Mld's vs thresholds and flips

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Abstract

Minimal log discrepancies (mld's) are related not only to termination of log flips [27] but also to the ascending chain condition (ACC) of some global invariants and invariants of singularities in the Log Minimal Model Program (LMMP). In this paper, we draw clear links between several central conjectures in the LMMP. More precisely, our main result states that the LMMP, the ACC conjecture for mld's and the boundedness of canonical Mori-Fano varieties in dimension $\le d$ imply the following: the ACC conjecture for a-lc thresholds, in particular, for canonical and log canonical (lc) thresholds in dimension $\le d$; the ACC conjecture for lc thresholds in dimension $\le d+1$; and termination of log flips in dimension $\le d+1$ for effective lc pairs. In particular, when $d=3$ we can drop the assumptions on LMMP and boundedness of canonical Mori-Fano varieties.

1 Introduction

One of the main open problems in the LMMP is the termination of log flips. Existence of log flips in the klt (hence $\mathbb{Q}$-factorial dlt) case has recently been settled [6]. It is well-known that the termination follows from two local (even formal) problems on mld's [27]: the ACC conjecture (see Conjecture 1.3 below), and the semicontinuity conjecture due to Ambro [3, Conjecture 2.4]. Recently, the first author [5] reduced a weaker termination in dimension $d+1$

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(e.g., when the log Kodaira dimension is nonnegative), in particular, termination of log flips in the relative birational case, to the LMMP in dimension d and the ACC conjecture for lc thresholds in dimension $d + 1$ (see Conjecture 1.7) which in its turn follows from V. Alexeev's, and brothers' A. and L. Borisov conjecture in dimension d (see [20, Conjecture 3.8] and cf. Conjecture 1.2).

We use the terminology of [25] [26] [12][15]; see also Notation and terminology below. However we need certain modifications or generalizations of some well-known notions and conjectures.

Definition 1.1 (cf. [12, Definition 1.6(v)) A proper contraction $X \to Z$ of normal varieties is called a *Mori-Fano fibration* if the following conditions hold:

a) $\dim Z < \dim X$;

b) X has only $\mathbb{Q}$-factorial lc singularities;

c) $\rho(X/Z) := \rho(X) - \rho(Z) = 1$, where $\rho(\ )$ is the Picard number; and

d) the anticanonical divisor $-K = -K_X$ is ample/Z.

If $Z =$ pt. is a point, X is called a *Mori-Fano variety*. We say that X is a *canonical* Mori-Fano variety if X has only canonical (cn) singularities.

Note that by the Kleiman projectivity criterion, any Mori-Fano fibration and variety are projective.

Conjecture 1.2 (Weak BAB) *The canonical d-dimensional Mori-Fano varieties are* bounded*, that is, a coarse moduli space of such varieties is well-defined and of finite type.*

BAB abbreviates V. Alexeev, and brothers A. and L. Borisov. The conjecture is a very special case of their conjecture (see [20]). Conjecture 1.2 is established in dimension ≤ 3 in characteristic zero [17] (the case $d = 2$ is classical). Actually, we need a much weaker version of this conjecture, namely, the boundedness of canonical d-dimensional Mori-Fano varieties X such that $K + B \equiv 0$ for some boundary $B \in \Gamma$ where Γ is a fixed set of real boundary multiplicities satisfying the descending chain condition (DCC).

Conjecture 1.3 (ACC for mld's) *Suppose that $\Gamma \subseteq [0, 1]$ satisfies the DCC. Then the following holds:*

(ACC) *The following subset of real numbers* $\mathbb{R}$

$$\left\{ \operatorname{mld}(P, X, B) \middle| \ (X, B) \text{ is lc, } \dim X = d, \ P \in X, \text{ and } B \in \Gamma \right\}$$

satisfies the ACC.

A point P can be nonclosed. Equivalently, we can consider only closed points $P \in X$, and assume that $\dim X \leq d$.

This conjecture is established in dimension $d \leq 2$ [1] [22], and for some special cases in higher dimensions [7] [25][4].

Definition 1.4 (a-lc thresholds) Let $a \geq 0$ be a real number, (X, B) be a log pair, and M be an $\mathbb{R}$-Cartier divisor on X. Then the real number or $+/-\infty$:

$$t = \mathrm{th}_a(M, X, B) := \sup\{\lambda \in \mathbb{R} \mid (X, B + \lambda M) \text{ is } a\text{-lc in codimension } \geq 2\}$$

is called the *a-lc threshold of M with respect to* (X, B); for the definition of a-lc see Notation and terminology below. In particular, if $a = 0$ or $a = 1$, the a-lc threshold is the *lc threshold* or the *cn threshold* respectively. By $\mathbb{Q}$-*factorial* thresholds we mean that we consider only $\mathbb{Q}$-factorial varieties. Similarly, we define the a-lc threshold *at a point* P (possibly not closed) if the a-lc condition in codimension ≥ 2 is replaced by the a-lc condition in codimension ≥ 2 at P.

Remark 1.5 Note that if $a \leq \mathrm{ldis}(X, B) < +\infty$, and $M > 0$, then $t \geq 0$, sup = max, and is a nonnegative real number (that is, not $+\infty$, cf. [12, Remark 1.4,(ii)]). In this situation, either $(X, B + tM)$ is precisely a-lc in codimension 2, that is $\mathrm{ldis}(X, B + tM) = a$, or $B + tM$ has a reduced component. Behavior of thresholds in codimension 1 (at divisorial points) is easy. However, when we consider thresholds at a point, the situation is more complicated (see Example 1.6 or cf. the proof of Proposition 2.5).

Note that we only need the case $a \leq 1$ if $\dim X \geq 2$. Indeed, $\mathrm{ldis}(X, B) \leq 1$ always when $\dim X \geq 2$, and $\mathrm{ldis}(X, B) = +\infty$ when $\dim X \leq 1$ (see Notation and terminology).

Example 1.6 The a-lc threshold at P may not be attained at P nor on the boundary. For example: take three planes S_1, S_2, S_3 in the space $\mathbb{P}^3$ passing through a line L. Take a closed point $P \in L$ and define $B = \frac{2}{3}S_1 + \frac{2}{3}S_2 + \frac{2}{3}S_3$. L is a lc centre for $(\mathbb{P}^3, B)$ but easy computations show that $\mathrm{mld}(P, \mathbb{P}^3, B) = 1$. On the other hand, $\lfloor B \rfloor = 0$.

So, in general, for the a-lc threshold at a point P, either we get a lc centre *passing* through P or the mld a is attained at P.

Conjecture 1.7 (ACC for a-lc thresholds) *Suppose that $d \geq 2$ is a natural number, $a \geq 0$ is a real number, $\Gamma \subset [0,1]$ satisfies the DCC and $S \subset \mathbb{R}$ is a set of nonnegative numbers satisfying the DCC. Then the following holds:*

(ACC) *The subset $\mathcal{T}_{a,d}(\Gamma, S)$ of $\mathbb{R}^+ \cup \{+\infty\}$ defined by*

$$\mathcal{T}_{a,d}(\Gamma, S) = \{\mathrm{th}_a(M, X, B) |\ (X, B) \text{ is } a\text{-lc in codimension} \geq 2, \dim X = d,$$
$$B \in \Gamma,\ M \text{ is an } \mathbb{R}\text{-Cartier divisor on } X, \text{ and } M \in S\}.$$

satisfies the ACC where $\mathbb{R}^+ = \{r \in \mathbb{R} \mid r \geq 0\}$; $+\infty$ corresponds to the case $M = 0$.

Moreover, the ACC holds for the similar set of a-lc thresholds at points, that is, when $\mathrm{th}_a(M, X, B)$ is replaced by the a-lc thresholds at some point $P \in X$ (see Definition 1.4). The latter set is larger. Thus ACC for thresholds at points implies ACC for thresholds on a variety.

From now on, by LMMP we mean the LMMP for $\mathbb{Q}$-factorial dlt pairs unless stated otherwise. We are ready to state the main result of this paper.

Main Theorem 1.8 *The* LMMP*,* ACC *for mld's and Conjecture 1.2 in dimension $\leq d$ imply the following:*

(i) ACC *for a-lc thresholds in dimension $\leq d$;*
(ii) ACC *for lc thresholds in dimension $\leq d + 1$; and*
(iii) *termination of log flips in dimension $\leq d + 1$ for effective lc pairs.*

See the proof in Section 5. For generalizations of statements (i-ii) in the Main Theorem and related problems see Section 2.

Corollary 1.9 *ACC for mld's and Conjecture 1.2 for* 4*-folds imply:*

(i) ACC *for a-lc thresholds in dimension* 4*;*
(ii) ACC *for lc thresholds in dimension* 5*; and*
(iii) *termination of log flips in dimension* 5 *for effective lc pairs.*

Proof ACC for mld's for 4-folds implies termination of 4-fold log flips [27, Corollary 5] and the LMMP in dimension 4 [26, Corollary 1.8]. Thus, we are done by the Main Theorem 1.8 □

Corollary 1.10 ACC *for mld's for* 3*-folds implies:*

(i) ACC *for a-lc thresholds in dimension* 3*;*
(ii) ACC *for lc thresholds in dimension* 4*; and*
(iii) *termination of 4-fold log flips for effective lc pairs.*

Proof Immediate by Main Theorem 1.8, and by [17], [13] for $\mathbb{Q}$-boundaries and [25] in general. □

ACC for mld's for algebraic surfaces gives a new proof of the following well-known and new results.

Corollary 1.11 *The following hold:*
(i) ACC *for a-lc thresholds of surfaces;*
(ii) ACC *for lc thresholds of* 3*-folds; and*
(iii) *termination of* 3*-fold log flips for effective lc pairs.*

Proof Immediate by Main Theorem 1.8, and [1][22] and [2]. □

Note that ACC for mld's for surfaces in [22] is established for $\mathbb{R}$-boundaries and without using classification. Thus, for the first time, termination in (iii) is proved without classification (cf. [5][13] [25, proof of 5.1.3 for 3-folds]).

Cn thresholds and, in particular, their ACC played a crucial role in the Sarkisov program for 3-folds [9] [18]. Another similar important invariant, the Sarkisov degree or its inverse – the anticanonical threshold – can be included into more general ones: Fano indices (see Corollary 2.13 below) and boundary multiplicities of log pairs in S_d(global) (see Definition 2.6 (v) and Weak finiteness 4.1). These invariants and results about them are important in the proof of our Main Theorem and will be discussed in Sections 2–4. Here we give a sample.

