset of $\mathbb{A}_k^n$ is then the complement of an affine algebraic set, that is, $U = \mathbb{A}_k^n - X$ where X is an affine algebraic set. This induces a topology on every subset of the affine space and we always refer to it as the Zariski topology.

Recall that a topological space T is called irreducible if $T = T_1 \cup T_2$ implies $T = T_1$ or $T = T_2$ where T_i are closed subsets.

Definition 2.12 (Affine algebraic variety) An irreducible affine algebraic set (with respect to the Zariski topology) is called an affine algebraic variety or just an affine variety.

Theorem 2.13. *Let X be an affine algebraic set. Then, X is an affine algebraic variety iff I_X is a prime ideal.*

Proof. Suppose that $X \subseteq \mathbb{A}_k^n$ is an affine algebraic variety. Assume that $fg \in I_X$ for $f, g \in k[t_1, \dots, t_n]$. Then,

$$X = \{x \in X \mid f(x) = 0\} \cup \{x \in X \mid g(x) = 0\}$$

where the two latter sets are closed subsets of X. So, X should be equal to one of them and so $f \in I_X$ or $g \in I_X$.

Conversely, suppose that I_X is prime. If $X = X_1 \cup X_2$ where X_i are closed subsets then $I_{X_1} \cap I_{X_2} \subseteq I_X$ which implies that $I_{X_1} \subseteq I_X$ or $I_{X_2} \subseteq I_X$ since I_X is prime. Therefore, $X \subseteq X_1$ or $X \subseteq X_2$. ☐

The ring $k[t_1, \ldots, t_n]$ considered as an ideal of itself is not defined to be prime, so in the theorem or other places we can exclude the case $X = \emptyset$.

Theorem 2.14 (Decomposition into irreducible components). *Every affine algebraic set is a union of finitely many affine algebraic varieties.*

Proof. If an affine algebraic set is not irreducible then $X = X_1 \cup X_2$ where X_1, X_2 are closed subsets of X. If X_2 is not irreducible, then $X = X_1 \cup X_3 \cup X_4$ and so on. This stops otherwise we find a decreasing sequence $\cdots X'' \subsetneqq X' \subsetneqq X$ of closed subsets. Therefore, we find an increasing sequence of ideals $I_X \subsetneqq I_{X'} \subsetneqq \cdots$ which contradicts the fact that $k[t_1, \ldots, t_n]$ is a noetherian ring. ☐

To see the relation between affine algebraic sets and ideals more precisely we need to borrow the following important theorem.

Theorem 2.15 (Hilbert Nullstellensatz). *Let I be an ideal of $k[t_1, \ldots, t_n]$. Then,*
(i) if $X = V(I)$, then $I_X = \sqrt{I}$,
(ii) any maximal ideal of $k[t_1, \ldots, t_n]$ is of the form $\langle t_1 - x_1, \ldots, t_n - x_n \rangle$,
(iii) $V(I) = \emptyset$ iff $I = k[t_1, \cdots, t_n]$.

See [1] for a proof.

Theorem 2.16. *An affine algebraic set X is a single point iff I_X is a maximal ideal.*

Proof. Use the Nullstellensatz theorem. ☐

Remark 2.17 From the above theorems we can see that we have a one-to-one correspondence between the prime ideals of $k[t_1, \ldots, t_n]$ and the affine algebraic varieties $X \subseteq \mathbb{A}^n_k$. We have a one-to-one correspondence between the maximal ideals of $k[t_1, \ldots, t_n]$ and the points of $\mathbb{A}^n_k$. We also have a one-to-one correspondence between the radical ideals of $k[t_1, \ldots, t_n]$ and the affine algebraic sets in $\mathbb{A}^n_k$.

Example 2.18 (Hypersurface) An irreducible non-constant polynomial $f \in k[t_1, \cdots, t_n]$ defines a prime ideal $I = \langle f \rangle$ and so defines an affine algebraic variety $X = V(f) \subseteq \mathbb{A}^n_k$ which is called a hypersurface. If $n = 2$, this is also called a curve.

Example 2.19 Any non-constant $f \in k[t_1, \cdots, t_n]$ can be decomposed into irreducible factors $f = f_1 \cdots f_m$. Therefore,

$$V(f) = V(f_1) \cup \cdots \cup V(f_m)$$

is the decomposition into irreducible components.

Example 2.20 (Subvarieties of $\mathbb{A}^2_k$) From commutative algebra, we know that a prime ideal in $k[t_1, t_2]$ is either 0, $\langle f \rangle$ for an irreducible f, or a maximal ideal $\langle t_1 - x_1, t_2 - x_2 \rangle$. Therefore, an affine algebraic variety in $\mathbb{A}^2_k$ is either $\emptyset$, $\mathbb{A}^2_k$, a curve or a point.

Example 2.21 (Product of affine algebraic sets) The set-theoretic product $\mathbb{A}^n_k \times \mathbb{A}^m_k$ may be identified with $\mathbb{A}^{n+m}_k$ by identifying the point $(x_1, \dots, x_n), (y_1, \dots, y_m)$ with $(x_1, \dots, x_n, y_1, \dots, y_m)$. Now if $X \subseteq \mathbb{A}^n_k$ and $Y \subseteq \mathbb{A}^m_k$ are affine algebraic sets, then $X \times Y \subseteq \mathbb{A}^{n+m}_k$ is an affine algebraic set in the obvious way.

Definition 2.22 (Coordinate ring) For an affine algebraic set $X \subseteq \mathbb{A}^n_k$, its coordinate ring is defined as $k[X] = k[t_1, \dots, t_n]/I_X$. This ring is a finitely generated k-algebra. Elements of this ring are called regular functions on X, they can be viewed as functions $X \to k$ and each one can be represented by a polynomial but not in a unique way.

Theorem 2.23. *An affine algebraic set $X \subseteq \mathbb{A}^n_k$ is an affine algebraic variety iff $k[X]$ is an integral domain. Moreover, X is a point iff $k[X] = k$.*

Proof. Clear by Theorems 2.13 and 2.16. □

Definition 2.24 (Regular map) Let $X \subseteq \mathbb{A}^n_k$ and $Y \subseteq \mathbb{A}^m_k$ be affine algebraic sets. A regular map $\phi\colon X \to Y$ is a map given by $\phi = (f_1, \dots, f_m)$ where f_i are regular functions on X. ϕ is an isomorphism if it has an inverse which is also a regular map.

Every regular map $\phi\colon X \to Y$ of affine algebraic sets gives rise to a homomorphism of k-algebras $\phi^*\colon k[Y] \to k[X]$ by combining regular functions on Y with ϕ.

For an affine algebraic set X and an ideal I of $k[X]$ we define

$$V_X(I) = \{x \in X \mid f(x) = 0, \forall f \in I\}$$

Theorem 2.25. *Let $\phi\colon X \to Y$ be a regular map of affine algebraic sets. Then,*
(i) ϕ is a continuous map with respect to the Zariski topology,
(ii) ϕ^* is injective iff $\phi(X)$ is dense in Y,
(iii) ϕ is an isomorphism iff $\phi^*\colon k[Y] \to k[X]$ is an isomorphism of k-algebras.

(iv) ϕ^* is surjective iff ϕ induces an isomorphism of X with a closed subset of Y.

Proof. (i) This is the case because for an ideal I of $k[Y]$, we have $\phi^{-1}V_Y(I) = V_X(Ik[X])$. In other words, inverse of closed subsets are closed and so ϕ is continuous.

(ii) Suppose that $\phi(X)$ is dense in Y and $\phi^*(f) = 0$. Then, $\phi(X) \subseteq V_Y(f)$ which is possible only if $f = 0$. Conversely, suppose that ϕ^* is injective and assume that $\phi(X) \subseteq V_Y(I)$ for some ideal I of $k[Y]$. Obviously, $I \subseteq \ker \phi^* = 0$, so $\phi(X)$ must be dense in Y.

(iii) Exercise.

