

ALGEBRAIC GEOMETRY PART III EXAMPLES AND EXERCISES 1

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As usual k is an algebraically closed field unless stated otherwise.

Exa: Let $f \in k[t_1, t_2]$ be of degree ≤ 1 , that is, $f = at_1 + bt_2 + c$ where $a, b, c \in k$. We call such f , linear. When f is not a constant, $X = V(f) \subset \mathbb{A}_k^2$ is called a line. We have $X = \{(x_1, \frac{ax_1+c}{-b}) \in \mathbb{A}^2 \mid x_1 \in k\}$ if $b \neq 0$. We get a similar description of X if $a \neq 0$.

Exa: Let $X = V(t_1 t_2) \subset \mathbb{A}_k^2$. Then, $X = V(t_1) \cup V(t_2)$, that is, X is the union of two lines. X is an affine algebraic set but not a variety because it is not irreducible.

Exa: Let $X = V(t_1, t_2) \subset \mathbb{A}_k^2$. Then, $X = \{(0, 0)\}$ which is a variety.

Exe: Let $X = V(t_1^2 + t_2^2 - 1, t_1 - 1) \subset \mathbb{A}_k^2$. What is I_X ? Is I_X a prime ideal?

Exe: Let $X = V(t_1^2 + t_2^2 + t_3^2) \subset \mathbb{A}_k^3$. Determine I_X when the characteristic of k is 2. Determine I_X when the characteristic of k is not 2.

Exe: In the lecture, we proved that every affine algebraic set X is the union of finitely many varieties, that is, $X = X_1 \cup X_2 \cup \dots \cup X_m$ where X_i are irreducible. If $X_i \not\subseteq X_j$ for $i \neq j$, we call X_i the irreducible components of X . Now decompose $X = V(t_1^2 - t_2 t_3, t_1 t_3 - t_1) \subset \mathbb{A}_k^3$ into its irreducible components.

Exe: Let X be a quasi projective algebraic set and $X = \bigcup U_j$ a cover of X by open subsets U_j . Prove that only finitely many of U_j would be enough to cover X .

Exa: Let $X = \{P_1, \dots, P_m\} \subset \mathbb{A}_k^n$ be finitely many distinct points. We determine the coordinate ring $k[X]$. We know that $I_X = \bigcap_{j=1}^m I_{\{P_j\}}$.

Date: 22 Oct 2007.

By Hilbert Nullstellensatz, $I_{\{P_j\}}$ is a maximal ideal of $k[t_1, \dots, t_n]$. So,

$$k[X] = k[t_1, \dots, t_n]/I_X \simeq \bigoplus_{j=1}^m k[t_1, \dots, t_n]/I_{\{P_j\}} \simeq \bigoplus_{j=1}^m k$$

Exe: Let $\phi: \mathbb{A}_k^1 \rightarrow \mathbb{A}_k^1$ be an isomorphism. Prove that ϕ is given by a linear polynomial.

Exe: Let $\phi: \mathbb{A}_k^2 \rightarrow \mathbb{A}_k^2$ be given by $\phi = (t_1, t_1 t_2)$. Is $\phi(\mathbb{A}_k^2)$ open, closed or dense?

Exa: Let $X = V(t_1 t_2 - 1) \subseteq \mathbb{A}_k^2$. The regular map to \mathbb{A}_k^1 given by $\phi = t_1$ is injective but not surjective so not an isomorphism. However, $\phi(X) = \mathbb{A}_k^1 - \{0\}$ which is dense in \mathbb{A}_k^1 .

Exa-exe: The regular map $\phi: \mathbb{A}_k^1 \rightarrow Y$ given by $\phi = (t^2, t^3)$ where $Y = V(s_2^2 - s_1^3) \subseteq \mathbb{A}_k^2$ (cusp singularity) is a 1-1 regular map which is not an isomorphism. In fact, the dual homomorphism $\phi^*: k[Y] = k[s_1, s_2]/\langle s_2^2 - s_1^3 \rangle \rightarrow k[t]$ is determined by $\phi^*(s_1) = t^2$ and $\phi^*(s_2) = t^3$. So t cannot be in the image of ϕ^* , so ϕ^* is not an isomorphism. If one tries to get an inverse it should be as $\psi = s_2/s_1$ which is not regular but rational. This means that \mathbb{A}_k^1 and Y are not isomorphism but birational.

