ALGEBRAIC GEOMETRY (PART III)
EXAMPLE SHEET 4

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(1) Let X be a topological space and $x \in X$. Show that the functor $F\colon \mathcal{S}h(X) \to \mathcal{A}b$ defined by $F(\mathcal{F}) = \mathcal{F}_x$ is a left exact functor. What are the right derived functors of F?

(2) Let $\{\mathcal{F}_i\}$ be a directed system of flasque sheaves on a Noetherian topological space X. Prove that $\varinjlim \mathcal{F}_i$ is a flasque sheaf.

(3) Let $\mathcal{F} = \bigoplus_{i \in I} \mathcal{F}_i$ where $\mathcal{F}_i$ are sheaves on a Noetherian topological space X. Show that $H^p(X, \mathcal{F}) \simeq \bigoplus_{i \in I} H^p(X, \mathcal{F}_i)$ for every p.

(4) Let $f\colon X \to Y$ be a continuous map of topological spaces. Let $\mathcal{F}$ be a flasque sheaf in $\mathcal{S}h(X)$. Show that the sheaf $f_*\mathcal{F}$ is flasque.

(5) Let X be a closed subset of a topological space Y and $f\colon X \to Y$ the inclusion map. Prove that for any sheaf $\mathcal{F}$ on X, we have $H^p(Y, f_*\mathcal{F}) \simeq H^p(X, \mathcal{F})$ for every p.

(6) Let X be an irreducible topological space and $\mathcal{F}$ be the constant sheaf on X defined by $\mathbb{Z}$. Compute the cohomology groups $H^p(X, \mathcal{F})$.

(7) Give an example of an affine scheme X and a sheaf $\mathcal{F}$ on X such that $H^p(X, \mathcal{F}) \neq 0$ for some $p > 0$.

(8) Let $\mathcal{F}$ be the constant sheaf on $X = \mathbb{A}^1_{\mathbb{C}}$ defined by $\mathbb{Z}$, and let $x, y \in X$ be distinct closed points, $U = X \setminus \{x\}$ and $V = X \setminus \{x, y\}$. Compute the Čech cohomology groups of $\mathcal{F}$ on X, U and V by choosing various coverings of these spaces.

(9) Give an example of a topological space X, an open covering $\mathcal{U} = (U_i)_{i \in I}$ and an exact sequence $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{E} \to 0$ of sheaves on X such that we do not get a long exact sequence of the Čech cohomology groups.

(10) Give an example of a topological space X, an open covering $\mathcal{U} = (U_i)_{i \in I}$ and a sheaf $\mathcal{F}$ on X such that $\check{H}^1(\mathcal{U}, \mathcal{F}) \neq H^1(X, \mathcal{F})$.

(11) Let $X = \mathbb{P}^1_k$ where k is an algebraically closed field. Let $\mathcal{K}$ be the constant sheaf on X defined by the function field K of X and let $\mathcal{O}_X \to \mathcal{K}$ be the induced

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natural morphism. Show that the exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{K} \to \frac{\mathcal{K}}{\mathcal{O}_X} \to 0$$

gives a flasque resolution of $\mathcal{O}_X$. Compute the cohomology groups of $\mathcal{O}_X$ using this resolution.

(12) Let X be a Noetherian scheme and let $\mathcal{Q}(X)$ be the category of quasi-coherent sheaves on X. Show that $\mathcal{Q}(X)$ has enough injectives.

(13) A morphism $f\colon X \to Y$ of schemes is said to be affine if for any open affine subscheme $U \subseteq Y$, $f^{-1}U$ is affine. Now let $f\colon X \to Y$ be an affine morphism of Noetherian separated schemes. Prove that for any quasi-coherent sheaf $\mathcal{F}$ on X, we have $H^p(Y, f_*\mathcal{F}) \simeq H^p(X, \mathcal{F})$ for every p.

(14) Let $X = \mathbb{A}^2_k = \operatorname{Spec} k[t_1, t_2]$ where k is a field and let $U = X \setminus \{x\}$ where x is the ideal $< t_1, t_2 >$. Calculate $H^1(U, \mathcal{O}_U)$.

(15) Let X be the closed subscheme of $\mathbb{P}^2_k$ defined by the ideal of a homogeneous polynomial F of degree d where k is a field. Show that $\dim_k H^0(X, \mathcal{O}_X) = 1$ and $\dim_k H^1(X, \mathcal{O}_X) = \frac{1}{2}(d-1)(d-2)$.

(16) Let X be a topological space, $\mathcal{U} = (U_i)_{i\in I}$ a finite open cover, and $\mathcal{F}$ a sheaf on X. Assume that for each $p \geq 0$ and each $i_0 < \cdots < i_p$ we have $H^l(U_{i_0,\dots,i_p}, \mathcal{F}|_{U_{i_0,\dots,i_p}}) = 0$ whenever $l > 0$. Show that there are isomorphisms $H^l(X, \mathcal{F}) \simeq \check{H}^l(\mathcal{U}, \mathcal{F})$ for any $l \geq 0$.

(17) Let A be a Noetherian ring and X a closed subscheme of $\mathbb{P}^n_A$. Show that $H^p(X, \mathcal{F}) = 0$ for any $p > n$ and any quasi-coherent sheaf $\mathcal{F}$ on X.

(18) Give another proof of Theorem 6.25 along the following lines. Let X be a Noetherian separated scheme, $\mathcal{U} = (U_i)_{i\in I}$ a finite open covering by open affine subschemes, $f_{i_0,\dots,i_p}\colon U_{i_0,\dots,i_p} \to X$ the inclusion morphism, and $\mathcal{F}_{i_0,\dots,i_p} = (f_{i_0,\dots,i_p})_*\mathcal{F}|_{U_{i_0,\dots,i_p}}$.

(i) Show that $(f_{i_0,\dots,i_p})_*$ sends any exact sequence of quasi-coherent sheaves on $U_{i_0,\dots,i_p}$ to an exact sequence of quasi-coherent sheaves on X.
(ii) Show that $H^l(X, \mathcal{F}_{i_0,\dots,i_p}) \simeq H^l(U_{i_0,\dots,i_p}, \mathcal{F}|_{U_{i_0,\dots,i_p}}) = 0$ for any $l > 0$.
(iii) Show that any $\mathcal{C}^p(\mathcal{U}, \mathcal{F})$ in the exact sequence

$$0 \to \mathcal{F} \to \mathcal{C}^0(\mathcal{U}, \mathcal{F}) \to \mathcal{C}^1(\mathcal{U}, \mathcal{F}) \to \dots$$

given in the course is acyclic, that is, $H^l(X, \mathcal{C}^p(\mathcal{U}, \mathcal{F})) = 0$ if $l > 0$.
(iv) Conclude that $H^l(X, \mathcal{F}) \simeq \check{H}^l(\mathcal{U}, \mathcal{F})$ for any $l \geq 0$.