(iv) Assume that there is a closed subset $Z \subseteq Y$ such that $\phi(X) \subseteq Z$ and that the map $\psi\colon X \to Z$ induced by ϕ is an isomorphism. If $\pi\colon Z \to Y$ is the inclusion map, then $\phi = \pi\psi$. By (iii), $\psi^*\colon k[Z] \to k[X]$ is an isomorphism. Since Z is a closed subset of Y, $k[Z] = k[Y]/I$ for some ideal I and π^* is just the natural surjection $k[Y] \to k[Z]$. Now the claim follows from $\phi^* = \psi^*\pi^*$.

Conversely, if ϕ^* is surjective, then we get an isomorphism $k[Y]/I \to k[X]$ for some ideal I and another application of (iii) shows that ϕ induces an isomorphism of X with the closed subset of Y defined by I. □

The theorem suggests that affine algebraic sets are uniquely determined by their coordinate rings. So, one should be able to discover all the properties of an affine algebraic set from its coordinate ring. This led to the revolution by Grothendieck which completely transformed algebraic geometry and related subjects. The idea is that why not look at any commutative ring and define some space for it and study its geometry. Such spaces are called schemes.

Example 2.26 Regular functions on an affine algebraic set X are regular maps from X to $k = \mathbb{A}^1_k$.

Example 2.27 The map $\phi\colon \mathbb{A}^n_k \to \mathbb{A}^m_k$ defined by $\phi = (t_1, \dots, t_m)$, where $m \leq n$, is a regular map which we refer to as a projection. If $X \subseteq \mathbb{A}^n_k$ is an affine algebraic set, the restriction map $\phi|_X\colon X \to \mathbb{A}^m_k$ is also a regular map.

Example 2.28 Let $X = V(t_1t_2 - 1) \subseteq \mathbb{A}^2_k$. The map to $\mathbb{A}^1_k$ given by $\phi = t_1$ is regular and injective but not surjective so not an isomorphism. However, $\phi(X) = \mathbb{A}^1_k - \{0\}$ which is dence in $\mathbb{A}^1_k$.

Example 2.29 The regular map $\phi\colon \mathbb{A}^1_k \to Y$ given by $\phi = (t^2, t^3)$ where $Y = V(s_2^2 - s_1^3) \subseteq \mathbb{A}^2_k$ (cusp singularity) is a 1-1 regular map however, it is not an isomorphism. In fact, the corresponding homomorphism $\phi^*\colon k[Y] = k[s_1, s_2]/\langle s_2^2 - s_1^3\rangle \to k[t]$ is determined by $\phi^*(s_1) = t^2$ and $\phi^*(s_2) = t^3$. So t cannot be in the image of ϕ^* hence it is not an isomorphism. If one tries to get an inverse it should be as $\psi = s_2/s_1$ which is not a regular function. Soon we will see that ψ is a rational function.

Definition 2.30 (Rational function) Let $X \subseteq \mathbb{A}^n_k$ be an affine algebraic variety. Since $k[X]$ is an integral domain, its field of fractions $k(X)$ can be constructed which is called the field of rational functions of X or the function field of X. Each element of $k(X)$ can be represented by f/g such that $g \neq 0$ and $f, g \in k[X]$. Equivalently, we can write it as f/g where $f, g \in k[t_1, \dots, t_n]$ such that $g \notin I_X$. It can be considered as a "function" $\pi\colon X \dashrightarrow k$ which may not be defined everywhere. We say that $\pi = f/g$ is defined or regular at $x \in X$ if we can find h, e such that $\pi = f/g = h/e$ and $e(x) \neq 0$.

Example 2.31 (i) Regular functions are also rational functions. (ii) When $X = \mathbb{A}^n_k$, any element of the field $k(X) = k(t_1, \dots, t_n)$ is a rational function on X. (iii) The function ψ defined in Example 2.29 is a rational function on $Y = V(s_2^2 - s_1^3) \subseteq \mathbb{A}^2_k$.

Theorem 2.32. *Let X be an affine algebraic variety. A rational function $\pi\colon X \dashrightarrow k$ is regular everywhere (i.e. at every point) iff it is a regular function.*

Proof. Suppose that π is a rational function which is regular everywhere. Then, for each $x \in X$ we can write $\pi = f_x/g_x$ such that $g_x(x) \neq 0$. Now if I is the ideal in $k[X]$ generated by all g_x, then $V_X(I) = \emptyset$ and so $I = k[X]$ by Hilbert Nullstellensatz theorem. Thus, $\sum_{i=1}^m h_i g_{x_i} = 1$ for some $h_i \in k[X]$ and finitely many points $x_1, \dots, x_m \in X$. Now

$$\pi = \pi \sum h_i g_{x_i} = \sum \pi h_i g_{x_i} = \sum h_i f_{x_i}$$

where $\pi = f_{x_i}/g_{x_i}$ hence $\pi \in k[X]$. □

Let X be an affine algebraic variety and $\pi \in k(X)$ a rational function. The domain of π is the set of points of X at which π is regular.

Theorem 2.33. *Let X be an affine algebraic variety. Then, the domain of a rational function $\pi\colon X \dashrightarrow k$ is a nonempty open subset of X.*

Proof. Since we can write $\pi = f/g$ such that $g \neq 0$, the domain U is not empty. Now if I is the ideal in $k[X]$ generated by all g of different ways of writing $\pi = f/g$, then $U = X \setminus V_X(I)$. □

Definition 2.34 (Rational map) Let $X \subseteq \mathbb{A}_k^n$ be an affine algebraic variety and $Y \subseteq \mathbb{A}_k^m$ an affine algebraic set. A rational map $\pi\colon X \dashrightarrow Y$ is given by $\pi = (\pi_1, \dots, \pi_m)$ where $\pi_i \in k(X)$ and such that

$$\pi(x) = (\pi_1(x), \dots, \pi_m(x)) \in Y$$

if all the π_i are defined at x, and we say that π is defined or regular at x. In particular, $\pi(X) = \{\pi(x) \mid \pi \text{ regular at } x\}$.

If $\pi(X)$ is dense in Y, then we get a homomorphism of the function fields $\pi^*\colon k(Y) \to k(X)$. π is called a birational isomorphism if Y is also a variety and such that there is π^{-1} which is the inverse of π where they are defined. We also call X and Y birational.

Example 2.35 Let $\pi\colon \mathbb{A}_k^1 \to Y$ be the regular map given by $\pi = (t^2, t^3)$ where $Y = V(s_2^2 - s_1^3) \subseteq \mathbb{A}_k^2$. We proved that π is not an isomorphism. Now $\theta = s_2/s_1$ is the inverse of π which is not regular but rational. So, π is a birational isomorphism.

Example 2.36 (Conics in the plane) Let $Y = V(f) \subset \mathbb{A}_k^2$ be a variety defined by an irreducible polynomial f of degree 2 and after a linear change of variables we can assume that $(0,0) \in Y$. Let $L_s = V(t_2 - st_1)$ where $s \in k$. Now L_s intersects Y at $(0,0)$. Since f has degree 2, it intersects Y at another point y_s except for finitely many values of s. To see this one needs to solve the equation $g(t_1) = f(t_1, st_1) = 0$. Except for finitely many $s \in k$, $g(t_1)$ is of degree 2 so $g(t_1) = t_1 h(t_1)$ for some polynomial $h(t_1) = A(s)t_1 + B(s)$. Thus, the other root of g is given by a rational function $\pi_1 = -B(s)/A(s)$ which determines the first coordinate of y_s. The other coordinate of y_s is given by the rational function $\pi_2 = s\pi_1$. In short, we have defined a rational map $\pi\colon \mathbb{A}_k^1 \dashrightarrow Y$ given by $\pi = (\pi_1, \pi_2)$. Now define $\theta\colon Y \dashrightarrow \mathbb{A}_k^1$ by $\theta = t_2/t_1$. We see that π and θ are inverse of each other so Y and $\mathbb{A}_k^1$ are birationally isomorphic.

Definition 2.37 (Quasi-affine algebraic set) A quasi-affine algebraic set $X \subseteq \mathbb{A}_k^n$ is an open subset of an affine algebraic set. A regular function on X is a function $\phi\colon X \to k$ such that for every $x \in X$, there is a neighborhood U of x, and $f, g \in k[t_1, \dots, t_n]$ such that on U, ϕ and f/g are equal, in particular, g has no zero on U. The set of regular functions on X is denoted by $k[X]$ which is a k-algebra.