Exe: Let $M_{n \times m}$ be the set of $n \times m$ matrices over k . Prove that there is a bijection from $M_{n \times m}$ to \mathbb{A}^{nm} . Prove that the set of non-invertible matrices in $M_{n \times n}$ corresponds to a hypersurface in \mathbb{A}^{n^2} .

*Exe:** Let k be any field not necessarily algebraically closed, and let $f, g \in k[t_1, t_2]$ be non-constant polynomials without common factor in their factorisations into irreducible factors. Prove directly that $V(f) \cap V(g) \subset \mathbb{A}_k^2$ is finite. (Consider f, g as polynomials over the ring $k(x)[y]$ or similarly $k(y)[x]$)

Exe: Let $X = V(t_1^2 + t_2^2 - 1) \subset \mathbb{A}_k^2$ and let π be the rational function of X defined by $\pi = (1 - t_2)/t_1$. Determine the domain of π .

Exe: Let $Y = V(t_2^2 - t_1^2 - t_1^3) \subset \mathbb{A}_k^2$ and let $\phi: \mathbb{A}_k^1 \rightarrow Y$ be the regular map defined by $\phi = (s^2 - 1, s(s^2 - 1))$. Describe ϕ^* and prove that ϕ is not an isomorphism but it is a birational isomorphism.

Exa: Let $Y = V(f) \subset \mathbb{A}_k^2$ be a variety defined by f of degree 2 and after a linear change of variables we can assume that $(0, 0) \in Y$. Let L_s be the line defined by $V(t_2 - st_1)$ where $s \in k$. This line intersects Y at $(0, 0)$. Since f has degree 2, it intersects Y at another point y_s except if s is in a fixed finite set $\Lambda \subset k$. To see this one needs to solve the equation $g(t_1) = f(t_1, st_1) = 0$. g is of degree 2 if $s \notin \Lambda$ and 0 is one of its solutions, so $g(t_1) = t_1 h(t_1)$ for some polynomial $h(t_1) = A(s)t_1 + B(s)$ which is linear in t_1 . Thus, the other root of g is given by a rational function $\pi_1 = -B(s)/A(s)$. Since $t_2 - st_1$ defines the line L_s , the other coordinate of y_s is given by another rational function $\pi_2 = s\pi_1$ in s . In short, we have defined a rational map $\pi: \mathbb{A}_k^1 \dashrightarrow Y$ given by $\pi = (\pi_1, \pi_2)$. Define also $\theta: Y \dashrightarrow \mathbb{A}_k^1$ by $\theta = t_2/t_1$. Then, π and θ are inverse of each other. This example shows that Y is birational to \mathbb{A}_k^1 .

Exe: Let $Y = V(t_1^2 + (t_2 - 1)^2 - 1) \subset \mathbb{A}_k^2$ and define the rational maps π and θ in the previous example explicitly.

Exa: Let $X = \mathbb{P}_k^n$ and let U_0 be the open subset defined by $s_0 \neq 0$. By theorems in the lecture, $k(X) = k(U_0)$. On the other hand, U_0 is the affine space \mathbb{A}_k^n . So, $k(X) = k(U_0) = k(t_1, \dots, t_n)$ where $t_i = s_i/s_0$.

Exa: Let $X = \mathbb{P}_k^n$, we prove that $k[X] = k$. Let $f: X \rightarrow k$ be a regular function. Then, for $x \in X$ there is a neighborhood $x \in U$ and F, G homogeneous of the same degree in $k[s_0, \dots, s_n]$ such that $f = F/G$ on U , and G has no zero on U . We can assume that F, G have no common factors. Suppose that G is not a constant and let $x' \in V(G) - V(F)$. Then, there is a neighborhood $x' \in U'$ and F', G' homogeneous of the same degree in $k[s_0, \dots, s_n]$ such that $f = F'/G'$ on U' , and G' has no zero on U' , and F', G' have no common factor. So, F/G and F'/G' give the same values on $U \cap U' \neq \emptyset$. Since X is irreducible, this is possible only if $FG' - F'G = 0$. This is a contradiction because $F(x')G'(x') \neq 0$ but $F'(x')G(x') = 0$.