Corollary 1.12 *Let* $\Gamma \subset [0,1]$ *be a DCC set. Then, there is a finite subset* $\Gamma_f \subset \Gamma$ *such that* $\mathcal{S}_3^0(\Gamma, \mathrm{global}) = \mathcal{S}_3^0(\Gamma_\mathrm{f}, \mathrm{global})$ *(see Definition 2.6).*

In other words, the set of boundary multiplicities which occur on the following log pairs is finite: the 3-fold proper log pairs (X,B) with $B \in \Gamma$, $K + B \equiv 0$, (X,B) is lc but not klt.

Proof Immediate by Theorems 2.10 and 2.12 (vi). □

If in the corollary, the set Γ has only rational numbers, it implies a stronger finiteness: there exists a positive integer n such that $n(K+B) \sim 0$ for each of the related pairs (see [21, Corollary 1.9 and its proof 9.9]). The same is expected in the nonklt case in any dimension. If in addition X is of Fano type (FT, see Notation and terminology below) and the pair (X,B) is ε-lc for some fixed $\varepsilon > 0$, then according to general BAB [21, Conjecture 1.1], it is expected that such pairs are bounded, in any given dimension d, e.g., by Conjecture 1.2 above for $\varepsilon = 1$.

Remark 1.13 Recently, the first author, Cascini, Hacon and M^{c}Kernan [6] have announced a proof of existence of klt (hence $\mathbb{Q}$-factorial dlt) log flips, existence of minimal models for varieties of general type, and that the LMMP holds with respect to any divisor if one starts from a dlt log Fano variety. So, in many situations in this paper, one could remove the assumption of the LMMP. For example, if X is FT we can omit the LMMP.

Notation and terminology

In this paper, a log pair $(X/Z, B)$ consists of normal algebraic varieties X, Z over a base field k of characteristic 0, e.g, $k = \mathbb{C}$, where X/Z is proper, and an $\mathbb{R}$-boundary B (i.e., a divisor with multiplicities in $[0, 1]$) such that $K + B$ is $\mathbb{R}$-Cartier where K stands for the canonical divisor. Of course, some results hold or are expected over any field, e.g., ACC for a-lc thresholds holds in Corollary 1.11 (i). When Z is a point or $X \to Z$ is the identity we usually drop Z.

An *effective* log pair is a log pair $(X/Z, B)$ [5] such that $K + B \equiv M/Z$ for some $\mathbb{R}$-Cartier divisor $M \geq 0$. This property is preserved under any log flip or divisorial contraction.

A variety X is of *Fano type* (FT) if there is an $\mathbb{R}$-boundary B such that (X, B) is a klt weak log Fano, i.e. $-(K + B)$ is nef and big.

A property holds at a point $P \in X$ means that that property holds at the point P but not necessarily in a neighbourhood of P. On the other hand, a property holds near a subset $Z \subseteq X$ means that that property holds in an open neighbourhood of Z.

If (X, B) is lc, then

$$1 - \mathrm{mld}(P, X, B) = \max\{\mathrm{mult}\, E \text{ in } B_W \mid f(E) = \overline{P}\}$$

where E runs through the prime divisors on W for any log resolution $f \colon W \to X$ with a nonempty set of such E, $K_W + B_W = f^*(K + B)$, mult stands for the multiplicity function on divisors, and $\overline{P}$ is the closure of P in X. We define

$$\mathrm{ldis}(X, B) = \min\{\mathrm{mld}(P, X, B) \mid P \in X \text{ is of codimension} \geq 2\}$$

which obviously satisfies $\mathrm{ldis}(X,B) \le 1$ if $\dim X \ge 2$, and $\mathrm{ldis}(X,B) = +\infty$ if $\dim X \le 1$ since there is no such resolution in this case.

For $a \ge 0$, we say that (X,B) is a-lc at $P \in X$ if $\mathrm{mld}(P,X,B) \ge a$. This implies, in particular, that (X,B) is lc near P [25, Corollary 1.5]. We say that (X,B) is a-lc if it is a-lc at any $P \in X$, and we say that (X,B) is a-lc in codimension ≥ 2 if the same holds for all P of codimension ≥ 2.

For a set $\Gamma \subseteq \mathbb{R}$ and an $\mathbb{R}$-divisor D, the inclusion $D \in \Gamma$ means that the nonzero multiplicities of D are in Γ. In particular, the zero divisor is always in Γ.

2 ACC for mld's and thresholds

For $\mathbb{R}$-divisors on X, we have the well-known *order*: $D_1 \ge D_2$ if $D_1 - D_2 \ge 0$. On the other hand, the topology and the following natural norm, the *maximal absolute value norm*, are also well-known: if $D = \sum d_i D_i$, where $d_i \in \mathbb{R}$, and D_i are distinct prime divisors on X, set $\parallel D \parallel = \max\{|d_i|\}$. In particular, limits of *divisors* are limits with respect to this norm.

Main Proposition 2.1 *We assume the* ACC *for mld's in dimension d. Let $\Gamma \subset [0,1]$ be a DCC set, and $a > 0$ be a real number. Then, there exists a real number $\tau > 0$ (depending on d, Γ, and a) satisfying the following* upper approximation property*: if (X,B) and (X,B') are two log pairs with a point $P \in X$ (not necessarily closed) such that*

(1) $\dim X = d$*;*

(2) $B \le B'$, $\parallel B - B' \parallel < \tau$, $B' \in \Gamma$; and

(3) $\mathrm{mld}(P,X,B) \ge a$*, that is, (X,B) is a-lc at P; and*

(4) (X,B') is lc in a neighborhood of P;

then $\mathrm{mld}(P,X,B') \ge a$ *and so (X,B') is also a-lc at P.*

To prove the proposition we need the following general fact.

Lemma 2.2 (Continuity) *Suppose that the pairs (X,B) and (X,B') are lc in a neighborhood of a point $P \in X$ where $B \le B'$. Then, $a' =$* $\mathrm{mld}(P,X,B')$ *and $a =$* $\mathrm{mld}(P,X,B)$ *are real numbers ≥ 0, and, for any real number x in the interval $[a',a]$ there exist two real numbers $\alpha, \beta \ge 0$ such that $\alpha + \beta = 1$, and* $\mathrm{mld}(P,X,\alpha B + \beta B') = x$.

Proof This follows from the definition of mld and the linear properties of the mult function (see Notation and terminology above). □

Proof (of Main Proposition 2.1) Suppose that the proposition does not hold. Then, there exists a sequence of positive real numbers $\tau_1 > \tau_2 > \dots$ with $\lim_{i \to +\infty} \tau_i = 0$, and a sequence of d-dimensional log pairs $(X_i, B_i), i = 1, 2, \dots$, such that the proposition does not hold for τ_i on (X_i, B_i) at a point $P_i \in X_i$. In other words, there exists $B'_i \in \Gamma$ on X_i satisfying (2-4) of 2.1 with $\tau = \tau_i$, and

$$\mathrm{mld}(P_i, X_i, B_i) \geq a \quad \text{but} \quad \mathrm{mld}(P_i, X_i, B'_i) < a$$

We now construct a new sequence of d-dimensional log pairs (T_i, A_i) and points $Q_i \in T_i$ such that $a_i = \mathrm{mld}(Q_i, T_i, A_i) < a$ is strictly increasing with i and such that Ω, the set of multiplicities of all boundaries A_i, satisfies the DCC.

By ACC for mld's the set $\{a'_i = \mathrm{mld}(P_i, X_i, B'_i)\}$ has a maximum which is less than a. We can assume that this maximum is equal to $\mathrm{mld}(P_1, X_1, B'_1)$. Put $(T_1, A_1) = (X_1, B'_1)$, $Q_1 = P_1$, and let $a_1 = \mathrm{mld}(Q_1, T_1, A_1)$. Note that a_1 is a real number ≥ 0 by (4).

Suppose that we have already constructed (T_j, A_j) for $1 \leq j \leq i-1$. Since Γ satisfies the DCC, we can choose τ_k such that there are no multiplicities of A_j, for $1 \leq j \leq i-1$, in $(r - \tau_k, r)$ for any $r \in \Gamma$. Now by Lemma 2.2 we can choose $(T_i, A_i) := (X_k, \alpha B_k + \beta B'_k)$, for some $\alpha, \beta > 0$ with $\alpha + \beta = 1$, and $Q_i = P_k$ such that

$$\frac{a_{i-1} + a}{2} < a_i = \mathrm{mld}(Q_i, T_i, A_i) < a.$$

Also by construction for every real number $\varepsilon > 0$, almost all (except for finitely many) multiplicities of Ω belong to intervals $(r - \varepsilon, r]$ where $r \in \Gamma$. This implies that Ω satisfies the DCC because so does Γ. On the other hand, the set of mld's $\{a_i\}$ does not satisfy the ACC which contradicts the ACC for mld's. □

Proposition 2.3 *Assume the ACC for mld's in dimension d and let $a > 0$ be a real number and $\Gamma \subset [0, 1]$ a DCC set. Moreover, let τ be as in the Main Proposition 2.1, and (X, B) and (X, B') log pairs satisfying (1-4) of 2.1. Let $Y \to X$ be an extremal divisorial contarction such that*

(1) $K_Y + B'_Y + (1-a)E$ is $\mathbb{R}$-Cartier,

where E is the exceptional reduced divisor (but not necessarily irreducible), and B'_Y is the birational transform of B' on X. Then $K_Y + B'_Y + (1-a)E$ is antinef/X.

Example 2.4 The typical situation where one can apply Proposition 2.3 is as follows (see also step 8 in the proof of Proposition 4.1). Let $(X_i, B_i), i = 1, 2, \dots,$ be a sequence of d-dimensional klt log pairs such that

a) $a_1 = \mathrm{ldis}(X_1, B_1) \geq \cdots \geq a_i = \mathrm{ldis}(X_i, B_i) \geq \cdots > 0$ with

b) $a = \lim_{i\to\infty} a_i > 0$; and

c) $B_1 \leq \cdots \leq B_i \leq \dots$ with

d) $B = \lim_{i\to\infty} B_i \in R$ where $R \subset [0,1]$ is a fixed finite set.

Conditions c) and d) mean that there exist prime divisors $D_{i,k}, k = 1, \dots, n$, on each X_i such that for every $i \geq 1$ we have

$$B_i = \sum_{k=1}^{n} b_{i,k} D_{i,k};$$

and, for every $k = 1, \dots, n$,

c') $b_{1,k} \leq \cdots \leq b_{i,k} \leq \dots$ with

d') $b_k = \lim_{i\to\infty} b_{i,k} \in R$

(cf. types in Definition 2.6 below) (Note that c) and d) use a slightly more general order and topology than the one introduced at the beginning of this section).