If X is irreducible, a rational function $\pi\colon X \dashrightarrow k$ on X is the equivalence class of a regular function on some open subset of X in the sense that if ϕ_U and ϕ_V are regular functions on the open subsets U and V respectively, then ϕ_U is equivalent to ϕ_V if $\phi_U|_{U\cap V} = \phi_V|_{U\cap V}$. The set of rational functions on X is denoted by $k(X)$ which is a field and is called the function field of X.

When $X \subseteq \mathbb{A}_k^n$ is an affine algebraic set, we defined regular and rational functions on X in Definitions 2.22 and 2.30. It is not too difficult to show that those notions coincide with those in the previous definition (see exercises below).

Exercises

Exercise 2.38 For an affine algebraic set $X = V(I)$, prove that $I \subseteq I_X$.

Exercise 2.39 Let $X', X'' \subseteq \mathbb{A}_k^n$ be affine algebraic sets. Prove that if $X' \subseteq X''$ then $I_{X''} \subseteq I_{X'}$. Prove that $I_{X' \cup X''} = I_{X'} \cap I_{X''}$. Show that it may happen that $I_{X' \cap X''} \neq I_{X'} + I_{X''}$.

Exercise 2.40 Let $X = V(t_1^2 + t_2^2 - 1, t_1 - 1) \subset \mathbb{A}_k^2$. What is I_X? Is I_X a prime ideal?

Exercise 2.41 Let $X = V(t_1^2 + t_2^2 + t_3^2) \subset \mathbb{A}_k^3$. Determine I_X when the characteristic of k is 2. Determine I_X when the characteristic of k is not 2.

Exercise 2.42 Decompose $X = V(t_1^2 - t_2 t_3, t_1 t_3 - t_1) \subset \mathbb{A}_k^3$ into its irreducible components.

Exercise 2.43 * Let k be any field not necessarily algebraically closed, and let $f, g \in k[t_1, t_2]$ be non-constant polynomials without common factor in their factorisations into irreducible factors. Prove directly that $V(f) \cap V(g) \subset \mathbb{A}_k^2$ is finite. (Consider f, g as polynomials over the ring $k(x)[y]$ or similarly $k(y)[x]$)

Exercise 2.44 Let $X = \{P_1, \dots, P_m\} \subset \mathbb{A}_k^n$ be finitely many distinct points. Show that

$$k[X] = k[t_1, \dots, t_n]/I_X \simeq \bigoplus_{j=1}^{m} k[t_1, \dots, t_n]/I_{\{P_j\}} \simeq \bigoplus_{j=1}^{m} k$$

Exercise 2.45 Let $\phi\colon \mathbb{A}_k^1 \to \mathbb{A}_k^1$ be an isomorphism. Prove that ϕ is given by a polynomial of degree 1.

Exercise 2.46 Let $\phi\colon \mathbb{A}_k^2 \to \mathbb{A}_k^2$ be given by $\phi = (t_1, t_1 t_2)$. Is $\phi(\mathbb{A}_k^2)$ open, close or dense?

Exercise 2.47 Let $M_{n\times m}$ be the set of $n \times m$ matrices over k. Prove that there is a bijection from $M_{n\times m}$ to $\mathbb{A}^{nm}$. Prove that the set of non-invertible matrices in $M_{n\times n}$ corresponds to a hypersurface in $\mathbb{A}^{n^2}$.

Exercise 2.48 Let $X = V(t_1^2 + t_2^2 - 1) \subset \mathbb{A}_k^2$ and let π be the rational function of X defined by $\pi = (1 - t_2)/t_1$. Determine the domain of π.

Exercise 2.49 Let X be an affine algebraic variety. Prove that
(i) the intersection of finitely many nonempty open subsets of X is open and nonempty,
(ii) if U is a nonempty open subset of X, then it is dense in X,
(iii) the domain of finitely many rational functions is also open and nonempty.

Exercise 2.50 Prove that if two rational functions of an affine algebraic variety are equal on a nonempty open subset, they should be equal.

Exercise 2.51 Prove that a rational map $\pi\colon X \dashrightarrow Y$ of affine varieties is a birational isomorphism iff π^* is an isomorphism of fields.

Exercise 2.52 For an affine algebraic variety X and an open subset $U \subseteq X$ we define

$$k[U] := \{\pi \in k(X) \mid \pi \text{ is regular at every point of } U\}$$

Now assuming $X = \mathbb{A}_k^1$ and $U = X \setminus \{0\}$ determine $k[U]$. Do the same with $X = \mathbb{A}_k^2$ and $U = X \setminus \{(0,0)\}$.

Exercise 2.53 Let X be an affine algebraic variety, $0 \neq f \in k[X]$, and $U_f := X \setminus V_X(f)$. Show that $k[U_f]$ is a k-algebra isomorphic to $k[X]_f$ (the latter object is the localisation of the ring $k[X]$ at the element f).

Exercise 2.54 Let $Y = V(t_2^2 - t_1^2 - t_1^3) \subset \mathbb{A}_k^2$ and let $\phi\colon \mathbb{A}_k^1 \to Y$ be the regular map defined by $\phi = (s^2-1, s(s^2-1))$. Describe ϕ^* and prove that ϕ is not an isomorphism but it is a birational isomorphism.

Exercise 2.55 Assume characteristic of k is not 2. Let $Y = V(t_1^2 + (t_2-1)^2 - 1) \subset \mathbb{A}_k^2$. Show that Y is birational to $\mathbb{A}_k^1$ by constructing rational maps $\pi\colon \mathbb{A}_k^1 \dashrightarrow Y$ and $\theta\colon Y \dashrightarrow \mathbb{A}_k^1$ explicitly so that π and θ are inverse of each other. Do the same for $Y = V(t_1^2+(t_2-1)^2-1) \subset \mathbb{A}_k^2$.

Exercise 2.56 * Show that the regular (and rational) functions on an affine algebraic set X are the same as the regular (and rational) functions on X considered as a quasi-affine algebraic set. (Hint: show that there is a natural injective homomorphism $k[X]_f \to k[U_f]$ for each $0 \neq f \in k[X]$ where $U_f = X \setminus V_X(f)$; next use appropriate coverings of X by such U_f.)

3. Quasi-projective varieties.

A (commutative) graded ring is a commutative ring $S = \bigoplus_{d=0}^{+\infty} S_d$ such that S_d are abelian groups and $S_d S_{d'} \subseteq S_{d+d'}$. Elements of S_d are called homogeneous of degree d. An ideal I of S is called homogeneous if $I = \bigoplus_{d=0}^{+\infty}(S_d \cap I)$. This is equivalent to I being generated by homogeneous elements. In particular, $S_+ = \bigoplus_{d=1}^{+\infty} S_d$ is a homogeneous ideal.

Example 3.1 The most important example for us is the polynomial ring $S = k[s_0, \dots, s_n]$ which has a natural graded structure: S_d is the vector space over k generated by all the monomials of degree d. By Hilbert basis theorem, every homogeneous ideal of this ring is finitely generated.

We say that $(x_0, \dots, x_n), (y_0, \dots, y_n) \in \mathbb{A}_k^{n+1}$ are equivalent if there is a nonzero $a \in k$ such that $(y_0, \dots, y_n) = (ax_0, \dots, ax_n)$. The equivalence class of $(x_0, \dots, x_n)$ is denoted by $(x_0 : \cdots : x_n)$.

Definition 3.2 (Projective space) The n-dimensional projective space is defined as

$$\mathbb{P}_k^n = \{x = (x_0 : \cdots : x_n) \mid x_i \in k \text{ and some } x_i \neq 0\}$$

We call $\mathbb{P}_k^1$ the projective line and call $\mathbb{P}_k^2$ the projective plane.

