

Mld's vs thresholds and flips

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Abstract

Minimal log discrepancies (mld's) are related not only to termination of log flips [22], and thus to the existence of log flips [11] but also to the ascending chain condition (acc) of some global invariants and invariants of singularities in the Log Minimal Model Program (LMMP). In this paper, we draw clear links between several central conjectures in the LMMP. More precisely, our main result states that the LMMP, the acc conjecture for mld's and the boundedness of canonical Mori-Fano varieties in dimension $\leq d$ imply the following: the acc conjecture for a -lc thresholds, in particular, for canonical and log canonical (lc) thresholds in dimension $\leq d$; the acc conjecture for lc thresholds in dimension $\leq d + 1$; termination of log flips in dimension $\leq d + 1$ for effective pairs; and existence of pl flips in dimension $\leq d + 2$. This also gives new proofs of some well-known and new results in the field in low dimensions: the acc conjecture holds for a -lc thresholds of surfaces; the acc conjecture holds for lc thresholds of 3-folds; termination of 3-fold log flips holds for effective pairs; and the existence of 4-fold pl flips holds.

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1 Introduction

Two main open problems in the Log Minimal Model Program (LMMP) are: existence and termination of log flips. Essentially, the latter one is the only problem: LMMP in dimension d , or inductively just termination in dimension d implies existence of log flips in dimension $d + 1$ [11]. Thus, if we establish termination of log flips in dimension $d + 1$, LMMP will be completed. On the other hand, the termination follows from two local (even formal) problems [22]: the ascending chain condition (ACC) conjecture for minimal log discrepancies (mld's; see Conjecture 1.3 below), and their semi-continuity conjecture due to Florin Ambro [4, Conjecture 2.4]. Recently, the first author [5] reduced a weaker termination in dimension $d + 1$ (e.g., when the log Kodaira dimension is nonnegative), in particular, termination of log flips in the relative birational case, to ACC conjecture for log canonical (lc) thresholds in dimension $d + 1$ (see Conjecture 1.7) which in its turn follows from V. Alexeev's, and brothers' A. and L. Borisov conjecture. This implies a weaker version, in particular, birational one, of LMMP in dimension $d + 1$. In this paper, we establish that the first of these conjectures, the ACC conjecture for mld's and a rather weak form of V. Alexeev's, and brothers' A. and L. Borisov conjecture (see Conjecture 1.2) in dimension d imply the weak version of LMMP in dimension $d + 1$. We hope that this version could be useful to resolve the two above local conjectures about mld's. We also show that the same conjectures are naturally related to some other similar problems in the field.

We use the terminology of [23] [24] [12][15]; see also Notation and terminology below. However we need certain modifications or generalizations of some well-known notions and conjectures.

Definition 1.1 (cf. [12, Definition 1.6(v)]) A proper contraction $X \rightarrow Z$ of normal varieties is called a *Mori-Fano fibration* if the following conditions hold:

- a) $\dim Z < \dim X$;

- b) X has only \mathbb{Q} -factorial log canonical (lc) singularities;
- c) $\rho(X/Z) := \rho(X) - \rho(Z) = 1$, where $\rho(\)$ is the Picard number; and
- d) $-K$ is ample on X/Z .

If $Z = \text{pt.}$ is a point, X is called a *Mori-Fano variety*. We say that X is a *canonical Mori-Fano variety* if X has only canonical (cn) singularities.

Note that by the Kleiman projectivity criterion, any Mori-Fano fibration and variety are projective.

Conjecture 1.2 (Weak BAB) *The canonical d -dimensional Mori-Fano varieties are bounded, that is, a coarse moduli space of such varieties is well-defined and of finite type.*

BAB abbreviates V. Alexeev, and brothers A. and L. Borisov. The conjecture is a very special case of their conjecture (see [20]). Conjecture 1.2 is established in dimension ≤ 3 in characteristic zero [KMMT] (the case $d = 2$ is classical). Actually, we need a much weaker version of this conjecture, namely, the boundedness of canonical d -dimensional Mori-Fano varieties X such that $K + B \equiv 0$ for some boundary $B \in \Gamma$ where Γ is a fixed set of boundary multiplicities satisfying the descending chain condition (dcc).