In particular, $B = \sum b_k D_{i,k}$ approximates B_i on X_i, that is, for any $\tau > 0$, $\| B_i - B \| < \tau$ for all $i \gg 0$ (divisors B on X_i have the same type $(b_1, \dots, b_n)$ in the sense of Definition 2.6 below; this is why we use this ambiguous notation). Thus if we take $R = \{b_k \mid k = 1, \dots, n\}$, for all $i \gg 0$ and for given $\tau > 0$, we satisfy all assumptions of Main Proposition 2.1 for any (X_i, B_i) and (X_i, B) with X_i, B_i, B, and P_i instead of X, B, B', and P respectively, except for the $\mathbb{R}$-Cartier property of $K_{X_i} + B$, and the lc property in (4). The $\mathbb{R}$-Cartier property will hold if for example X is $\mathbb{Q}$-factorial. Moreover, if $a < 1$, then each $a_i < 1$ for $i \gg 0$, and there exists an extremal divisorial contraction $Y_i \to X_i$ with an exceptional prime divisor E_i with centre equal to the closure $\overline{P_i}$ on X_i and $a_i = \mathrm{mld}(P_i, X_i, B_i) = a(E_i, X_i, B_i)$ [25, Theorem 3.1]. Y_i is $\mathbb{Q}$-factorial too [25, Theorem 3.1], and $K_{Y_i} + B_{Y_i} + (1-a)E_i$ is $\mathbb{R}$-Cartier where B_{Y_i} denotes the birational transform of B.

Thus in the $\mathbb{Q}$-factorial case and under (4), for all $i \gg 0$, (X_i, B) is a-lc at P_i, and $K_{Y_i} + B_{Y_i} + (1-a)E_i$ is antinef/X_i.

Proof (of Proposition 2.3) Suppose that $K_Y+B'_Y+(1-a)E$ is not antinef$/X$. Then by property (1) of the proposition and the extremal property, it is numerically positive$/X$. On the other hand,

$$K_Y + B'_Y + \sum(1 - a(E_i, X, B'))E_i \equiv 0/X,$$

where by Main Proposition 2.1 the discrepancy $a(E_i, X, B') \geq a$ for each prime component E_i of $E = \sum E_i$. Thus the $\mathbb{R}$-Cartier divisor

$$\sum(a-a(E_i, X, B'))E_i = (K_Y+B'_Y+\sum(1-a(E_i, X, B'))E_i)-(K_Y+B'_Y+(1-a)E)$$

is numerically negative$/X$. According to the negativity lemma [23, 1.1], the divisor is effective and $\neq 0$, that is, for each E_i we have $a - a(E_i, X, B') > 0$, a contradiction. □

The following result is the big chunk of (i) in our Main Theorem 1.8, and it gives another application of Main Proposition when the support of B is not universally bounded.

Proposition 2.5 ACC *for mld's and lc thresholds in dimension d implies* ACC *for a-lc thresholds in the same dimension for all $a \geq 0$, in particular, for canonical thresholds.*

Proof It is enough to verify the ACC for a-lc thresholds at points. Suppose that we have a monotonic increasing sequence t_i of d-dimensional a-lc thresholds, that is, there exists a sequence (X_i, B_i) of d-dimensional log pairs with boundaries $B_i \in \Gamma$, points $P_i \in X_i$, and $\mathbb{R}$-Cartier divisors $M_i \in S$ on X_i such that

(1) (X_i, B_i) is a-lc at P_i; and

(2) t_i is the a-lc threshold of M_i at P_i with respect to (X_i, B_i);

in particular, for each i

(3) $\mathrm{mld}(P_i, X_i, B_i + t_iM_i) = a$; or

(4) t_i is the lc threshold of M_i with respect to (X_i, B_i) in a neighborhood of P_i (see Remark 1.5 and Example 1.6).

If for infinitely many i, (4) holds, then we are done since we are assuming ACC for lc thresholds. Thus after taking a subsequence, we can assume (3) for all i. Note that by the lc property of $(X_i, B_i + t_iM_i)$ at P_i the limit $t = \lim_{i\to\infty} t_i$ exists because t_i are bounded from above: $t_i \leq \frac{1}{m_0}$ where $m_0 = \min\{0 < m \in S\}$. We can apply Main Proposition 2.1 to each $X =$

$X_i, B = B_i, B' = B_i + tM_i$, and $P = P_i$. Indeed, (1) of the proposition holds because $\dim X = d$. The assumption (2) of the proposition follows from construction, in particular, the multiplicities of B' are of the form $b_{i,j} + tm_{i,j}$ and satisfy the DCC as their components $b_{i,j}$ and $m_{i,j}$ do. The assumption (3) of the proposition $\mathrm{mld}(P, X, B) \geq a$ holds by (1) above. Finally, $K_X + B'$ is $\mathbb{R}$-Cartier because each M_i is $\mathbb{R}$-Cartier, and the assumption (4) of the proposition, that is, (X, B') is lc in a neighborhood of P, follows from ACC for lc thresholds. Indeed, if the lc property does not hold for $i \gg 0$, then we get an increasing set of lc thresholds $t_i' = \mathrm{lct}(M_i, X_i, B_i)$ near P_i for infinitely many i, such that $t_i \leq t_i' < t$. This contradicts ACC for lc thresholds. Therefore, by Main Proposition 2.1 and (3) above $t = t_i$ for all $i \gg 0$ and t_i stabilizes. □

The Main Proposition 2.1 also gives some relations between different ACC versions besides the ones for mld's and thresholds in the introduction. Now we recall some of them.

Definition 2.6 (cf. [15, Section 18].) (i) The *type order* $\mathcal{B}$ is a direct sum of $\mathbb{R}$ countably many times, that is, the set of sequences $(b_1, \ldots, b_n)$ with $b_i \in \mathbb{R}, n \geq 0$, and the following order: $(b_1, \ldots, b_m) < (b_1', \ldots, b_n')$ if either $n < m$ or $n = m$ and each $b_i \leq b_i'$ with at least one strict inequality. The *maximal element* is the empty sequence with $n = 0$.

(ii) A *type* of an $\mathbb{R}$-divisor $D = \sum d_i D_i$ on X, where D_i are distinct prime divisors on X, is the sequence $(d_1, \ldots, d_n)$ of its nonzero multiplicities (in any possible ordering). We usually do not think of D with a specific ordering of the prime components in mind, so D can have several types. Even one can add finitely many zeros.

(iii) (Cf. [15, Definition 18.3].) A log pair (X, B) has *maximal a-lc type* $(b_1, \ldots, b_n)$ *near* a closed subset $Z \subseteq X$ and respectively *at a point* $P \in X$ (not necessarily closed) if $B = \sum b_i D_i$ where $D_1, \ldots, D_n$ are prime divisors, in particular, B has type $(b_1, \ldots, b_n)$, and if (X, B) is a-lc near Z and respectively at P, but (X, B') is not a-lc near Z and respectively at P for any $\mathbb{R}$-divisor $B' = \sum b_i' D_i$ of type $(b_1', \ldots, b_n')$ such that $K + B'$ is $\mathbb{R}$-Cartier and $B' > B$ in any neighborhood of Z and respectively at P.

(iv) (Cf. [15, 18.15.1].) $\mathcal{S}_d(\mathrm{Fano})$ is the set of types $(b_1, \ldots, b_n)$ such that there is a nonsingular Fano variety X of dimension at most d and a boundary B of type $(b_1, \ldots, b_n)$ such that $K + B \equiv 0$. (See Example 2.9, (1) below.)

(v) (Cf. [15, 18.15.$\bar{1}$].) $\mathcal{S}_d(\mathrm{global})$ is the set of types $(b_1, \ldots, b_n)$ such that there is a proper normal variety X of dimension at most d and a boundary

B of type $(b_1, \dots, b_n)$ such that (X, B) is lc, and $K + B \equiv 0$. (See Example 2.9, (2) below.) We denote by $\mathcal{S}_d^0(\text{global})$ its subset with non-klt (X, B).

(vi) (Cf. [15, 18.15.2].) $\mathcal{S}_{a,d}(\text{local})$ is the set of types $(b_1, \dots, b_n)$ such that there is a pointed $\mathbb{Q}$-factorial variety $P \in X$ of dimension at most d, and prime divisors $D_1, \dots, D_n$ on X such that $B = \sum b_i D_i$ is a boundary, and (X, B) has maximal a-lc type $(b_1, \dots, b_n)$ at P (See Example 2.9, (3) below). We denote by $\mathcal{S}_{a,d}^0(\text{local})$ its subset with non-klt (X, B) near P.

(vii) (Cf. [15, 18.15.$\overline{2}$].) $\overline{\mathcal{S}}_{a,d}(\text{local})$ is the set of types $(b_1, \dots, b_n)$ such that there is a $\mathbb{Q}$-factorial variety X of dimension at most d, prime divisors $D_1, \dots, D_n$ on X, and a closed subset $Z \subseteq X$ such that $B = \sum b_i D_i$ is a boundary, and (X, B) has maximal a-lc type $(b_1, \dots, b_n)$ near Z. We denote by $\overline{\mathcal{S}}_{a,d}^0(\text{local})$ its subset when Z is a subset of $\text{LCS}(X, B)$, the non-klt locus of (X, B).

(viii) (Cf. [15, 18.15.1].) $\mathcal{S}_d(\text{Mori-Fano})$ is the set of types $(b_1, \dots, b_n)$ such that there is a Mori-Fano variety X of dimension at most d, and a boundary B of type $(b_1, \dots, b_n)$ such that (X, B) is lc, and $K + B \equiv 0$. We denote by $\mathcal{S}_d^0(\text{Mori-Fano})$ its subset with non-klt (X, B).

(ix) (Cf. [15, 18.15.1].) $\mathcal{S}_d(\text{Mori-Fano cn})$ is the set of types $(b_1, \dots, b_n)$ such that there is a cn Mori-Fano variety X of dimension at most d and a boundary B of type $(b_1, \dots, b_n)$, and $K + B \equiv 0$.