There are good reasons to introduce the projective space. Projective algebraic sets (that will be introduced soon) are the analogue of compact topological spaces. Compact spaces behave better than the non-compact ones. For example later on we will see that any two projective curves in $\mathbb{P}_k^2$ intersect. This is clearly not true in $\mathbb{A}_k^2$. One can think of the projective space as a compactified version of the affine space. For example,

$$\mathbb{P}_k^1 = \{(1 : a) \mid a \in k\} \cup \{(0 : 1)\}$$

and we can identify $\{(1 : a) \mid a \in k\}$ with $\mathbb{A}_k^1$ which is like saying that $\mathbb{P}_k^1$ is obtained by adding "the point at infinity" to $\mathbb{A}_k^1$.

Now let $F \in k[s_0, \dots, s_n]$ and $x \in \mathbb{P}_k^n$. In general, if

$$x = (x_0 : \cdots : x_n) = (x'_0 : \cdots : x'_n)$$

then $F(x_0 : \cdots : x_n)$ and $F(x'_0 : \cdots : x'_n)$ are not equal so it does not make sense to talk about the value $F(x)$. However, if F is homogeneous, then $F(x_0 : \cdots : x_n) = 0$ iff $F(x'_0 : \cdots : x'_n) = 0$ so in this case at least the vanishing statement $F(x) = 0$ makes sense (but still $F(x)$ does not make sense).

Definition 3.3 (Projective algebraic set) Let I be a homogeneous ideal of $k[s_0, \ldots, s_n]$. Define

$$V(I) = \{x \in \mathbb{P}^n_k \mid \ F(x) = 0, \forall F \in I \text{ homogeneous}\}$$

A projective algebraic set X in $\mathbb{P}^n_k$ is defined to be $X = V(I)$ for some homogeneous ideal I of $k[s_0, \ldots, s_n]$. If $F_1, \ldots, F_m$ are homogeneous polynomials generating I, then $V(I) = V(F_1, \ldots, F_m)$.

Definition 3.4 (Zariski topology) Similar to the affine case $\mathbb{P}^n_k$ is a topological space by taking its closed subsets to be projective algebraic sets. This induces a topology on each subset of $\mathbb{P}^n_k$ which we again call the Zariski topology.

For a projective algebraic set $X \subseteq \mathbb{P}^n_k$, define its ideal I_X as the ideal generated by

$$\{F \in k[s_0, \ldots, s_n] \mid F \text{ is homogeneous, } F(x) = 0 \text{ for all } x \in X\}$$

Obviously, I_X is a homogeneous ideal. Now define the coordinate ring of X as $S[X] = k[s_0, \ldots, s_n]/I_X$ which is a graded ring.

Theorem 3.5. *Let I be a homogeneous ideal of $k[s_0, \ldots, s_n]$. Then,*
(i) $X = V(I) = \emptyset$ iff $S_+ \subseteq \sqrt{I}$,
(ii) If $X = V(I) \neq \emptyset$, then $I_X = \sqrt{I}$.

Proof. (i) Let $Y = V(I) \subseteq \mathbb{A}^{n+1}_k$. Then, the map $\alpha \colon Y - \{0\} \to X$ given by $\alpha(x_0, \ldots, x_n) = (x_0 : \cdots : x_n)$ is surjective. Thus, $X = \emptyset$ iff $Y \subseteq \{0\}$ iff $S_+ =< s_0, \ldots, s_n > \subseteq \sqrt{I}$.

(ii) Since I is homogeneous, $I_Y = \sqrt{I}$ is also homogeneous. Hence $I_Y \subseteq I_X$. On the other hand, since $X \neq \emptyset$, $I_X \neq S$ and since I_X is homogeneous, every element of I_X vanishes at every point of Y (including the origin) hence $I_X \subseteq I_Y$. $\square$

Definition 3.6 (Quasi-projective algebraic set) A quasi-projective algebraic set is an open subset of a projective algebraic set. A projective variety is an irreducible projective algebraic set. A quasi-projective variety is an open subset of a projective variety.

As mentioned earlier, a homogeneous polynomial $F \in k[s_0, \ldots, s_n]$ does not necessarily define a function on $\mathbb{P}^n_k$ or a subset of it. However, if we take two homogeneous polynomials $F, G \in k[s_0, \ldots, s_n]$ of the same degree, by putting

$$(F/G)(x_0 : \cdots : x_n) = F(x_0, \ldots, x_n)/G(x_0, \ldots, x_n)$$

if $G(x_0, \ldots, x_n) \neq 0$, we get a well-defined function $F/G \colon \mathbb{P}^n_k - V(G) \to k$.

Definition 3.7 (Regular and rational functions) Let X be a quasi-projective algebraic set. A function $\phi\colon X \to k$ is called a regular function if for every $x \in X$, there is a neighborhood U of x, and homogeneous polynomials F, G of the same degree such that on U, ϕ and F/G are equal, in particular, G has no zero on U. The set of regular functions on X is denoted by $k[X]$ which is a k-algebra.

If X is irreducible, a rational function $\pi\colon X \dashrightarrow k$ on X is the equivalence class of a regular function on some open subset of X in the sense that if ϕ_U and ϕ_V are regular functions on the open subsets U and V respectively, then ϕ_U is equivalent to ϕ_V if $\phi_U|_{U\cap V} = \phi_V|_{U\cap V}$. A rational function then is uniquely determined by some F/G where G is not identically zero on X. The set of rational functions on X is denoted by $k(X)$ which is a field and is called the function field of X.

Definition 3.8 (Regular and rational maps) Let X and Y be quasi-affine or quasi-projective algebraic sets. A regular map $\phi\colon X \to Y$ is a continuous map (with respect to the Zariski topologies) such that it sends a regular function on a open subset $V \subseteq Y$ to a regular function on $\phi^{-1}V$, that is, if $f \in k[V]$ then the composition $f\phi \in k[\phi^{-1}V]$. The regular map $\phi\colon X \to Y$ is called an isomorphism if it has an inverse which is also a regular map. When we say a quasi-affine or a quasi-projective algebraic set is affine, we mean that it is isomorphic to an affine algebraic set in some affine space.

When X is irreducible, a rational map $\pi\colon X \dashrightarrow Y$ is the equivalence class of a regular map $\phi_U\colon U \to Y$ for some open subset $U \subseteq X$. We say that π is a birational isomorphism if Y is also a variety and π has a rational inverse θ. In this case, we also say that X and Y are birational.

Let $X \subseteq \mathbb{P}^n_k$ be a quasi-projective algebraic set and $F_0 : \cdots : F_m$ homogeneous polynomials of the same degree such that at least one of them is not identically zero on X. Then, we can define a well-defined function

$$(F_0 : \cdots : F_m)\colon X - V(F_0, \dots, F_m) \to \mathbb{P}^m_k$$

by putting

$$(F_0 : \cdots : F_m)(x_0 : \cdots : x_n) = (F_0(x_0 : \cdots : x_n) : \cdots : F_m(x_0 : \cdots : x_n))$$

Theorem 3.9. *Let $X \subseteq \mathbb{P}^n_k$ and $Y \subseteq \mathbb{P}^m_k$ be quasi-projective algebraic sets and $\phi\colon X \to Y$ a map (set theoretic). Then, ϕ is a regular map iff for every $x \in X$, there is a neighborhood $x \in U$, and homogeneous polynomials F_i of the same degree such that $\phi = (F_0 : \cdots : F_m)$ on U.*

Proof. We take the rings $k[s_0, \ldots, s_n]$ and $k[r_0, \ldots, r_m]$ corresponding to $\mathbb{P}^n_k$ and $\mathbb{P}^m_k$, respectively. First assume that ϕ is a regular map. Let $x \in X$. Without loss of generality, we may assume $\phi(x) = (y_0 : \cdots : y_m)$ with $y_0 \neq 0$. Let $V = Y \setminus V_Y(r_0)$. Now r_i/r_0 is a regular function on V for any i, and by definition it gives a regular function ϕ_i on $\phi^{-1}V$. Moreover, the map $\phi^{-1}V \to V$ given by $(1 : \phi_1 : \cdots : \phi_m)$ is identical to ϕ on $\phi^{-1}V$. Each ϕ_i is expressed as F_i/G_i on some neighborhood of x. If we take U to be the intersection of all these neighborhoods, then ϕ is given by $(1 : F_1/G_1 : \cdots : F_m/G_m)$ on U. Now $(G_1 ... G_m : F_1G_2 \ldots G_m : \cdots)$ is the desired map.

