ALGEBRAIC GEOMETRY (PART III) EXERCISE SHEET 3

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As usual k is an algebraically closed field unless stated otherwise.

Exercise 1. *Show that* $\Omega^1[\mathbb{P}^n_k] = 0$.

Exercise 2. *Assume* $\operatorname{char} k \neq 2$, *and let* $X = V(t_1^2+t_2^2-1) \subset \mathbb{A}^2_k$. *Show that* $\omega = \frac{1}{t_2}dt_1$ *is a regular differential form on* X *and that* $\Omega^1[X] \simeq k[X]\omega$.

Exercise 3. *Let* $X = V(t_2^2 - t_1^3) \subset \mathbb{A}^2_k$. *Show that there is a non-zero regular differential form* ω *on* X *such that* $\omega(x) = 0$ *for every* $x \in X$.

Exercise 4. *Let* X *be a quasi-projective algebraic set and* $x \in X$. *Assume that there is an open affine neighbourhood* U *of* x *such that* $\Omega[U]$ *is a free* $k[U]$*-module. Show that* X *is smooth at* x.

Exercise 5. *Let* X *be a smooth projective curve and* $x \in X$. *Show that there is a rational function* $f \in k(X)$ *which is not regular at* x *but regular at any other point.*

Exercise 6. *Let* X *be a smooth projective curve of genus* g. *Show that there is a dominant regular map* $\phi\colon X \to \mathbb{P}^1_k$ *such that* $\phi^*\colon k(\mathbb{P}^1_k) \to k(X)$ *is an extension of degree at most* $g+1$ *(here degree means* $\dim_{k(\mathbb{P}^1_k)} k(X)$*).*

Exercise 7. *Let* $X = V(F) \subset \mathbb{P}^2_k$ *be a projective curve where* F *is irreducible of degree* l. *Show that* X *has at most* $\frac{1}{2}(l-1)(l-2)$ *singular points.*

Exercise 8. *Let* X *be a smooth projective curve,* $x \in X$, *and* $D = lx$. *Let* $n = \dim_k H^0(X, D) - 1$. *Recall from Exercise 24 of Example Sheet 2 that we have a rational map* $\pi_D\colon X \dashrightarrow \mathbb{P}^n_k$. *Show that* π_D *is a regular map if* $l \gg 0$.

Exercise 9. *Let* $X = V(F) \subset \mathbb{P}^2_k$ *be smooth where* F *is irreducible of degree 2. Pick six distinct points* $x_1, \dots, x_6 \in X$ *and let* $L_{i,j}$ *be the line passing through* x_i, x_j *when* $i \neq j$. *Now let* $Q = L_{1,2} \cap L_{4,5}$,

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$Q' = L_{2,3} \cap L_{5,6}$, and $Q'' = L_{3,4} \cap L_{6,1}$. Show that Q, Q', Q'' lie on a line (this is Pascal's theorem in classical projective geometry).

Exercise 10. *Let X be a quasi-projective algebraic set and $\{\mathcal{F}_i\}_{i\in\mathbb{Z}}$ a sequence of sheaves on X. For each open set $U \subseteq X$ let $\mathcal{G}(U) = \bigoplus_{i\in\mathbb{Z}} \mathcal{F}_i(U)$. Show that $\mathcal{G}$ is a sheaf on X (the sheaf $\mathcal{G}$ is usually denoted by $\bigoplus_{i\in\mathbb{Z}} \mathcal{F}_i$). Next, assuming that the $\mathcal{F}_i$ and $\mathcal{G}$ are quasi-coherent, show that $H^p(X, \mathcal{G}) \simeq \bigoplus_{i\in\mathbb{Z}} H^p(X, \mathcal{F}_i)$ for each p.*

Exercise 11. *Let $X = \mathbb{P}^n_k$, $H = V(s_0) \subset \mathbb{P}^n_k$, and $A = V(s_n) \subset \mathbb{P}^n_k$. Show that for each $m \in \mathbb{Z}$ there is a natural exact sequence*

$$0 \to \mathcal{O}_X((m-1)H) \to \mathcal{O}_X(mH) \to \mathcal{O}_A(mH_A) \to 0$$

where H_A is the divisor $H \cap A$ on A.

Exercise 12. *** Let $X = \mathbb{P}^n_k$, $H = V(s_0) \subset \mathbb{P}^n_k$, and let W_m be the k-vector space of homogeneous polynomials of degree m in $k[s_0, \dots, s_n]$. By taking an appropriate open affine covering of $\mathbb{P}^n_k$ and using Čech cohomology (try as much as you can to) show that the following statements hold for every $m \in \mathbb{Z}$ (the isomorphisms are as k-vector spaces):*
(i) $H^0(X, \mathcal{O}_X(mH)) \simeq W_m$,
(ii) $H^n(X, \mathcal{O}_X(-(n+1)H)) \simeq k$,
(iii) $H^n(X, \mathcal{O}_X(mH)) \simeq H^0(X, \mathcal{O}_X(-(n+1+m)H))^*$ where $*$ stands for dual,
(iv) $H^p(X, \mathcal{O}_X(mH)) = 0$ if $0 < p < n$.

[Hint 1: it is helpful to consider $\mathcal{G} := \bigoplus_{m\in\mathbb{Z}} \mathcal{O}_X(mH)$ and to try to calculate $H^p(X, \mathcal{G})$ assuming that $\mathcal{G}$ is quasi-coherent]
[Hint 2: to prove (iv) you need to use the exact sequences $0 \to \mathcal{O}_X((m-1)H) \to \mathcal{O}_X(mH) \to \mathcal{O}_A(mH_A) \to 0$]

Exercise 13. *Let $X = \mathbb{A}^{n+1}_k \setminus \{0\}$, $G = k^* = k \setminus \{0\}$, and $a_0, \dots, a_n \in \mathbb{N}$. Each $g \in G$ can be identified with the automorphism $\phi_g \colon X \to X$ given by $\phi_g(x_0, \dots, x_n) = (g^{a_0}x_0, \dots, g^{a_n}x_n)$. Show that the quotient of X by G is the weighted projective space $\mathbb{P}(a_0, \dots, a_n)$.*

Exercise 14. *Let $X = \mathbb{A}^2_k$ and $G = \{1, \mu, \mu^2\}$ where μ is a 3-rd root of unity. Each $g \in G$ can be identified with the automorphism $\phi_g \colon X \to X$ given by $\phi_g(x_1, x_2) = (gx_1, gx_2)$. Let Y be the quotient of X by G. Find explicit equations of Y as a closed subset of some affine space. Is Y smooth?*

Exercise 15. *Let $X = \mathbb{A}^2_k$ and let $\phi \colon X \to X$ and $\psi \colon X \to X$ be given by $\phi(x_1, x_2) = (\mu x_1, -\mu x_2)$ and $\psi(x_1, x_2) = (x_2, -x_1)$ where μ is a 2-nd root of unity. Let G be the subgroup of $\mathrm{Aut}(X)$ generated by ϕ and ψ.*

Find explicit equations for the quotient of X by G, say Y. Find the singular points of Y.