Some of these definitions are slightly different from those in [15, 18.15]. We consider each of the above sets $\mathcal{S}_d(\text{Fano}), \dots, \mathcal{S}_d(\text{Mori-Fano cn})$ as a subset of the order $\mathcal{B}$. Thus it has ordering induced from $\mathcal{B}$. For $a = 0$, we set $\mathcal{S}_d = \mathcal{S}_{a,d}$, e.g., $\overline{\mathcal{S}}_d(\text{local}) = \overline{\mathcal{S}}_{0,d}(\text{local})$. Finally, we denote by $\mathcal{S}_d(\Gamma, \text{local})$ the types in $\mathcal{S}_d(\text{local})$ when $B \in \Gamma$. Similar notation for the other sets.

Conjecture 2.7 (cf. [15, Conjecture 18.16]) *Each set* $\mathcal{S}_d(\text{global})$, $\mathcal{S}_d^0(\text{global})$, $\mathcal{S}_{a,d}(\text{local})$, $\overline{\mathcal{S}}_{a,d}(\text{local})$, $\mathcal{S}_{a,d}^0(\text{local})$, $\overline{\mathcal{S}}_{a,d}^0(\text{local})$, $\mathcal{S}_d(\text{Mori} - \text{Fano})$, $\mathcal{S}_d^0(\text{Mori} - \text{Fano})$, $\mathcal{S}_d(\text{Mori} - \text{Fano cn})$, *satisfies the ACC.*

Remark 2.8 (1) In Definition 2.6, (vi-vii) the $\mathbb{Q}$-factorial assumption can be replaced by the $\mathbb{Q}$-Cartier property of the prime divisors D_i (cf. Example 2.9, (3) below).

(2) Since we omit the lc assumption for (X, B) in Definition 2.6, (ix), $S_d(\text{Mori-Fano cn})$ is not a subset of $S_d(\text{Mori-Fano})$.

(3) ACC for $S_d(\text{Mori-Fano cn})$ follows from Conjecture 1.2 in dimension d (cf. Example 2.9, (1) below).

Example 2.9 (1) $\mathcal{S}_d$(Fano) satisfies the ACC in any dimension d by the boundedness of nonsingular Fano varieties X of dimension $\leq d$ [16]. Indeed, there exists a generic curve $C \subset X$ which positively intersects each prime divisor D_i on X and with bounded $(-K \cdot C)$: C is a generic curve section for an embedding $X \subset \mathbb{P}^N$ of bounded degree in a fixed projective space $\mathbb{P}^N$. Then for positive integers $m_i = (D_i \cdot C)$, $\sum b_i m_i = (B \cdot C) = (-K \cdot C)$. On the other hand, for any increasing sequence of types $(b_1^l, \ldots, b_{n_l}^l), l = 1, 2, \ldots,$ we can suppose that their sizes stabilize: $n_l = n$ for all $l \gg 0$, and each $b_i^l > 0$. Therefore, after taking a subsequence, the multiplicities $m_i^l = (D_i^l \cdot C)$ stabilize too: for each $i = 1, \ldots, n$, $m_i^l = m_i > 0$ for all $l \gg 0$. Hence the types stabilize: for each $i = 1, \ldots, n$, $b_i^l = b_i > 0$ for all $l \gg 0$ (cf. the proof of [23, Second Termination 4.9]). These arguments show that the ACC holds for $\mathcal{S}_d$(Fano) even if we remove the assumption $b_i \leq 1$.

On the other hand, the condition on the dimension is necessary for all the sets in Conjecture 2.7. For example, let $X = \mathbb{P}^d$ be the projective space of dimension d, and D a generic hypersurface in $\mathbb{P}^d$ of degree $d + 2$. Then

$$K_{\mathbb{P}^d} + \frac{d+1}{d+2} D \equiv 0.$$

Thus $((d + 1)/(d + 2)) \in \mathcal{S}_d$(Fano) and obviously $\bigcup_{d \geq 1} S_d$(Fano) does not satisfy the ACC.

(2) For $\mathcal{S}_d$(global) and $\mathcal{S}_d$(Mori-Fano), the assumption that (X, B) is lc is very important. Let $Q_n \subset \mathbb{P}^{(n+1)}$ be the cone over a rational normal curve of degree n with a line generator L. Then for a generic hyperplane section H,

$$-K = (n + 2)L \equiv 3L + \frac{n-1}{n} H.$$

If we replace $3L$ by $L_1 + L_2 + L_3$ or $L_1/2 + \cdots + L_6/2$ with distinct generators L_i, we construct strictly increasing sequences of types $(1, 1, 1, (n-1)/n)$ and $(1/2, 1/2, 1/2, 1/2, 1/2, 1/2, (n-1)/n)$ respectively. However, they are not in $\mathcal{S}_d$(global) because $(Q_n, L_1 + L_2 + L_3 + (n - 1)H/n)$ and $(Q_n, L_1/2 + \cdots + L_6/2 + (n - 1)H/n)$ are not lc (at the vertex of Q_n).

(3) The $\mathbb{Q}$-factorial property in Definition 2.6, (vi-vii) is very important, too. Let $f\colon Y \to X$ be a contraction of a nonsingular rational curve C on a nonsingular 3-fold Y, and D_1, D_2 two nonsingular prime divisors on Y with intersection only along C and with normal crossings. Set $-n = C^2$ on D_1. For any $n \geq 2$ there exists such a contraction, e.g., toric one. Then $K + B \equiv 0/X$ for $B = D_1 + (n-2)D_2/n$. Thus we have a strictly increasing

sequence of types $(1, (n-2)/n)$ which does not belong (entirely) to any set in Conjecture 2.7 if it satisfies the ACC. That is in Definition 2.6, (v) the *proper* assumption is very important. The same types correspond to the image (X, f_*B). However it does not belong to the sets in Definition 2.6, (vi-vii) because X is not $\mathbb{Q}$-factorial.

Let D_3 be a divisor which transversally intersects D_1, D_2 in a single point (again such a divisor exists in the toric case). Then, for $B' = D_1 + D_3 + (n-1)/nD_2$, $K + B' \equiv 0/X$, and (X, f_*B') is exactly lc near $P = f(C)$: $\mathrm{ldis}(X, f_*B') = 0$, but $(n-1)/n$ does not satisfy the ACC and is not a counter example to ACC for thresholds since f_*D_2 is not a $\mathbb{Q}$-Cartier divisor. Similar examples can be constructed for any a instead of 0.

(4) The ACC for $\mathcal{S}_d$(Mori-Fano cn) holds for $d \le 3$ by [17]. Moreover, the boundary property of B, namely that each $b_i \le 1$, and the condition $\rho(X) = 1$ are not necessary because canonical Fano 3-folds are bounded [17].

The main result of this section is the following.

Theorem 2.10 ACC *for mld's and lc thresholds in dimension* $\le d$ *imply*

(i) the ACC for $\mathcal{S}_{a,d}(\mathrm{local}), \overline{\mathcal{S}}_{\mathrm{a,d}}(\mathrm{local}), \mathcal{S}^0_{\mathrm{a,d}}(\mathrm{local})$*, and* $\overline{\mathcal{S}}^0_{a,d}(\mathrm{local})$ *with any* $a \ge 0$*;*

The ACC for $\mathcal{S}_d(\mathrm{Mori-Fano\ cn})$*, the* LMMP *and* ACC *for mld's in dimension* $\le d$ *imply*

(ii) the ACC for $\mathcal{S}_d(\mathrm{global})$*, and* $\mathcal{S}_d(\mathrm{Mori-Fano})$*;*

(iii) the ACC for $\mathcal{S}^0_{d+1}(\mathrm{Mori-Fano})$*;*

(iv) the ACC for $\mathcal{S}_{d+1}(\mathrm{local})$ *and* $\overline{\mathcal{S}}_{d+1}(\mathrm{local})$*; and*

(v) the ACC *for lc thresholds in dimension* $\le d+1$*;*

If in addition, the LMMP *holds in dimension* $d+1$*, then*

(vi) the ACC for $\mathcal{S}^0_{d+1}(\mathrm{global})$ *holds.*

Addendum 2.11 *The ACC for* $\mathcal{S}_d(\mathrm{Mori-Fano\ cn})$ *can be replaced by Conjecture 1.2 in dimension* $\le d$ *(everywhere!) because the latter implies the former. (Cf. Example 2.9, (1) and Remark 2.8, (3) above.)*

Theorem 2.12 *Let* $\Gamma \subset [0,1]$ *be a DCC set. Then Theorem 2.10 holds when each* $B \in \Gamma$*. Moreover, then* Γ *can be assumed to be finite, that is, there exists a finite subset* Γ_f *such that* $\mathcal{S}_{a,d}(\Gamma, \mathrm{local}) = \mathcal{S}_{a,d}(\Gamma_{\mathrm{f}}, \mathrm{local})$*,* $\mathcal{S}_d(\Gamma, \mathrm{global}) = \mathcal{S}_d(\Gamma_{\mathrm{f}}, \mathrm{global})$*, etc.*

We also have $\mathcal{S}_d(\Gamma,\ \mathrm{Fano}) = \mathcal{S}_d(\Gamma_f,\ \mathrm{Fano})$.

Corollary 2.13 (ACC for anticanonical (ac) thresholds) *Let* $\Gamma \subseteq [0,1]$ *be a DCC set. Assume the ACC for* $\mathcal{S}_d(\text{Mori}-\text{Fano cn})$*, the* LMMP *and* ACC *for mld's in dimension* $\le d$*. Then,*

(i) the ACC holds for $\mathcal{S}_{a,d}(\text{global FT})$ *where* $\mathcal{S}_{a,d}(\text{global FT})$ *is the set of types* $(b_1,\ldots,b_n)$ *such that there is a FT variety* X *of dimension at most* d*, an ample Cartier divisor* H *on* X*, and a boundary* B *of type* $(b_1,\ldots,b_n)$ *such that* (X,B) *is lc, and* $K+B+aH\equiv 0$*; in particular,* $\mathcal{S}_{0,d}(\text{global FT}) = \mathcal{S}_{\mathrm{d}}(\text{global FT})$ *the subset in* $\mathcal{S}_d(\text{global})$ *corresponding to FT varieties* X*;*

Moreover, there exists a finite subset Γ_f *such that* $\mathcal{S}_{a,d}(\Gamma,\text{global FT}) = \mathcal{S}_{a,d}(\Gamma_{\mathrm{f}},\text{global FT})$*.*

(ii) let S *be a DCC set of positive real numbers, then the ACC holds for the subset* $\overline{\mathcal{A}}_{a,d}(\Gamma,S)$ *of* $\mathbb{R}^+=\{r\in\mathbb{R}\mid r\ge 0\}$ *defined by*

$$\overline{\mathcal{A}}_{a,d}(\Gamma,S)=\{\mathrm{act}_a(M,X,B)|X \textit{ is proper, } \dim X=d,\ B\in\Gamma,\ (X,B) \textit{ is lc},$$

$$X \textit{ is FT, } K+B \textit{ is antinef, and } M \textit{ is an ample } S\textit{-Cartier divisor on } X\}.$$

where $t=\mathrm{act}_a(M,X,B)$ *means that* $K+B+tM+aH\equiv 0$ *for some ample Cartier divisor* H *on* X *and* $t\ge 0$*, and the* S*-Cartier property means that* M *is a linear combination of ample Cartier divisors with multiplicities in* S*.*

(iii) The log Fano indices, that is, the maximal positive real number t *such that* $K+B+tM\equiv 0$ *for some ample Cartier divisor* M*, satisfies the ACC for the lc pairs* (X,B)*, with FT variety* X *of dimension* $\le d$ *and* $B\in\Gamma$*.*

We expect that Corollary 2.13 holds when FT is omitted, that is, for $a>0$, (X,B) is just a lc Fano variety.