Now assume that ϕ is a map which is locally given as in the statement of the theorem. First we prove that it is continuous. For each $x \in X$ there is a neighborhood $x \in U$ on which ϕ is given by $\pi = (F_0 : \cdots : F_m)$. It is enough to prove that $\pi = \phi|_U \colon U \to Y$ is continuous. A closed subset Z of Y is as $Z = V_Y(H_1, \ldots, H_l)$. Now the composition of $H_1, \ldots, H_l$ with π give the defining equations of $\pi^{-1}Z$ which is a closed subset of U. Hence, π is continuous.

Now let V be any open subset of Y and ψ a regular function on V. By definition, ψ is locally given as F/G. Hence the composition $\psi\phi$ is locally given by some F'/G' which means that $\psi\phi$ is a regular function on $\phi^{-1}V$. □

Corollary 3.10. *Let $X \subseteq \mathbb{P}^n_k$ be a quasi-projective algebraic variety and $Y \subseteq \mathbb{P}^m_k$ a quasi-projective algebraic set. A rational map $\pi \colon X \dashrightarrow Y$ is uniquely determined by some $\pi = (F_0 : \cdots : F_m)$ on some non-empty open subset $U \subseteq X$.*

The following theorem clarifies the relation between the affine and the projective worlds.

Theorem 3.11. *Algebraic sets defined in the affine and projective spaces are related as follows:*

(i) Quasi-affine algebraic sets are quasi-projective.

(ii) Each quasi-projective algebraic set is covered by affine algebraic sets.

Proof. Let $U_i = \{(x_0 : \cdots : x_n) \in \mathbb{P}^n_k \mid x_i \neq 0\}$. Then, we have a bijection $\phi_i \colon \mathbb{A}^n_k \to U_i$ given by $\phi_i(a_1, \ldots, a_n) = (a_1 : \cdots : 1 : \cdots : a_n)$ where 1 appears as the i-th coordinate. Note that the inverse of ϕ_i is given by the function that takes $(x_0 : \cdots : x_n)$ to

$$(\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i})$$

Clearly, $\mathbb{P}^n_k$ is covered by the sets $U_0, \ldots, U_n$.

For $f \in k[t_1, \dots, t_n]$ of degree d, define the homogeneous polynomial F associated to f by $s_i^d f(\frac{s_0}{s_i}, \dots, \frac{s_n}{s_i})$ which is in $k[s_0, \dots, s_n]$ (note that here we are ignoring $\frac{s_i}{s_i}$). Conversely, for a homogeneous $G \in k[s_0, \dots, s_n]$, we can associate $g = G(t_1, \dots, 1, \dots, t_n) \in k[t_1, \dots, t_n]$ where 1 appears in the i-th coordinate.

(i) Let $X \subseteq \mathbb{A}_k^n$ be a quasi-affine algebraic set and let $X' = \phi_i(X)$. We will see that $\phi_i|_X \colon X \to X'$ is an isomorphism. It is easy to see that ϕ_i is a homeomorphism which implies that $\phi_i|_X$ is also a homeomorphism. Let $V \subseteq X$ be an open subset and $V' = \phi_i|_X(V)$. Then, a regular function on V is a function $V \to k$ which is locally given by a fraction f/g. Such a function corresponds to a function $V' \to k$ which is locally of the form $s_i^d F/G$ for some $d \in \mathbb{Z}$ where F, G are the homogeneous polynomials associated to f, g respectively.

Conversely, a regular function $V' \to k$ is locally of the form F/G for some homogeneous polynomials F, G of the same degree. Such a function corresponds to a function $V \to k$ which is locally of the form f/g where f, g are associated to F, G.

So we have a 1-1 correspondence between regular functions on V and V'. That is, we have an isomorphism.

(ii) Let X be a quasi-projective algebraic set. We will show that for every $x \in X$, there is a neighborhood $x \in U$ which is affine, that is, it is isomorphic to an affine algebraic set. Since x is in one of the affine pieces U_i covering the projective space, x has a neighborhood $U = U_i \cap X$ which is quasi-affine. Now by replacing X with U we can assume that $X \subseteq \mathbb{A}_k^n$, that is, it is quasi-affine. Let $Y = \overline{X} = V(I)$ be the closure in $\mathbb{A}_k^n$. Further shrinking X, we can assume that $X = Y \setminus V_Y(f)$ for some polynomial f. Now let $W = \mathbb{A}_k^n \setminus V(f)$ and let Z be the affine algebraic set in $\mathbb{A}_k^{n+1} = \mathbb{A}_k^n \times \mathbb{A}_k^1$ defined by $Z = V(ft_{n+1} - 1)$. The map $\phi \colon Z \to W$ given by $\phi = (t_1, \dots, t_n)$ is a regular map. Moreover, the map $\psi \colon W \to Z$ given by $\psi = (t_1, \dots, t_n, 1/f)$ is the inverse of ϕ. So, Z and W are isomorphic and this induces an isomorphism between $X = W \cap Y$ and a closed subset of Z. $\square$

We will now give a series of examples of quasi-projective algebraic sets and their maps.

Example 3.12 Let $M_{2\times 2}$ be the set of 2×2 matrices over k. Any nonzero element of $M_{2\times 2}$ determines a point of $\mathbb{P}_k^3$. Now all such matrices which are not invertible define a projective algebraic set in $\mathbb{P}_k^3$.

Example 3.13 (Projectivisation of an affine algebraic set) Let $X = V(I) \subseteq \mathbb{A}_k^n$ be an affine algebraic set. By identifying $\mathbb{A}_k^n$ with U_0 as in the proof of Theorem 3.11 we can take the closure $\overline{X}$ in $\mathbb{P}_k^n$ which is by definition the intersection of all projective algebraic sets in $\mathbb{P}^n_k$ containing X. Let J be the homogeneous ideal of $k[s_0, \dots, s_n]$ generated by all the F associated to $f \in I$. We prove that $\overline{X} = V(J) \subseteq \mathbb{P}^n_k$. Let $x \in X$ and $G \in J$ homogeneous. Then, $G = \sum P_i F_i$ where F_i is associated to $f_i \in I$, and so $G(x) = 0$ because $F_i(x) = 0$ since $f_i(x) = 0$. In other words, $X \subseteq \overline{X} \subseteq V(J)$.

If $\overline{X} \neq V(J)$, then $\overline{X} \subseteq V(\langle G \rangle + J) \subsetneq V(J)$ for some homogeneous polynomial G. Then $X = V(\langle g \rangle + I)$ where g is associated to G. Thus, $g \in \sqrt{I}$ and so $G^l \in J$ for some $l \in \mathbb{N}$. Therefore, $V(\langle G \rangle + J) = V(J)$, a contradiction.

Example 3.14 Let $X = \mathbb{P}^n_k$. Since $\mathbb{A}^n_k$ is a dense open subset of X, $k(X) = k(\mathbb{A}^n_k)$ hence $k(X) \simeq k(t_1, \dots, t_n)$ where we could identify t_i with s_i/s_0.

Example 3.15 Let $X = \mathbb{P}^n_k$, we prove that $k[X] = k$. Let $\phi\colon X \to k$ be a regular function. Then, for $x \in X$ there is a neighborhood $x \in U$ and F, G homogeneous of the same degree in $k[s_0, \dots, s_n]$ such that $\phi = F/G$ on U, and G has no zero on U. We can assume that F, G have no common factor. Suppose that G is not a constant and let $x' \in V(G) \setminus V(F)$. Then, there is a neighborhood $x' \in U'$ and F', G' homogeneous of the same degree in $k[s_0, \dots, s_n]$ such that $\phi = F'/G'$ on U', and G' has no zero on U', and F', G' have no common factor. So, F/G and F'/G' give the same values on $U \cap U' \neq \emptyset$. Since X is irreducible, this is possible only if $FG' - F'G = 0$. This is a contradiction because $F(x')G'(x') \neq 0$ but $F'(x')G(x') = 0$. So, F, G are constant and so is ϕ.