Conjecture 1.3 (ACC for mld's) *Suppose that $\Gamma \subseteq [0, 1]$ satisfies the dcc. Then the following is expected:*

(ACC) *The following subset of real numbers \mathbb{R}*

$$\{ \text{mld}(P, X, B) \mid (X, B) \text{ is lc, } \dim X = d, P \in X, \text{ and } B \in \Gamma \}$$

satisfies the ascending chain condition (acc)

A point P can be nonclosed. Equivalently, we can consider only closed points $P \in X$, and assume that $\dim X \leq d$.

This conjecture is established in dimension $d \leq 2$ [2] [26], and for some special cases in higher dimensions [8] [23][3].

Definition 1.4 (a -lc thresholds) Let $a \geq 0$ be a real number, (X, B) be a log pair, and H be an \mathbb{R} -Cartier divisor on X . Then the real number or $+\infty$:

$$t = \text{th}_a(M, X, B) = \sup\{\lambda \in \mathbb{R} \mid (X, B + \lambda H)$$

$\text{is } a\text{-lc in codimension } \geq 2 \text{ [23, 1.3]}\}$

is called the *a-lc threshold of H with respect to (X, B)* . In particular, if $a = 0$ or $a = 1$, the *a-lc threshold* is the *lc threshold* or *cn threshold* respectively. *\mathbb{Q} -factorial threshold* means that we consider only \mathbb{Q} -factorial varieties. Similarly, we get the *a-lc threshold at a point P* (possibly not closed) if the *a-lc condition* in codimension ≥ 2 is replaced by the *a-lc condition in P* .

Remark 1.5 Note that if $+\infty > \text{ldis}(X, D) \geq a$, and $H > 0$, then $t \geq 0$, $\text{sup} = \max$, and is a nonnegative real number (that is, not $+\infty$, cf. [12, Remark 1.4,(ii)]). In this situation, either $(X, D + tH)$ is precisely *a-lc* in codimension 2, that is $\text{ldis}(X, D + tH) = a$, or $(X, D + tH)$ has reduced components. Behavior of thresholds in codimension 1 (at divisorial points) is easy. However, when we consider thresholds at a point, the situation is more complicated (see Example 1.6 or cf. the proof of Proposition 2.5).

Note that we need only $a \leq 1$ if $\dim X \geq 2$. Indeed, $\text{ldis}(X, D) \leq 1$ always when $\dim X \geq 2$, and $\text{ldis}(X, D) = +\infty$ when $\dim X \leq 1$ because it corresponds to the empty set (see Notation and terminology: mld in codimension ≥ 2 . Cf. thresholds at a point in Definition 1.4 above). In contrast, for $\text{ldis}(X, B) < a$ and $H > 0$, $t < 0$ holds. Usually in applications, $\dim X \geq 2$, $1 \geq \text{ldis}(X, B) \geq a$, and $D = B$ is a boundary, e.g., $D = 0$ [12][15][9][10].

Example 1.6 The *a-lc threshold at P* may not be attained at P nor on the boundary. For example: take three planes S_1, S_2, S_3 in the space \mathbb{P}^3 passing through a line L . Take a closed point $P \in L$ and define $B = \frac{2}{3}S_1 + \frac{2}{3}S_2 + \frac{2}{3}S_3$. L is a lc centre for (\mathbb{P}^3, B) but easy computations show that $\text{mld}(P, \mathbb{P}^3, B) = 1$. On the other hand, $(b_1, b_2, b_3) \neq (1, 1, 1)$.

So, in general, for the *a-lc threshold at a point P* , either we get a lc centre passing through P or the mld a is attained at P .