Exa: Regular maps of affine algebraic sets are the same as their regular maps as quasi-projective algebraic sets.

Proof. Let $X \subseteq \mathbb{A}_k^n$ and $Y \subseteq \mathbb{A}_k^m$ be affine algebraic sets and $\phi: X \rightarrow Y$ a regular map of affine algebraic sets given by $\phi = (f_1, \dots, f_m)$ where $f_i \in k[t_1, \dots, t_n]$. We naturally identify \mathbb{A}_k^n with $U_0 \subset \mathbb{P}_k^n$ and \mathbb{A}_k^m with $V_0 \subset \mathbb{P}_k^m$. Under this identification, we use affine and projective coordinates for the same points without mention.

By previous theorems, ϕ is continuous.

On X the map ϕ corresponds to the map given by

$$\Phi = (s_0^{d_0} : s_0^{d_1} F_1 : \cdots : s_0^{d_m} F_m)$$

where $F_i \in k[s_0, \dots, s_n]$ is the homog. polynomial associated to f_i and d_i are in $\mathbb{N} \cup \{0\}$ so that all the components have the same degree. Moreover, if $V \subseteq Y$ is an open subset, a regular function on V (with quasi-projective definition) is locally given by F/G and when we compose it with Φ , it gives a function which is locally like F'/G' and so we get a regular function on $\Phi^{-1}V$. In other words, Φ is a regular map in the quasi-projective sense.

Now let $\Phi: X \rightarrow Y$ be a regular map in the sense of quasi-projective varieties and let $x \in X$. Taking the ring $k[r_0, \dots, r_m]$ corresponding to \mathbb{P}_k^m , any r_i/r_0 is a regular function on Y , and by definition it gives a regular function ϕ_i on X . Moreover, the function $X \rightarrow Y$ given by $(1 : \phi_1 : \cdots : \phi_m)$ is identical to Φ . So, $\phi = (\phi_1, \dots, \phi_m)$ corresponding to Φ is a regular map in the affine sense. \square

The proof also shows that rational maps of affine varieties are the same as their rational maps as quasi-projective varieties. The same is true for rational functions.

Exa: Let $X = V(s_1 s_2 - s_0^2) \subseteq \mathbb{P}_k^2$. Then, $\phi: X \rightarrow \mathbb{P}_k^1$ given by $\phi = (s_0 : s_1) = (s_2 : s_0)$ is a regular map. This is an isomorphism with the inverse $\theta: \mathbb{P}_k^1 \rightarrow X$ given by $\theta = (r_0 r_1 : r_1^2 : r_0^2)$. This example also shows that $X_1 = X - \{(0 : 1 : 0), (0 : 0 : 1)\}$ is isomorphic to $\mathbb{P}_k^1 - \{(1 : 0), (0 : 1)\}$. The first one is a closed affine set in \mathbb{A}_k^2 but the second one is an open subset of \mathbb{A}_k^1 .

Exa: $\phi: \mathbb{P}_k^n \rightarrow \mathbb{P}_k^n$ given by $\pi = (s_0^d, \dots, s_n^d)$ is a regular map. If $d > 1$, it is not an isomorphism because it is not 1-1.

Exa: $\pi: \mathbb{P}_k^2 \dashrightarrow \mathbb{P}_k^2$ given by $\pi = (s_1 s_2, s_0 s_2, s_0 s_1)$ is a rational map. It is not regular at the three points $(0 : 0 : 1)$, $(0 : 1 : 0)$, and $(1 : 0 : 0)$. However, it is birational its inverse being $\theta = (r_1 r_2, r_0 r_2, r_0 r_1)$. This is called a Cremona transformation.

Exa: let d be a natural number and $m = \binom{n+d}{d} - 1$. Define $\phi: \mathbb{P}_k^n \rightarrow \mathbb{P}_k^m$ by the m polynomial $s_0^{d_0} \cdots s_n^{d_n}$ where $\sum d_i = d$ and $d_i \in \mathbb{N} \cup \{0\}$. Then, ϕ is a regular map which is called the Veronese embedding.

Exa: Let $\phi: \mathbb{P}_k^n \rightarrow \mathbb{A}_k^m$ be a regular map. Prove that ϕ is constant, that is, $\phi(\mathbb{P}_k^n)$ is a single point.