Example 3.16 Let $X = V(s_1 s_2 - s_0^2) \subseteq \mathbb{P}^2_k$. Then, $\phi\colon X \to \mathbb{P}^1_k$ given by $\phi = (s_0 : s_1) = (s_2 : s_0)$ is a regular map. Now ϕ is actually an isomorphism with the inverse $\psi\colon \mathbb{P}^1_k \to X$ given by $\psi = (r_0 r_1 : r_1^2 : r_0^2)$. This example also shows that $X_1 = X \setminus \{(0:1:0), (0:0:1)\}$ is isomorphic to $\mathbb{P}^1_k \setminus \{(1:0), (0:1)\}$. The first one is a closed affine set in $\mathbb{A}^2_k$ but the second one is an open subset of $\mathbb{A}^1_k$.

Example 3.17 $\phi\colon \mathbb{P}^n_k \to \mathbb{P}^n_k$ given by $\phi = (s_0^d, \dots, s_n^d)$ is a regular map. If $d > 1$, it is not an isomorphism.

Example 3.18 (Cremona transformation) $\pi\colon \mathbb{P}^2_k \dashrightarrow \mathbb{P}^2_k$ given by

$$\pi = (s_1 s_2 : s_0 s_2 : s_0 s_1) = (1/s_0 : 1/s_1 : 1/s_2)$$

is a rational map. It is not regular at the three points $(0:0:1)$, $(0:1:0)$, and $(1:0:0)$. However, it is birational its inverse being $\theta = (r_1 r_2 : r_0 r_2 : r_0 r_1)$. This is called a Cremona transformation.

Example 3.19 (Veronese embedding) Let d be a natural number and $m = (\begin{smallmatrix} n+d \\ d \end{smallmatrix}) - 1$. Define $\phi\colon \mathbb{P}_k^n \to \mathbb{P}_k^m$ by the m monomials $s_0^{d_0} \cdots s_n^{d_n}$ where $\sum d_i = d$ and $d_i \in \mathbb{N} \cup \{0\}$. Then, ϕ is a regular map which is called the Veronese embedding.

Example 3.20 (Rational normal curve) The map $\phi\colon \mathbb{P}_k^1 \to \mathbb{P}_k^d$ given by

$$\phi = (s_0^d : s_0^{d-1}s_1 : s_0^{d-2}s_1^2 : \cdots : s_1^d)$$

is a regular map and its image, say X, is called the rational normal curve of degree d. We can also define X as the set given by the equations $r_ir_j - r_{i-1}r_{j+1}$ for $1 \leq i \leq j \leq d-1$. Indeed, every point in X satisfies those equations. Actually, $\phi\colon \mathbb{P}_k^1 \to X$ is an isomorphism and its inverse is given by the map

$$\psi = (r_0 : r_1) = (r_1 : r_2) = \cdots = (r_{d-1} : r_d)$$

Example 3.21 (Segre map) We would like to turn the set-theoretic product $\mathbb{P}_k^m \times \mathbb{P}_k^n$ into a projective algebraic set. We define a map

$$\phi\colon \mathbb{P}_k^m \times \mathbb{P}_k^n \to \mathbb{P}_k^{(m+1)(n+1)-1}$$

which is given by s_it_j at the (i,j)-th coordinate where s_i are the variables corresponding to $\mathbb{P}_k^m$ and t_j are variables corresponding to $\mathbb{P}_k^n$. The image of ϕ, say X, turns out to be a closed subset of $\mathbb{P}_k^{(m+1)(n+1)-1}$ given by the equations $r_{i,j}r_{p,q} - r_{i,q}r_{p,j}$. Moreover, ϕ is a bijection onto X so we can identify $\mathbb{P}_k^m \times \mathbb{P}_k^n$ with X and carry the topology on X onto $\mathbb{P}_k^m \times \mathbb{P}_k^n$ and again call it the Zariski topology. In addition, we can think of $\mathbb{P}_k^m \times \mathbb{P}_k^n$ as a projective algebraic set via ϕ.

Example 3.22 (Products) Let ϕ be the Segre map. Let

$$U_i = \{(x_0 : \cdots : x_m) \in \mathbb{P}_k^m \mid x_i \neq 0\}$$

and

$$V_j = \{(y_0 : \cdots : y_n) \in \mathbb{P}_k^n \mid y_j \neq 0\}$$

From the definition of ϕ we can see that $\phi(U_i \times V_j)$ is an open subset of $\phi(\mathbb{P}_k^m \times \mathbb{P}_k^n)$ and that $\phi(U_i \times V_j) \simeq \mathbb{A}_k^{m+n}$. Now if $X \subseteq \mathbb{P}_k^m$ and $Y \subseteq \mathbb{P}_k^n$ are closed subsets, then $X \times Y$ is also a closed subset in the Zariski topology of $\mathbb{P}_k^m \times \mathbb{P}_k^n$ induced by ϕ: to see this it is enough to observe that

$$\phi((X \times Y) \cap (U_i \times V_j))$$

is a closed subset of $\phi(U_i \times V_j)$. Similar reasoning shows that if X and Y are only assumed to be quasi-projective, then $\phi(X \times Y)$ is also naturally quasi-projective hence it induces a quasi-projective structure on $X \times Y$. Moreover, the topology on $X \times Y$ is finer than the product topology.

An important property of the product $X \times Y$ is that it comes with two natural regular maps $X \times Y \to X$ and $X \times Y \to Y$ called the first and second projections (not to be confused with other kinds of projections) where the first map sends a point (x, y) to x and the second map sends (x, y) to y.

Example 3.23 (Projection from a point) We can identify $\mathbb{P}_k^{n-1}$ with the closed subset of $\mathbb{P}_k^n$ consisting of points $(x_0 : \cdots : x_n)$ with $x_n = 0$. Pick a point $y \in \mathbb{P}_k^n$ outside $\mathbb{P}_k^{n-1}$. We can define a map

$$\phi\colon \mathbb{P}_k^n \setminus \{y\} \to \mathbb{P}_k^{n-1}$$

by sending a point x to the intersection $L \cap \mathbb{P}_k^{n-1}$ where L is the line passing through x and y. If we fix the coordinates $x = (x_0 : \cdots : x_n)$ and $y = (y_0 : \cdots : y_n)$, then by definition L is the set $\{ax + by \in \mathbb{P}_k^n \mid a, b \in k\}$ (obviously, $a \neq 0$ or $b \neq 0$).

We refer to ϕ as projection from the point y. If $X \subseteq \mathbb{P}_k^n$ is quasi-projective, the restriction of ϕ gives a regular map $X \setminus \{y\} \to \mathbb{P}_k^{n-1}$ which we again call projection from y. If we take $y = (0 : 0 : \cdots : 0 : 1)$, then ϕ can be simply described as $\phi = (s_0 : \cdots : s_{n-1})$.

Example 3.24 (Graph of a map) Let $\phi\colon X \to Y$ be a regular map. We define the graph of ϕ to be the closed subset $\Gamma_\phi \subset X \times Y$ consisting of those points (x, y) such that $y = \phi(x)$. Now assume that ϕ is just a rational map and let U be its domain of definition, that is, the set of points where ϕ is regular. We take the graph $\Gamma_{\phi|_U}$ and define the graph Γ_ϕ to be the closure of $\Gamma_{\phi|_U}$ inside $X \times Y$.

Example 3.25 (Blow up) Let $\phi\colon \mathbb{A}_k^n \dashrightarrow \mathbb{P}_k^{n-1}$ be the rational map which sends a point $(x_1, \ldots, x_n)$ to $(x_1 : \cdots : x_n)$. The map is defined everywhere except at the origin $0 = (0 : \cdots : 0)$. We denote the graph Γ_ϕ by $\mathrm{B}_0\mathbb{A}_k^n$ and call it the blow up of $\mathbb{A}_k^n$ at the origin 0. The first projection $\mathbb{A}_k^n \times \mathbb{P}_k^{n-1} \to \mathbb{A}_k^n$ gives us a regular map $\pi\colon \mathrm{B}_0\mathbb{A}_k^n \to \mathbb{A}_k^n$ which is also referred to as the blow up of $\mathbb{A}_k^n$ at the origin 0.