Conjecture 1.7 (ACC for *a-lc thresholds*) *Suppose $d \geq 2$ is a natural number, $a \geq 0$, $\{0\} \subseteq \Gamma \subseteq [0, 1]$ satisfies the dcc and $\{0\} \subsetneq S \subset \mathbb{R}$ is a finite (even dcc) set of nonnegative numbers. Then the following is expected:*

(ACC) *The subset*

$$\mathcal{T}_{a,d}(\Gamma, S) = \{\text{th}_a(M, X, B) \mid (X, B) \text{ is } a\text{-lc in codimension } \geq 2, \dim X = d,$$

$$B \in \Gamma, M \text{ is an } \mathbb{R}\text{-Cartier divisor on } X, \text{ and } M \in S\}.$$

of $\mathbb{R}^+ \cup \{+\infty\}$ satisfies the acc; $+\infty$ corresponds to the case $M = 0$.

It is also expected that ACC holds for a -lc thresholds at a point P , that is, for the set with the a -lc in $P \in X$ (see Definition 1.4). The latter set is larger. Thus ACC for thresholds at a point implies ACC for thresholds on a variety, and in what follows, ACC for thresholds means at a point.

The case when $d = 1$ is obvious: the set of $\text{mld}(P, X, B) = 1 - \text{mult}_P B$, for prime divisors P on X , satisfies the acc if and only if the multiplicities of possible B satisfy the dcc. Similarly, ACC for a -lc thresholds at prime divisors can be easily verified (cf. Example 4.2 below).

Main Theorem 1.8 *LMMP, ACC for mld's and Conjecture 1.2 in dimension $\leq d$ imply the following:*

- (i) ACC for a -lc thresholds in dimension $\leq d$;
- (ii) ACC for lc thresholds in dimension $\leq d + 1$;
- (iii) termination of log flips in dimension $\leq d + 1$ for effective pairs; and
- (iv) existence of pl flips in dimension $\leq d + 2$.

See the proof in Section 5. For generalizations of statements (i-ii) in the Main Theorem and of the corollaries below, see Section 2.

Corollary 1.9 *ACC for mld's and Conjecture 1.2 for 4-folds imply:*

- (i) ACC for a -lc thresholds in dimension 4;
- (ii) ACC for lc thresholds in dimension 5;
- (iii) termination of log flips in dimension 5 for effective pairs; and
- (iv) existence of pl flips in dimension 6.

Proof ACC for mld's for 4-folds implies termination of 4-fold log flips [22, Corollary 5] and thus LMMP for 4-folds [24, Corollary 1.8]. \square

Corollary 1.10 *ACC for mld's of 3-folds implies:*

- (i) ACC for a -lc thresholds in dimension 3;
- (ii) ACC for lc thresholds in dimension 4;
- (iii) termination of 4-fold log flips for effective pairs; and
- (iv) existence of pl flips in dimension 5.

Proof Immediate by Theorem 1.8, and by [17], [13] for \mathbb{Q} -boundaries and [23] in general. \square

ACC for mld's of algebraic surfaces gives a new proof of the following well known and new results.

Corollary 1.11 *The following hold:*

- (i) ACC for a -lc thresholds of surfaces;
- (ii) ACC for lc thresholds of 3-folds;
- (iii) termination of 3-fold log flips for effective pairs; and
- (iv) the existence of 4-fold pl flips.

Proof Immediate by Theorem 1.8, and [2][26] and [1]. \square

Note that ACC for mld's of surfaces in [26] is established for \mathbb{R} -boundaries and without using classification. Thus, for the first time, termination in (iii) is proved without classification (cf. [13] [19, proof of 5.1.3 for 3-folds]). However, this termination for 3-folds is still partial.