Remark 2.14 If $\rho(X)=1$, then the *a-anticanonical* (*a-ac*) *threshold* is well defined for any fixed ample Cartier divisor H if $K+B+aH$ is antinef. In this case, there exists a (unique) real number t such that

$$K+B+tM+aH\equiv 0.$$

where M is an ample S-Cartier divisor and S is a set of positive real numbers. For $a=0$, we get the ac threshold [12, p. 47].

In Corollary 2.13 we can suppose that a is varying in a DCC set. Then it is expected that the corresponding thresholds t in dimension $\le d$ satisfies the ACC. This is clear from the proof of Corollary 2.13 (below).

Proof (of Corollary 2.13) The case (i) follows from its counterpart in Theorem 2.10 (ii). To apply the theorem we replace the boundary B with $B+aH$ with an appropriate choice of H (see proof of (ii) below). The type of B will be extended by that of aH. Since the latter has finitely many possible multiplicities, the ACC for B is equivalent to the ACC of extended types to which we apply Theorem 2.10. When $B \in \Gamma$, the existence of Γ_f follows similarly from Theorem 2.12.

(ii) Suppose that such thresholds do not satisfy the ACC. Let Ω be an infinite set of such thresholds, which satisfies the DCC. Now take any $t \in \Omega$. Also take X, B, H and $M \neq 0$ corresponding to t. Since M is S-Cartier, there are $s_j \in S$ and ample Cartier divisors H_j such that $M = \sum_j s_j H_j$. By anticanonical boundedness [24], ts_j is bounded from above (see also Example 2.9, (1)). By effective base point freeness [14], there is h, a natural number (not depending on X, H, H_j but depending only on the dimension d), such that hH_j and hH are free divisors and $(a/h), (ts_j/h) \in [0,1]$.

Write $M \equiv \sum_j (s_j/h)H'_j$ where $H'_j \in |hH_j|$ is a general member. For $d \geq 2$, a general member H'_j is irreducible. For $d = 1$, the number of components in H'_j is bounded. Thus we can assume that

$$(X, B + \sum_j (ts_j/h)H'_j + (a/h)H')$$

is lc where $H' \in |hH|$ is a general member. In particular, the possible multiplicities of $B + \sum_j (ts_j/h)H'_j + (a/h)H'$ satisfy the DCC. Now Ω is finite by Theorem 2.12, (ii) a contradiction.

(iii) If $S = \{1\}$ and $a = 0$, then any ample S-Cartier divisor M is just an ample Cartier divisor and the ac threshold satisfies $K + B + tM \equiv 0$. Thus the possible t satisfy the ACC by (ii). This implies the ACC for the Fano indices. □

Proof (of Theorem 2.10) We will see that each statement follows from the same statement under the assumption $B \in \Gamma$ for some Γ under the DCC. Thus Theorem 2.10 follows from Theorem 2.12.

For (v) such a set Γ is given by assumptions. In the other cases, we need to verify that each increasing sequence of types $(b^l_1, \ldots, b^l_{n_l}), l = 1, 2, \ldots,$ stabilizes. By definition of the ordering, their sizes stabilize: $n_l = n$ for all $l \gg 0$. Then for the corresponding pairs (X_l, B_l), $B_l = \sum_{i=1}^{n_l} b^l_i D^l_i \in \Gamma$ where

$$\Gamma = \{b^l_i \mid i = 1, \ldots, n, \text{ and } l = 1, 2, \ldots\}.$$

Since each sequence $b_i^l, l = 1, 2, \dots,$ increases, Γ satisfies the DCC.

Proof (of Addendum 2.11) Use the boundedness as in Example 2.9, (1). $\square$

Proof (of Theorem 2.12) (i) By the inclusions

$$\mathcal{S}^0_{a,d}(\Gamma, \text{local}) \subset \mathcal{S}_{a,d}(\Gamma, \text{local}) \text{ and } \overline{\mathcal{S}}^0_{a,d}(\Gamma, \text{local}) \subset \overline{\mathcal{S}}_{a,d}(\Gamma, \text{local})$$

it is enough to prove the ACC for the ambient sets. On the other hand, by definition, for any type $(b_1, \dots, b_n)$ in $\overline{\mathcal{S}}_{a,d}(\Gamma, \text{local})$ and any of its components b_i there exists a type $(b'_1, \dots, b'_m)$ in $\mathcal{S}_{a,d}(\Gamma, \text{local})$ with a component $b'_{i'} = b_i$: take a point $P \in \operatorname{Supp} D_i \cap Z$ such that (X, B) has maximal a-lc type at P and then take the type given by those components of B which contain P. Thus it is enough to prove the ACC for $\mathcal{S}_{a,d}(\Gamma, \text{local})$.

Let $(b_1^l, \dots, b_{n_l}^l), l = 1, 2, \dots,$ be an increasing sequence of types in the set $\mathcal{S}_{a,d}(\Gamma, \text{local})$ with $n_l \geq 1$. Since this sequence is increasing, we can assume that there is $n \geq 1$ such that $n_l = n$ for any l. By definition, we have a sequence of pointed $\mathbb{Q}$-factorial varieties $P_l \in X_l$ of dimension $\leq d$ and prime divisors $D_1^l, \dots, D_n^l$ on X_l such that $B_l = \sum b_i^l D_i^l$ is a boundary and $P_l \in \bigcap_{i=1}^n \operatorname{Supp} D_i^l$. Moreover, for each l, $\operatorname{mld}(P_l, X_l, B_l) = a$ or every D_i^l contains a lc centre of (X_l, B_l) passing through P_l. If the latter case happens for every l, then $(b_1^l, \dots, b_n^l)$ satisfies the ACC by ACC for lc thresholds in dimension $\leq d$ (see the arguments below). Thus taking a subsequence we can suppose that the first case happens for every l, i.e. $\operatorname{mld}(P_l, X_l, B_l) = a$, and $a > 0$.

We can choose a subsequence such that the limits below exist (e.g., unique)(see Example 2.4 above)

$$b_i = \lim_{l \to \infty} b_i^l \text{ for } i = 1, \dots, n, \quad B'_l = \sum_{i=1}^n b_i D_i^l, \text{ and } R := \{b_i \mid i = 1, \dots, n\}.$$

Then for any $\tau > 0$, $\| B_l - B'_l \| < \tau$ for all $l \gg 0$. Note that $K_{X_l} + B'_l$ is $\mathbb{R}$-Cartier because X_l is $\mathbb{Q}$-factorial. By ACC for lc thresholds and Main Proposition 2.1 for $X = X_l, B = B_l, P = P_l$, and every $l \gg 0$, we can assume that (X_l, B'_l) is lc near P_l, and a-lc at P_l. Therefore, $\operatorname{mld}(P_l, X_l, B'_l) \geq a = \operatorname{mld}(P_l, X_l, B_l)$.

We can derive the lc property, (4) of Main Proposition 2.1, of (X_l, B'_l) from the assumptions as follows (cf. proof of Proposition 2.5). If (X_l, B'_l) is not lc near P_l for $l \gg 0$, then (since X_l is $\mathbb{Q}$-factorial), for infinitely many l there is $G_l = \sum_{i=1}^{n} g_i^l D_i^l$ such that $B_l \leq G_l \leq B_l'$ and such that (X_l, G_l) has maximal 0-lc type $(g_1^l, \ldots, g_n^l)$ at P_l. The set of multiplicities of those G_l satisfies the DCC and is not finite. We can assume that $\{g_1^l\}$ is not finite but increasing and that D_1^l contains a lc centre. So, g_1^l is the lc threshold of D_1^l with respect to $(X_l, G_l - g_1^l D_1^l)$. This contradicts ACC for lc thresholds.

Now by Monotonicity of mld's (see [23, 1.3.3]) and since $B_l' \geq B_l$, the sequence stabilizes: $B_l' = B_l$ for every $l \gg 0$. This proves the ACC.

(ii) This will be established in the weak finiteness section (Proposition 4.1) modulo (v) in dimension d which can be assumed by induction. Note that $\mathcal{S}_d$(global) includes $\mathcal{S}_d$(Mori-Fano).

(iii) Let (X_l, B_l) be a pair of dimension $\leq d+1$ for each l such that $B_l = \sum_{i=1}^{n_l} b_i^l D_i^l$ has a type $(b_1^l, \ldots, b_{n_l}^l) \in \mathcal{S}_{d+1}^0(\Gamma, \text{Mori-Fano})$ such that these types are strictly increasing with respect to l. We can assume that $\{b_1^l\}$ is a strictly increasing sequence. By assumptions, (X_l, B_l) is lc but not klt. We can take a $\mathbb{Q}$-factorial dlt blow up (Y_l, B_{Y_l}) for (X_l, B_l); this only needs special termination of $\mathbb{Q}$-factorial dlt log flips in dimesion $\leq d+1$ which follows from the LMMP in dimension $\leq d$ [6][5, Construction 3.1]. Here we have a birational morphism $Y_l \to X_l$ and $K_{Y_l} + B_{Y_l}$ is the crepant pullback of $K + B_l$.

Suppose that D_1^l intersects $\mathrm{LCS}(X_l, B_l)$ for infinitely many l. For each such l the birational transform of D_1^l intersects $\lfloor B_{Y_l} \rfloor$ becuase all the exceptional divisors of $Y_l \to X_l$ are components of $\lfloor B_{Y_l} \rfloor$. Then using adjunction, restrict to a component T_l of $\lfloor B_{Y_l} \rfloor$ which intersects D_1^l. The boundary multiplicities that we get on T_l are of the following type

$$b' = \frac{m-1}{m} + \sum \frac{c_i}{m} b_i^l \leq 1$$

with natural numbers m, c_i [23, 3.10]. It is easy to verify the following fact.