Put $V = \mathbb{A}_k^n \setminus \{0\}$. Then the restricton of π gives an isomorphism $\pi^{-1}V \to V$: the inverse of this map is the map which sends a point v to $(v, \phi(v))$. On the other hand, $\pi^{-1}\{0\} \simeq \mathbb{P}_k^{n-1}$ which we can verify as follows. Pick a point $0 \neq x \in \mathbb{A}^n$ and let L be the line in $\mathbb{A}^n$ which passes through x and the origin, and let $L' = L \setminus \{0\}$ (unlike in the projective case the line through $v, v' \in \mathbb{A}_k^n$ is defined as $\{av + (1-a)v' \mid a \in k\}$). Every point of L' is mapped to the same point of $\mathbb{P}_k^{n-1}$ by ϕ, say y. Thus

$$L' \times \{y\} \subseteq \Gamma_\phi \cap (L \times \{y\})$$

and since Γ_ϕ is a closed subset of $\mathbb{A}^n_k \times \mathbb{P}^{n-1}_k$, the intersection $\Gamma_\phi \cap (L \times \{y\})$ is a closed subset of $L \times \{y\}$ hence it is equal to $L \times \{y\}$ which in particular means that $(0, y) \in \pi^{-1}\{0\}$. Therefore, we have $\pi^{-1}\{0\} = \{0\} \times \mathbb{P}^{n-1}_k \simeq \mathbb{P}^{n-1}_k$.

EXERCISES

Exercise 3.26 Prove that intersection, sum, product and radical of homogeneous ideals in a graded ring are again homogeneous.

Exercise 3.27 For homogeneous ideals of $k[s_0, \dots, s_n]$, prove that $V(IJ) = V(I \cap J) = V(I) \cup V(J)$ and $V(\sum I_l) = \bigcap V(I_l)$.

Exercise 3.28 Let $X', X'' \subseteq \mathbb{P}^n_k$ be projective algebraic sets. Prove that if $X' \subseteq X''$ then $I_{X''} \subseteq I_{X'}$. Prove that $I_{X' \cup X''} = I_{X'} \cap I_{X''}$.

Exercise 3.29 Let X be a quasi-projective algebraic set and $X = \bigcup U_j$ a cover of X by open subsets U_j. Prove that only finitely many of U_j would be enough to cover X.

Exercise 3.30 Show that regular maps of affine algebraic sets are the same as their regular maps as quasi-projective algebraic sets. Prove the same for rational maps of affine varieties.

Exercise 3.31 Let $U \neq \emptyset$ be an open subset of a quasi-projective variety X. Prove that $k(U) = k(X)$.

Exercise 3.32 Let X be a closed subset of $\mathbb{P}^m_k \times \mathbb{P}^n_k$ and assume that s_i and t_j are the variables corresponding to $\mathbb{P}^m_k$ and $\mathbb{P}^n_k$ respectively. Show that there are polynomials $F_1, \dots, F_l \in k[s_0, \dots, s_m, t_0, \dots, s_n]$ that are homogenuous in the s_i and also homogenuous in the t_j such that X is the set of zeros of $F_1, \dots, F_l$.

Exercise 3.33 Find the equations which determine the blow up $\mathrm{B}_0\mathbb{A}^n_k$ as a closed subset of $\mathbb{A}^n_k \times \mathbb{P}^{n-1}_k$.

Exercise 3.34 Let X and Y be quasi-projective varieties. Show that the following are equivalent:

- X and Y are birational,
- there are open subsets $U \subseteq X$ and $V \subseteq Y$ such that U and V are isomorphic,
- there is an isomorphism of fields $k(X) \to k(Y)$ over k.

Exercise 3.35 The variety $X = V(s_0s_3 - s_1s_2) \subset \mathbb{P}^3_k$ is called the quadric surface. Show that

- $X \simeq \mathbb{P}^1_k \times \mathbb{P}^1_k$,
- X is birational to $\mathbb{P}^2_k$.

4. Dimension

Let X be a topological space. The dimension of X, that is $\dim X$, is the sup of $l \in \mathbb{Z}$ such that there is a sequence $Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_l$ of nonempty closed irreducible subsets. Dimension of a quasi-projective algebraic set is defined with respect to its Zariski topology. Obviously, dimension of a single point is zero and dimension of $\mathbb{A}^1_k$ is 1. Dimension of $\emptyset$ by convention is -1.

For any commutative ring R one can also define the (Krull) dimension as the sup of length of sequences of prime ideals $P_l \subsetneq \cdots \subsetneq P_0$. When X is an affine algebraic set, then $\dim X = \dim k[X]$ because irreducible closed subsets of X correspond to prime ideals of $k[X]$. We borrow the following theorem from commutative algebra and develop the dimension theory upon it.

Theorem 4.1. *Let R be a finitely generated k-algebra integral domain. Then, $\dim R < +\infty$. Moreover, if $0 \neq a \in R$ is not a unit, then $\dim R/\langle a \rangle = \dim R - 1$.*

Theorem 4.2. *Let X be a quasi-projective algebraic variety. Then,*

(i) if X is affine and $f \in k[X]$ such that $\emptyset \neq V_X(f) \neq X$, then $\dim V_X(f) = \dim X - 1$;

(ii) if $U \subseteq X$ is a non-empty open subset, then $\dim X = \dim U$;

(iii) if X and f are as in (i), then every irreducible component of $V_X(f)$ has dimension $\dim X - 1$.

Proof. (i) From the previous theorem we get

$$\dim V_X(f) = \dim k[X]/\sqrt{\langle f \rangle} = \dim k[X]/\langle f \rangle = \dim k[X] - 1$$

(ii) Clearly, $\dim U \leq \dim X$. Cover X by open affine subsets U_i. Then, $\dim X = \max\{\dim U_i\}$ so by replacing X with some U_i and replacing U with $U \cap U_i$ we can assume that X is affine.

We may assume that $Y := X - U \neq \emptyset$. Since X is irreducible, $\dim Y < \dim X$. Now let f be a polynomial such that $V_X(f) \cap U \neq \emptyset$ and such that $V_X(f)$ does not contain any irreducible component of Y. Hence,

$$\dim V_X(f) = \dim X - 1 = \dim Z$$

for some irreducible component Z of $V_X(f)$ (see Exercise 5.2). This implies that Z is not inside Y, otherwise Y should have dimension more than $\dim X - 1$. Now by induction

$$\dim U > \dim Z \cap U = \dim Z = \dim X - 1$$

therefore $\dim U = \dim X$.

(iii) Let Z be an irreducible component of $V_X(f)$. Let U be an affine open subset of X which intersects Z but not the other components of $V_X(f)$. Now f is also a regular function on U and $V_U(f) = Z \cap U$. So, dimension of $Z \cap U$ is $\dim U - 1 = \dim X - 1$. Thus, $\dim Z = \dim X - 1$. $\square$

Example 4.3 We show that $\dim \mathbb{A}_k^n = \dim \mathbb{P}_k^n = n$. Let $H \subset \mathbb{A}_k^n$ be the affine algebraic variety defined by $t_n = 0$. Then, $\dim H = \dim \mathbb{A}_k^n - 1$. On the other hand, $H \simeq \mathbb{A}_k^{n-1}$, so by induction $\dim \mathbb{A}_k^n = n$. Since $\mathbb{A}_k^n$ is an open subset of $\mathbb{P}_k^n$, $\dim \mathbb{P}_k^n = n$.

Theorem 4.4. *Let $F_1, F_2 \in R = k[s_0, s_1, s_2]$ be two non-constant homogeneous polynomials and let $X_i = V(F_i) \subset \mathbb{P}_k^2$. Then, $X_1 \cap X_2 \neq \emptyset$, i.e. any two projective curves in $\mathbb{P}_k^2$ intersect.*

Proof. We can assume that the F_i are irreducible and that $\langle F_1 \rangle \neq \langle F_2 \rangle$. Let $Z = V(F_1, F_2) \subseteq \mathbb{A}_k^3$. Note that since F_1 and F_2 are homogeneous, Z contains the origin and so it is not empty. We prove that $\dim Z = 1$. Let $R' = R/\langle F_1 \rangle$ and $R'' = R'/\langle F_2 \rangle$. By previous theorems, $\dim R = 3$, $\dim R' = 2$ and finally $\dim R'' = 1$. On the other hand, $R'' \simeq R/\langle F_1, F_2 \rangle$ and $\dim R'' = \dim Z$, so $\dim Z = 1$.