Cn thresholds and, in particular, their ACC is crucial for the Sarkisov program [7] [18]. Another similar important invariant, the Sarkisov degree or its inverse – the anticanonical threshold – can be included into more general ones: Fano indices (see Cor. 2.14 below) and boundary multiplicities of log pairs for $S_d(\text{global})$ (see Def. 2.6 (v) and Weak finiteness 4.1). These invariants and results about them are important in the proof of our Main Theorem and will be discussed in Sections 2–4. Here we give a sample.

Corollary 1.12 *Let $\Gamma \subset [0, 1]$ be a dcc set. Then, there is a finite subset $\Gamma_f \subset \Gamma$ such that $\mathcal{S}_3^0(\Gamma, \text{global}) = \mathcal{S}_3^0(\Gamma_f, \text{global})$ (see Definition 2.6).*

In other words, the set of boundary multiplicities which occur on the following log pairs is finite: the 3-fold projective log pairs (X, B) with $B \in \Gamma$, $K + B \equiv 0$, (X, B) is lc but not klt.

Proof Immediate by Theorem 2.10 (vi). \square

Notation and terminology

In this paper, a log pair $(X/Z, B)$ consists of normal algebraic varieties X, Z over a base field k of characteristic 0, e.g. $k = \mathbb{C}$, where X/Z is a projective morphism, and an \mathbb{R} -boundary B (i.e., a divisor with multiplicities in $[0, 1]$) such that $K + B$ is \mathbb{R} -Cartier. Of course, some results hold or are expected over any field, e.g., ACC for a -lc thresholds holds in Corollary 1.11 (i). We consider the log minimal model program (LMMP)[23, 5.1] in dimension d in the category of lc pairs of dimension d .

An *effective* log pair is a log pair $(X/Z, B)$ [5] such that $K + B \equiv M/Z$ for some \mathbb{R} -divisor $M \geq 0$. This property is preserved under any log flip or divisorial contraction. A variety X is of *Fano type* (FT) if there an \mathbb{R} -boundary B such that (X, B) is a klt weak log Fano.

A property holds at a point $P \in X$ means that that property holds at the point P but not necessarily in a neighbourhood of P . On the other hand, a property holds near P means that that property holds in an open neighbourhood of P .

If (X, B) is lc, then

$$1 - \text{mld}(P, X, B) = \max\{\text{mult } E \text{ in } B_W \mid E \text{ is a prime divisor on } W \text{ and } f(E) = P\}$$

for any (crepant) log resolution $W \rightarrow X$ where $K_W + B_W = f^*(K + B)$ and mult stands for the multiplicity function on divisors. We define

$$\text{ldis}(X, B) = \min\{\text{mld}(P, X, B) \mid P \in X \text{ is of codimension } \geq 2\}$$

We say (X, B) is *a-lc* at $P \in X$ if $\text{mld}(P, X, B) \geq a$. This implies, in particular, that (X, B) is lc near P [23, Corollary 1.5].

For a set $\Gamma \subset \mathbb{R}$ and an \mathbb{R} -divisor D on a variety, by $D \in \Gamma$ we mean that the (nonzero) multiplicities of D are in Γ .

2 Acc of mld's and thresholds

For \mathbb{R} -divisors on X , we have the well-known *order*:

$D_1 \geq D_2$ if $D_1 - D_2 \geq 0$, that is effective.

On the other hand, the topology and the following natural norm, the *maximal absolute value norm*, are well-known: if $D = \sum d_i D_i$, where $d_i \in \mathbb{R}$, and D_i are distinct prime divisors on X , set

$$\| D \| = \max \| d_i \| .$$

In particular, limits of *divisors* are limits in the norm.

Main Proposition 2.1 *We assume ACC for mld's in dimension d . Let $\Gamma \subset [0, 1]$ be a dcc subset, and a be a positive real number. Then, there exists a real number $\tau > 0$ (depending also on d) satisfying the following upper*

approximation property: if (X, B) and (X, B') are two log pairs with a point $P \in X$ (not necessarily closed) such that

- (1) $\dim X = d$;
 - (2) $B \leq B'$, $\|B - B'\| < \tau$, $B' \in \Gamma$; and
 - (3) $\text{mld}(P, X, B) \geq a$, that is, (X, B) is a -lc at P , and $K + B'$ is \mathbb{R} -Cartier; we can omit the last assumption when X is \mathbb{Q} -factorial; and
 - (4) (X, B') is lc in a neighborhood of P ;
- then $\text{mld}(P, X, B') \geq a$ and (X, B') is also a -lc at P .