Lemma 2.15 *Any set of such b' satisfies the DCC where $b_i^l \in \Gamma$ (cf. [23, Second termination 4.9] and [15, 18.21.4]). Moreover, if it is finite, then the corresponding set of b_i^l is finite.*

By Theorem 2.12 in dimension d we have that the set of all possible b' is finite. By Lemma 2.15, the set of b_1^l is finite. This is a contradiction. Therefore, we can assume that D_1^l does not intersect $\mathrm{LCS}(X_l, B_l)$ for any l.

There is an extremal ray R_l on Y_l such that the birational transform of D_1^l intersects R_l positively. If R_l is of fibre type, then by restricting to the general fibre and using induction on d we get a contradiction. So assume otherwise. The reduced part of B_{Y_l} intersects R_l, otherwise R_l corresponds to a flipping or divisorial type extremal ray R'_l on X_l. This is not possible since $\rho(X_l) = 1$.

Let $(Y_l^+, B_{Y_l}^+)$ be the model after operating on R_l (i.e., after a flip or divisorial contraction). Thus, the birational transform of D_1^l on Y_l^+ intersects the reduced part of $B_{Y_l}^+$. We get a contradiction as above by restricting to a component of the reduced part of $B_{Y_l}^+$.

(iv) As noted above it is enough to verify the ACC for $\mathcal{S}_{d+1}(\Gamma, \text{local})$. Suppose that there are (X_l, B_l) such that $B_l = \sum_{i=1}^{n_l} b_i^l D_i^l$ has a maximal 0-lc type $(b_1^l, \dots, b_{n_l}^l)$ in $\mathcal{S}_{d+1}(\Gamma, \text{local})$ at $P_l \in X_l$ such that these types are strictly increasing with respect to l. We can assume that the set $\{b_1^l\}$ is strictly increasing. If for infinitely many l, D_1^l contains a lc centre of (X_l, B_l) of codimension $\leq d$ (i.e. a lc centre which is not a closed point) passing through P_l, then by taking hyperplane sections, we reduce the problem to dimension $\leq d$ for which we may assume that the theorem is already proved.

So, we assume that none of D_1^l contains a lc centre of (X_l, B_l) of dimension ≥ 1 passing through P_l. Now, take a $\mathbb{Q}$-factorial dlt blow up of each (X_l, B_l). Then using adjunction, restrict to an appropriate exceptional divisor in the reduced part of the boundary which intersects the birational transform of D_1^l. The exceptional divisor is complete over P_l by the above property of lc centers. The multiplicities that we get are as in Lemma 2.15. We get a contradiction by (ii).

(v) This is proved exactly as in (iv) using induction on d (part (ii)).

(vi) Let (X_l, B_l) be a pair of dimension $d+1$ for each l such that $B_l = \sum_{i=1}^{n_l} b_i^l D_i^l$ has a type $(b_1^l, \dots, b_{n_l}^l) \in \mathcal{S}_{d+1}^0(\Gamma, \text{global})$ such that these types are strictly increasing with respect to l. We can assume that $\{b_1^l\}$ is a strictly increasing sequence. By assumptions, (X_l, B_l) is lc but not klt. By taking $\mathbb{Q}$-factorial dlt blow ups, we may assume that (X_l, B_l) is $\mathbb{Q}$-factorial dlt. As in step 3 of the proof of Proposition 4.1, run the anti-LMMP on D_1^l. After finitely many steps, either we get a fibration or the Mori-Fano case. For the former case we use induction on d and for the latter case use (iii).

The statement about Γ_f follows from Lemma 2.16 below. The abridged property for $S_{a,d}(\Gamma, \text{local})$, $S_{a,d}^0(\Gamma, \text{local})$, $\overline{S}_{a,d}(\Gamma, \text{local})$, $\overline{S}_{a,d}^0(\Gamma, \text{local})$, $\mathcal{S}_{d+1}(\text{local})$, and $\overline{\mathcal{S}}_{d+1}(\text{local})$ follows from [15, Theorem 18.22]. For the Mori-Fano varieties we can use Theorem 3.5. The global case is proved in Proposition 4.1 or we can reduce it to Mori-Fano varieties by the LMMP. $\square$

Lemma 2.16 *Any suborder $\mathcal{S} \subset \mathcal{B}$ satisfies the ACC if each $(b_1, \ldots, b_n) \in \mathcal{S}$ is in R, that is, each $b_i \in R$, for a fixed finite set of real numbers R. Conversly, if $\mathcal{S}$ satisfies the ACC and there is a DCC set Γ such that any $(b_1, \ldots, b_n) \in \mathcal{S}$ is in Γ, and for each $(b_1, \ldots, b_n) \in \mathcal{S}$ some abridged type $(b'_1, \ldots, b'_{n'})$ with bounded n' is in $\mathcal{S}$, then there is a finite set R such that each $(b_1, \ldots, b_n) \in \mathcal{S}$ is in R. Abridged means that both types have the same set of nonzero components.*

Proof First suppose that each $(b_1, \ldots, b_n) \in \mathcal{S}$ is in R for a fixed finite set of real numbers R. If $\mathcal{S}$ does not satisfy the ACC, then we can find a strictly increasing set of elements $\beta_1, \beta_2, \ldots$ in $\mathcal{S}$. We can assume that they all have the same size, that is, there is n such that $\beta_l = (b_1^l, \ldots, b_n^l)$. Since R is finite, there are only finitely many such types, a contradiction.

Now suppose that we have $\mathcal{S}$ satisfying the ACC and the other assumptions of the lemma. Let $R \subset \Gamma$ be the set of all real numbers appearing as a component in some type in $\mathcal{S}$. It is enough to prove that R is finite. If R is not finite, then there is a strictly increasing sequence $\{r_l\}_{l\in\mathbb{N}} \subset R$ and an infinite set of types $\beta_1, \beta_2, \ldots$ in $\mathcal{S}$ such that r_l is a component of β_l. Replacing each β_l by an abridged one $\beta_l = (b_1^l, \ldots, b_{n'}^l)$, we can assume that $b_1^l = r_l$. If $n' = 1$, then we get a contradiction. Otherwise, consider types $\lambda_l = (b_2^l, \ldots, b_{n'}^l)$ and use induction on size and the DCC property of Γ to get an infinite increasing subsequence of $\{\beta_l\}_{l\in\mathbb{N}}$. By construction it is strictly increasing. This is a contradiction, because the set $\{\beta_l\}_{l\in\mathbb{N}}$ does not satisfy the ACC. $\square$

3 Log twist

In this section, we introduce a construction which is crucial for us and which generalizes (resembles) Sarkisov links of Type I and II [18, Theorem 13-1-1], and we establish its basic properties.

Construction 3.1 (Log twist) Assume the LMMP in dimension d. Let X be a d-dimensional Mori-Fano variety, and B be a boundary such that (X, B) is klt and noncanonical in codimension ≥ 2 (noncn for short), and $K + B \equiv 0$. Fix a prime exceptional divisor E such that $a := 1 - e := \mathrm{ldis}(X, B) = a(E, X, B)$. Then there exists (and is unique for the fixed E)

the following transformation of X which we call a *log twist*:

$$\begin{array}{ccccccc} Y = Y_1 & \dashrightarrow & Y_2 & \dashrightarrow & \cdots & \dashrightarrow & Y' = Y_n \\ \downarrow{\scriptstyle f} & & & & & & \downarrow{\scriptstyle f'} \\ X & & & & & & X' \end{array}$$

where $f\colon Y = Y_1 \to X$ is an extremal divisorial contraction with E being its exceptional divisor, and all horizontal modifications $Y_i \dashrightarrow Y_{i+1}, i = 1, \ldots, n-1$, are extremal $-E$-flips such that for the crepant boundary $B_{Y'}$

(1) $(Y', B_{Y'})$ is klt, and $K_{Y'} + B_{Y'} \equiv 0$.

Moreover, $f'\colon Y' = Y_n \to X'$ is either a Mori-Fano fibration with $\dim X' \geq 1$ or an extremal divisorial contraction of a divisor E' onto a Mori-Fano variety X' with the crepant boundary $B_{X'}$ such that

(2) $(X', B_{X'})$ is klt, and $K_{X'} + B_{X'} \equiv 0$.

In addition, the following two facts hold:

(3) If D is an effective divisor on Y which is antinef over X then its birational transform D' on Y' is nef over X', and strictly positive when $D \neq 0$.

(4) Thus, if in (3) D' is also antinef over X', then $D = D' = 0$.

Definition 3.2 (cf. [18, Theorem 13-1-1]) *We say that the twist has Type I if $Y' \to X'$ is a fibration. Otherwise the twist has Type II.*

Proof (of Construction 3.1) There is an extremal divisorial contraction $f\colon Y = Y_1 \to X$ with E being its exceptional divisor by the LMMP assumption or by [6, Corollary 1.4.3]. Let B_Y be the crepant pullback of B on Y. According to our assumptions, B_Y is a boundary. Moreover,

(5) $a < 1$ and $0 < e < 1$; and

(6) $\mathrm{ldis}(Y, B_Y) \geq \mathrm{ldis}(X, B) = a$.

By construction $K_Y + B_Y \equiv 0$, and Y is $\mathbb{Q}$-factorial. Now we run the LMMP starting from Y with respect to $K_Y + B_Y - eE \equiv -eE$. By (5) this is the same as D-MMP with respect to $D = -E$. Instead of using the LMMP assumption we could use [6, Corollary 1.3.1]. Since E is always positive on the generic member of some covering family of curves, after finitely many flips $Y_i \dashrightarrow Y_{i+1}$, we get an extremal contraction $f'\colon Y' = Y_n \to X'$ which is not a flipping, that is, f' is a Mori-Fano fibration or a divisorial contraction, contracting E'. The first case gives a twist of Type I, and the second one gives a Type II twist.

In both cases, E is positive with respect to f', and so is E with respect to the flipping contraction of each flip $Y_i \dashrightarrow Y_{i+1}$. In particular, E is a divisor on X' if f' has Type II. In both cases, the flips are log flops with respect to $K_{Y_i} + B_{Y_i}$, and all B_{Y_i} are (crepant) boundaries. Thus, both Type I and Type II twists satisfy property (1), and in addition, the Type II also satisfies (2). By (6) in both cases,

(6') $\operatorname{ldis}(Y', B_{Y'}) \geq \operatorname{ldis}(X, B) = a$.