Now any point of Z which is not the origin gives a point of $X_1 \cap X_2$, therefore $X_1 \cap X_2 \neq \emptyset$. $\square$

Theorem 4.5. *Let X be a quasi-projective variety. Then, $\dim X$ is the transcendence degree of $k(X)$ over k.*

Proof. If $U \neq \emptyset$ is an open subset, then $k(U) = k(X)$ so we can assume that $X \subseteq \mathbb{A}_k^n$ is an affine variety. Let d be the transcendence degree of $k(X)$ over k. Then, there are elements $z_1, \dots, z_d \in k(X)$ which are algebraically independent over k, and they generate a subfield $L = k(z_1, \dots, z_d) \subseteq k(X)$ such that $k(X)$ is a finite extension of L.

First suppose that $L = k(X)$. Then, using the z_i, we get a rational map $\pi\colon X \dashrightarrow \mathbb{A}_k^d$ given by $\pi = (z_1, \dots, z_d)$. Since we can recover all the variables t_i on X, π is birational. On the other hand, $\mathbb{A}_k^d$ can be considered as a hypersurface Y in $\mathbb{A}_k^{d+1}$.

Now assume that $L \neq k(X)$ for any set $z_1, \dots, z_d$ of algebraically independent elements. Then by [5, Appendix 5, Proposition 1], we can find algebraically independent elements $z_1, \dots, z_d$ and an element $z_{d+1} \in k(X) - L$ such that $k(X) = L(z_{d+1})$ and $k(X)$ is a separable extension of L. There is an irreducible polynomial $f \in k[s_1, \dots, s_{d+1}]$ such that $f(z_1, \dots, z_{d+1}) = 0$. Define $Y = V(f)$ in the affine space $\mathbb{A}_k^{d+1}$. We can define a rational map $\pi\colon X \dashrightarrow Y$ by $\pi = (z_1, \dots, z_{d+1})$ which induces an isomorphism $k(Y) \simeq k(X)$ over k hence π is a birational isomorphism (see exercises to Section 3). So, again X is birational to a hypersurface in $\mathbb{A}_k^{d+1}$.

Since X and Y are birational, they have isomorphic open subsets U and V. Therefore, $\dim X = \dim Y$. Finally, since Y is a hypersurface in $\mathbb{A}_k^{d+1}$, its dimension is d. ☐

In particular, we also proved the following

Corollary 4.6. *Any quasi-projective algebraic variety of dimension d is birational to an irreducible hypersurface in $\mathbb{A}_k^{d+1}$.*

For a regular map $\phi\colon X \to Y$, we define the fibre of ϕ over $y \in Y$ to be $X_y := \phi^{-1}\{y\}$.

Theorem 4.7. *Let $\phi\colon X \to Y$ be a regular map of quasi-projective algebraic varieties which is dominant, i.e. $\phi(X)$ is dense in Y. Let $\dim X = d$ and $\dim Y = d'$. Then,*

(i) for any $y \in \phi(X)$, $\dim X_y \geq d - d'$,
(ii) there is a non-empty open set $U \subseteq Y$ such that if $y \in U \cap \phi(X)$, then $\dim X_y = d - d'$,
(iii) if ϕ is surjective, then for each l,

$$P_l = \{y \in Y \mid \dim X_y \geq l\}$$

is a closed subset of Y.

Proof. (i) Let $y \in \phi(X)$. By replacing Y and X with appropriate open affine subsets, we can assume that X, Y are affine. Pick $f \in k[Y]$ so that $y \in V_Y(f) \neq Y$. By previous theorems, every irreducible component of $\phi^{-1}V_Y(f) = V_X(f\phi)$ is of dimension $d-1$ and $X_y \subseteq V_X(f\phi)$. Choose an irreducible component S of $V_X(f\phi)$ which contains an irreducible component of X_y with maximal dimension. Let T be the closure of $\phi(S)$. By induction applied to $S \to T$, we see that

$$\dim X_y \geq \dim S - \dim T \geq d - 1 - (d' - 1) = d - d'$$

(ii) As in (i), we can assume that X, Y are affine. Since ϕ is dominant, the induced map $\phi^*\colon k[Y] \to k[X]$ is injective by Theorem **??**. Moreover, the induced map $k(Y) \to k(X)$ on the function fields is also injective and $d - d'$ is the transcendence degree of $k(X)$ over $k(Y)$. Let $f_1, \ldots, f_r \in k[X]$ so that they generate $k[X]$ as a k-algebra. We can assume that $f_1, \ldots, f_{d-d'}$ are algebraically independent over $k(Y)$ and each of the $f_{d-d'+1}, \ldots, f_r$ is algebraically dependent over the field generated by $k(Y)$ and $f_1, \ldots, f_{d-d'}$. Thus, for each $d - d' < i \leq r$, there is a polynomial $\Omega_i \in k[Y][u_1, \ldots, u_{d-d'}, u]$ such that $\Omega_i(f_1, \ldots, f_{d-d'}, f_i) = 0$.

Now

$$\Omega_i(f_1,\dots,f_{d-d'},u)\in k[Y][f_1,\dots,f_{d-d'}][u]$$

and we can assume that this is not zero. Let α_i be the set of coefficients of $\Omega_i(f_1,\dots,f_{d-d'},u)$ (so by construction $\alpha_i\subset k[Y][f_1,\dots,f_{d-d'}]\subseteq k[X]$).

Applying induction to the irreducible components of $V_X(\alpha_i)$ shows that there is an open non-empty set $U\subseteq Y$ such that if $y\in U\cap\phi(X)$ and if M is a component of X_y having maximal dimension (among the irreducible components of X_y) then M is not inside $V_X(\alpha_i)$ for any i. Therefore, if $g_i=f_i|_M$, then $k[M]=k[g_1,\dots,g_r]$ and from the equations $\Omega_i(g_1,\dots,g_{d-d'},g_i)=0$ we deduce that each of the $g_{d-d'+1},\dots,g_r$ is algebraically dependent over the field generated by k and the $g_1,\dots,g_{d-d'}$. Therefore, the transcendence degree of $k(M)$ over k is at most $d-d'$ hence $\dim X_y=\dim M\leq d-d'$ and from (i) we get $\dim X_y=d-d'$ for any $y\in U\cap\phi(X)$.

(iii) Assume ϕ surjective. From (i) and (ii) we see that $P_{d-d'}=Y$. Let U the maximal open set which satisfies (ii) and let T be an irreducible component of $Y\setminus U$. Pick an irreducible component S of $\phi^{-1}T$ such that $S\to T$ is dominant. Applying induction to all such $S\to T$ shows that $\dim X_y\geq d-d'+1$ for any $y\in T$. So, $P_{d-d'+1}=Y\setminus U$ is a closed subset. Continuing this type of argument and by applying induction on dimension and induction on l one proves that each P_l is closed. □

5. EXERCISES

Exercise 5.1 Let X be a non-empty Hausdorff topological space (eg, a metric space). Show that $\dim X=0$. So, our definition of dimension does not suit other types of geometry.[5.1]

Exercise 5.2 Let X be a quasi-projective algebraic set. Prove that if $Y\subseteq X$ is a closed subset of X, then $\dim Y\leq\dim X$. Prove that if $X=X_1\cup\cdots\cup X_l$ where X_i are closed subsets, then $\dim X=\max\{\dim X_i\}$.

Exercise 5.3 Let X be a quasi-projective algebraic set and $X=\bigcup U_i$ an open cover. Show that $\dim X=\max\{\dim U_i\}$.

Exercise 5.4 Let $\phi\colon\mathbb{P}^n_k\to\mathbb{A}^m_k$ be a regular map. Prove that ϕ is constant, that is, $\phi(\mathbb{P}^n_k)$ is a single point.

Exercise 5.5 Let $\phi\colon\mathbb{P}^n_k\to\mathbb{P}^m_k$ be a regular map where $n>m$. Prove that ϕ is constant.

[5.1] This was provided by Tom Sutton.

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