By construction, $\rho(X') = \rho(Y') - 1 = \rho(Y_i) - 1 = \rho(Y) - 1 = \rho(X) = 1$, and, for Type II, X' is $\mathbb{Q}$-factorial. Hence, for this case, since E is not exceptional over X' and by (5), $-K_{X'}$ is ample which means that X' is a Mori-Fano variety.

Now let D be an effective divisor on Y which is antinef/X. According to the previous paragraph, each $\rho(Y_i) = 2$. Let R_1 be the extremal ray corresponding to the contraction $Y \to X$, and R_2 be the other extremal ray. By our assumption, $D \cdot R_1 \leq 0$. Since $D \geq 0$, $D \cdot R_2 \geq 0$. Thus the first flip $Y_1 \dashrightarrow Y_2$ is a $-D$-flip or $-D$-flop. Similarly, each next flip $Y_i \dashrightarrow Y_{i+1}$ is a $-D_i$-flip or $-D_i$-flop where D_i denotes the birational transform of D. Therefore, $D' := D_n$ is nef/X'. If D' is in addition antinef over X', then $D' \equiv 0/X'$ which implies that D' is antinef hence $D' = 0$. So, we proved (3) and (4). Uniqueness of the log twist follows from the construction. $\square$

Definition 3.3 A log twist is called *final*, if

(a) $Y' \to X'$ is a fibration, that is, it is of Type I; or

(b) $Y' \to X'$ is of Type II, X' is noncn, and $\operatorname{ldis}(X', B_{X'}) = 1 - e'$ where $e' = \operatorname{mult}_{E'} B_{Y'}$; or

(c) $Y' \to X'$ is of Type II, and X' is canonical (cn for short).

Therefore, if a log twist is not final, it is of Type II with noncn X'. Thus we can take a log twist of $(X', B_{X'})$. Moreover, we expect that a sequence of log twists:

$$(7) \qquad (X, B) \dashrightarrow (X', B_{X'}) \dashrightarrow \cdots \dashrightarrow (X^{(i)}, B_{X^{(i)}}) \dashrightarrow \ldots$$

terminates, where each log twist is nonfinal, except possibly for the last one.

Proposition 3.4 (Termination of log twists) *(i) Suppose that for a sequence as in (7), there exists a real number $a_0 < 1$ such that*

$$\text{(UBD)} \qquad a^{(i)} = \operatorname{ldis}(X^{(i)}, B_{X^{(i)}}) \leq a_0$$

for each $i \geq 1$ except possibly for the last i. Then, assuming the LMMP *in dimension $d = \dim X$, the sequence terminates and* universally *with respect to a_0, that is, the sequence is finite and the number of twists in it is bounded whereas the bound depends only on a_0 and d.*

(ii) The ACC for mld's near 1 with $\Gamma = \{0\}$ in dimension d implies (UBD) *for any sequence as in (7), for some $a_0 < 1$ where a_0 depends only on d.*

By ACC for mld's in dimension d near 1 with $\Gamma = \{0\}$, we mean that 1 is not an upper limit in the mld spectrum (1.3) in dimension d when $B = 0$. This is a very special case of Conjecture 1.3.

Theorem 3.5 *Let $(X/Z, B = \sum b_i D_i)$ be a log pair of dimension d such that:*

a) $X \to Z$ is a proper contraction;
b) (X, B) is lc; and
c) $K + B$ is antinef/Z.
Then the LMMP in dimension d implies:

$$\rho^W(X/Z) \geq -d + \sum b_i$$

where ρ^W is the Weil number, that is, the rank of Weil divisors on X modulo numerical equivalence. In particular, $\sum b_i \leq \dim X + 1$ when $\rho^W(X/Z) = 1$.

Proof See [19, Theorem 2.3]. □

Proof (of Proposition 3.4) (i) Note that if a log twist is not final then (6') implies

$$(6") \qquad a' = \operatorname{ldis}(X', B_{X'}) = \operatorname{ldis}(Y', B_{Y'}) \geq \operatorname{ldis}(X, B) = a$$

and

(8) for the prime divisor E', $1 \geq a(E', X, B) = a(E', X', B_{X'}) > a'$.
Similarly, for any nonfinal twist $X^{(i)} \dashrightarrow X^{(i+1)}$,
(8') the prime divisor $E^{(i+1)}$ contracted by $X^{(i)} \dashrightarrow X^{(i+1)}$ satisfies

$$1 \geq a(E^{(i+1)}, X^{(i)}, B_{X^{(i)}}) = a(E^{(i+1)}, X^{(i+1)}, B_{X^{(i+1)}}) > a^{(i+1)}$$

where $a^{(i+1)} = \text{ldis}(X^{(i+1)}, B_{X^{(i+1)}})$. Thus the sequence of mld's $a, a', \dots, a^{(i)}, \dots$ is increasing:

$$a \le a' \le \dots \le a^{(i)} \le \dots,$$

or equivalently,

$$(9) \qquad e = 1 - a \ge e' = 1 - a' \ge \dots \ge e^{(i)} = 1 - a^{(i)} \ge \dots.$$

Lets define the *difficulty* $d^{(i)}$ of $(X^{(i)}, B_{X^{(i)}})$ to be the number of prime components D_k of $B_{X^{(i)}}$ with $b_k = \text{mult}_{D_k} B_{X^{(i)}} \ge e^{(i)}$.

The difficulty increases: for any nonfinal twist $X^{(i)} \dashrightarrow X^{(i+1)}$,

$$(10) \qquad d^{(i+1)} \ge d^{(i)} + 1.$$

Indeed, $e^{(i+1)} \le e^{(i)}$ by (9), and by (6")(8') none of the prime boundary components D_k of $B_{X^{(i)}}$ with $b_k = 1 - a(D_k, X^{(i)}, B_{X^{(i)}}) \ge e^{(i)} \ge e^{(i+1)} = 1 - a^{(i+1)}$ is contracted. On the other hand, any nonfinal twist $X^{(i)} \dashrightarrow X^{(i+1)}$ blows up a new prime component with multiplicity $e^{(i)} \ge e^{(i+1)}$ which adds 1 in the inequality (10).

Note now, that by Theorem 3.5 there exists a natural number N such that, on each Mori-Fano variety of dimension d, a boundary B has at most N boundary components with multiplicities $\ge e_0 = 1 - a_0 > 0$ if $K + B$ is antinef, and (X, B) is lc. More precisely, we can take any $N \ge (d+1)/e_0$.

By UBD each $e^{(i)} \ge e_0 = 1 - a_0$ for any $i \ge 1$ except for the last i (if the sequence is finite). Thus we have at most N nonfinal twists where N depends only on d and a_0.

(ii) By the ACC we can find a_0 as in UBD by putting

$$a_0 = \max\{\text{ldis}(X^{(i)}, 0) \mid i \ge 1\} \cap [0, 1)$$

which satisfies $\text{ldis}(X^{(i)}, B_{X^{(i)}}) \le \text{ldis}(X^{(i)}, 0) \le a_0 < 1$ where $X^{(i)}$ is in a sequence as in (7). □

Addendum 3.6 *Assume the LMMP and the ACC for mld's in dimension* d *and let* $\Gamma \subset [0, 1]$ *be a DCC set. Put* $\Gamma^{(0)} = \Gamma$ *and for* $i \ge 1$ *define*

$$\Gamma^{(i)} = \Gamma^{(i-1)} \cup \{1 - \text{ldis}(X^{(j)}, B_{X^{(j)}}) \mid 0 \le j \le i - 1\}$$

where the pairs $(X^{(j)}, B_{X^{(j)}})$ *come from all the possible sequences as in (7) in dimension* d *such that* $B \in \Gamma$ *for their starting pairs* (X, B)*. Then, the increasing sequence*

$$\Gamma \subseteq \Gamma' \subseteq \dots \subseteq \Gamma^{(i)} \subseteq \dots$$

stabilizes, and satisfies the DCC, that is, there exists N such that

$$\Gamma^\infty = \bigcup_{i\geq 0} \Gamma^{(i)} = \Gamma^{(N)}$$

and Γ^∞ satisfies the DCC.

Proof By Proposition 3.4, the length of any sequence as in (7) in dimension d is bounded by a number N. So, $\Gamma^{(i)} = \Gamma^{(N)}$ if $i > N+1$. Moreover, for any $i \geq 1$, $\Gamma^{(i)}$ satisfies the DCC because by induction $\Gamma^{(i-1)}$ satisfies the DCC and the set

$$\{1 - \operatorname{ldis}(X^{(j)}, B_{X^{(j)}}) \mid 0 \leq j \leq i-1\}$$

also satisfies the DCC by ACC for mld's where $(X^{(j)}, B_{X^{(j)}})$ comes from all the possible sequences as in (7) in dimension d starting with pairs (X, B) such that $B \in \Gamma$. Note that here we use the full ACC conjecture not just near 1. □

Lemma 3.7 *Let $\Gamma \subset [0,1]$ be a DCC set, and $X_i \dashrightarrow X'_i$ be a sequence of birational, (e.g., nonfinal) log twists in dimension d such that*

a) $B_i \in \Gamma$, *and*

b) *the set of multiplicities of all boundaries B_i is infinite.*

Then ACC *for mld's and lc thresholds in dimension d imply that the set of multiplicities of all boundaries $B_{X'_i}$ is infinite too.*

Proof By our assumptions each X'_i is a birational modification of X_i with a divisorial contraction $Y'_i \to X'_i$. In particular, all crepant boundaries $B_{Y'_i}$ and $B_{X'_i}$ are well defined.

By the DCC of Γ and after taking a subsequence, we can suppose that there exists a sequence of prime divisors D_i on X_i such that the corresponding sequence of boundary multiplicities $b_i = \operatorname{mult}_{D_i} B_i$ is strictly increasing. Moreover, by Theorem 3.5 the number of components of B_i is bounded hence we may assume that other multiplicities of prime divisors D_j on X_i are increasing (not necessarily strictly).

If infinitely many members of the sequence D_i are nonexceptional over X'_i, the lemma holds. If not, after taking a subsequence, we can suppose that each D_i is contracted over X'_i, that is, $D_i = E'_i$ on Y'_i, and it is numerically negative on Y'_i over X'_i. Thus by the property (3) of twists, D_i is numerically positive on Y_i over X_i. Hence each D_i contains $P_i = C_{X_i}E_i$, the centre of E_i on X_i. This implies that the set of new boundary multiplicities $e_i = 1 - \text{ldis}(X_i, B_i)$ is not finite, otherwise by taking a subsequence we may assume that e_i is independent of i and now apply Proposition 2.5 with $a = 1 - e_i$ to get a contradiction (cf. Theorem 2.10 (i)). $\square$

Addendum 3.8 *We can omit* ACC *for lc thresholds in Lemma 3.7 if we assume the* LMMP *and Conjecture 1.2 in dimension* $d-1$.

Proof Clear from Theorem 2.10 (v). $\square$

4 Weak finiteness

In this section we prove Theorem 2.12, (ii).

Proposition 4.1 (Weak finiteness) *Assume the* LMMP *and* ACC *for mld's in dimension* $\leq d$ *and the ACC for* $\mathcal{S}_d(\text{Mori} - \text{Fano cn})$, *and let* $\Gamma \subset [0,1]$ *be a DCC set. Then,* $\mathcal{S}_d(\Gamma, \text{global})$ *satisfies the ACC and there is a finite subset* $\Gamma_f \subseteq \Gamma$ *such that* $\mathcal{S}_d(\Gamma, \text{global}) = \mathcal{S}_\text{d}(\Gamma_\text{f}, \text{global})$.

Proof We use induction on d.

Step 1. The case $d = 1$ is an easy exercise. Assume $d \geq 2$ and that the theorem holds in dimension $\leq d-1$. Suppose that there exists a sequence of proper lc pairs (X_i, B_i), $i = 1, 2, \ldots$, of dimension $\leq d$ such that B_i has a type in $\mathcal{S}_d(\Gamma, \text{global})$, in particular, $K + B_i \equiv 0$, and such that the set of boundary multiplicities $M = \{b_{i,k}\}$, for boundaries $B_i = \sum b_{i,k}D_{i,k}$ is infinite. Since M satisfies the DCC we can assume that the sequence $b_{i,1}, i = 1, 2, \ldots$ is strictly increasing. Below we derive a contradiction (see Step 8).

Step 2. *We can suppose that each* X_i *is projective* $\mathbb{Q}$*-factorial, and, in particular, each divisor* $D_{i,1}$ *is* $\mathbb{Q}$*-Cartier*. By the LMMP each (X_i, B_i) has a projective $\mathbb{Q}$-factorial dlt blow up (Y_i, B_{Y_i}) [5, Construction 3.1]. Now replace (X_i, B_i) with (Y_i, B_{Y_i}). We need to extend Γ by 1 since all the new components of B_{Y_i}, i.e. components which are not birational transforms of the components of B_i, have coefficient 1.

Step 3. *We can suppose that each* X_i *is a Mori-Fano variety*. Indeed, by our assumptions we can apply the LMMP to $(X_i, B'_i = B_i - b_{i,1}D_{i,1})$.

Note that $K_{X_i} + B_i - b_{i,1}D_{i,1} \equiv -b_{i,1}D_{i,1}$ where $b_{i,1} > 0$. Thus $D_{i,1}$ is positive on each extremal ray in the process. In particular, the divisor $D_{i,1}$ will never be contracted. At the end we get a Mori-Fano fibration $Y_i \to Z_i$ for $K_{X_i} + B_i - b_{i,1}D_{i,1} \equiv -b_{i,1}D_{i,1}$. So, by replacing (X_i, B_i) with (Y_i, B_{Y_i}), where B_{Y_i} is the birational transform of B_i, we may assume that we have a Mori-Fano fibration $X_i \to Z_i$. Note also that $D_{i,1}$ is ample/Z_i.

If $\dim Z_i \geq 1$ for infinitely many i, then by restriction to a general fibre of X_i/Z_i we get a contradiction by induction (c.f Lemma 2.15). Thus, replacing with a subsequence, we can assume that each Z_i is a point, and X_i is a Mori-Fano variety of dimension d.

Step 4. *We can suppose that only finitely many varieties X_i are cn,* and thus, replacing by a subsequence, *we can suppose that all varieties X_i are noncn.* Otherwise, we can suppose that each X_i is cn. Then the ACC for $\mathcal{S}_d$(Mori-Fano cn) gives a contradiction.

So, replacing by a subsequence, we can suppose that each X_i has a noncn point (it may be nonclosed) of codimension ≥ 2. We can assume also that each (X_i, B_i) is klt by Theorem 2.12 (iii). Now we can construct a log twist $X_i \dashrightarrow X_i'$ as in Construction 3.1.

Step 5. *We can suppose that each log twist $X_i \dashrightarrow X_i'$ is final.* Indeed, if it is not final, replace (X_i, B_i) with $(X_i', B_{X_i'})$, and take another twist, etc. According to our assumptions, Proposition 3.4, Addendum 3.6, Theorem 2.10 (v), and Lemma 3.7, we can suppose that each twist $X_i \dashrightarrow X_i'$ is final. Let $f_i' \colon Y_i' \to X_i'$ be the corresponding contraction as in Construction 3.1.

As in Step 1 and after taking a subsequence, we can still suppose that there exists a sequence of prime divisors $D_{i,1}$ on X_i with strictly increasing boundary multiplicities $b_{i,1}$.

Step 6. *Infinitely many f_i' are divisorial contractions,* that is, of Type II (see 3.2). Otherwise infinitely many $Y_i' \to X_i'$ are Mori-Fano fibrations. Then, replacing by a subsequence, we can assume that all of them are fibrations. On the other hand, it is impossible by induction that infinitely many birational transforms of $D_{i,1}$ on Y_i' are strictly positive over X_i', because then they intersect general fibres (cf. Step 3). Therefore, by (3) in Construction 3.1, we can assume that each $D_{i,1}$ is strictly positive over X_i.

Let E_i be the exceptional divisor of $Y_i \to X_i$ and let $P_i \in X_i$ such that $C_{X_i}E_i = \overline{P}_i$ and $a_i := \mathrm{ldis}(X_i, B_i) = \mathrm{mld}(P_i, X_i, B_i) = a(E_i, X_i, B_i)$ where $C_{X_i}E_i$ denotes the center of E_i on X_i. Put $e_i = 1 - a_i$. Then by ACC for mld's and the DCC for Γ, the set $\Gamma' = \Gamma \cup \{e_i\}$ satisfies the DCC. Moreover, after taking a subsequence we may assume that the numbers e_i form a (not necessarily strict) increasing sequence. Note that the crepant boundaries B_{Y_i} on Y_i and $B_{Y'_i}$ on Y'_i belong to the set Γ'.

Since $D_{i,1}$ is strictly positive over X_i, $D_{i,1}$ contains $C_{X_i}E_i$. Then, by Proposition 2.5 the set $\{e_i\}$ is not finite, so we can suppose that e_i is strictly increasing (cf. Step 8 below and the proof of Lemma 3.7). This again gives a contradiction because, by (3) in Construction 3.1, the birational transform of each E_i on Y'_i is positive over X'_i.

Thus we can assume that each $Y'_i \to X'_i$ is divisorial with X'_i a Mori-Fano variety, and some divisor E'_i is contracted over X'_i. Take $P'_i \in X'_i$ such that $C_{X'_i}E'_i = \overline{P'_i}$ and put $a'_i = a(E'_i, X'_i, B_{X'_i})$ and $e'_i = 1 - a'_i$.

Step 7: *We can assume that* $a'_i = \mathrm{mld}(P'_i, X'_i, B_{X'_i})$ *for infinitely many* i, *so we can assume that for all* i. Otherwise, since the twists are final, X'_i is canonical for infinitely many i. This is a contradiction, because such varieties are bounded (see Step 4).

Step 8. *Contradiction:* $M = \{b_{i,1}\}$ *is finite.* By the DCC of Γ and since the support of B_i has a bounded number of components (Theorem 3.5), we can assume that each sequence $b_{i,k}, i = 1, 2, \ldots,$ is increasing with respect to i where we write $B_i = \sum_{k=1}^{n} b_{i,k}D_{i,k}$. The crepant divisors

$$B_{Y_i} = \sum_{k=1}^{n+1} b_{i,k}D_{i,k} = e_iE_i + \sum_{k=1}^{n} b_{i,k}D_{i,k}$$

satisfy the same property where $b_{i,n+1} := e_i$. Now we define the set $R = \{r_k | k = 1, 2, \ldots, n+1\}$ as the set of limits (not necessarily distinct)

$$r_k = \lim_{i\to\infty} b_{i,k}.$$

First suppose that $a = \lim_{i\to\infty} a_i > 0$ and $a' = \lim_{i\to\infty} a'_i > 0$. Let τ be a positive real number constructed in Main Proposition 2.1. We can assume that each $b_{i,k} \in [r_k - \tau, r_k]$. Hence by Proposition 2.3 each $K_{Y_i} + B^\tau_{Y_i}$ is antinef over X_i, and so is its birational transform $K_{Y'_i} + B^\tau_{Y'_i}$ over X'_i. Here and for the rest of the proof, the superscript τ stands for the limit, that is, for example $B^\tau_{Y_i} = \lim_{j\to\infty} B_{Y_j}$ on Y_i in the sense of Example 2.4. By construction

$$B^\tau_{Y_i} \geq B_{Y_i}$$

hence (4) in Construction 3.1 and

$$0 \leq B^\tau_{Y_i} - B_{Y_i} \equiv K_{Y_i} + B^\tau_{Y_i} = K_{Y_i} + B_{Y_i} + (B^\tau_{Y_i} - B_{Y_i})$$

imply that $B^\tau_{Y_i} - B_{Y_i} = 0$. This means that all limits are stabilized. This contradicts infinity of M, in particular strict monotonicity of $b_{i,1}$.

Now if $a = 0$, then to get the antinef property of $K_{Y_i} + B^\tau_{Y_i}$ over X_i, we can use Theorem 2.12 (iv). In fact, if $K_{Y_i} + B^\tau_{Y_i}$ is not antinef over X_i, then $K_{X_i} + A_i$ is maximally 0-lc at some point of X_i for some $B_i \lneq A_i \lneq B^\tau_i$ which contradicts Theorem 2.12 (iv). We have a similar argument when $a' = 0$. The rest of the proof is exactly as in the $a, a' > 0$ case. □

5 Proof of Main Theorem

Proof (of Main Theorem 1.8) (i) By induction, we can assume ACC for lc thresholds in dimension $\leq d$. Now we can use Proposition 2.5. (ii) This follows from Theorem 2.10 (v). (iii) We can use (ii) and the main result of [5]. □

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