Note that by [23, Lemma 1.4] the assumption (4) of the proposition is equivalent to the lc property of (X, B') at P , that is, to the inequality $\text{mld}(P, X, B') \geq 0$. To prove the proposition we need the following general fact.

Lemma 2.2 (Continuity) *Suppose that the pairs (X, B) and (X, B') are lc in a neighborhood of a point P . Then, $a' = \text{mld}(P, X, B')$ and $a = \text{mld}(P, X, B)$ are real numbers ≥ 0 , and, for any real number x in the interval $[a', a]$ there exist two real numbers $\alpha, \beta \geq 0$ such that $\alpha + \beta = 1$, and $\text{mld}(P, X, \alpha B + \beta B') = x$.*

Proof Let $D = B' - B$, and $a \geq a'$.

By the lc property, both mld's are real numbers ≥ 0 . Then, the last statement holds for the pairs $(X, B + tD)$ with $t = 0$ and $t = 1$ for which respectively $B + tD = B$ and B' . By the convexity in [25, (1.3.2)] the same holds for any t in the interval $[0, 1]$: $(X, B + tD)$ is lc near P , and $\text{mld}(P, X, B + tD)$ is a real number ≥ 0 .

Let $s = \sup\{t \in [0, 1] \mid \text{mld}(P, X, B + tD) \geq x\}$. The set of such s is not \emptyset because $a = \text{mld}(P, X, B) \geq x \geq a'$. We claim that $\text{mld}(P, X, B + sD) = x$. Put $\text{mld}(P, X, B + sD) = y$. Since the last mld is a real number (not $-\infty$) there is a log resolution $f: W \rightarrow X$ of $(X, B + sD)$ on which the mld is attained on divisors:

$$1 - y = \max\{\text{mult } E \text{ in } B_W + sf^*D \mid E \text{ is a prime divisor on } W \text{ and } f(E) = P\},$$

where B_W denotes the crepant pull-back of B , that is, $K_W + B_W = f^*(K + B)$. Then for any t , $\text{mld}(P, X, B + tD) \leq 1 - m(t)$ with

$$m(t) := \max\{\text{mult } E \text{ in } B_W + tf^*D \mid E \text{ is a prime divisor on } W \text{ and } f(E) = P\}$$

because to calculate the mld one needs to consider the inf (of log discrepancies) for all resolutions. Note that $m(t)$ is a piecewise linear and continuous real-valued function of t . This follows from the linear property of $\text{mult}_E(B_W + tf^*D)$ with respect to t .

If $y < x$, then for any t sufficiently close to s , and in particular, for such $t < s$, $\text{mld}(P, X, B + tD) < x$ too, which contradicts our constructions when $s > 0$. Thus $s = 0$ or $y \geq x$, and actually $y = x$ in both cases by the following stability property: *if $\text{mld}(P, X, B + sD) > x$, and $(P, X, B + tD)$ is lc in a small neighborhood of s in $[0, 1]$, then $\text{mld}(P, X, B + tD) > x$ too in some small neighborhood of s in $[0, 1]$.* Note that $s < 1$ for $y > x$. After taking a log resolution: the stability follows from its log nonsingular version: for a normal crossing \mathbb{R} -subboundary C on X and any point P , if $\text{mld}(P, X, C) > x \geq 0$, then the same holds for any small perturbation of C in the nonreduced part, that is, the perturbation of only multiplicities of $C < 1$. In fact let $C = [C] + \sum c_i D_i$ and we may assume that all the components pass through P . Then, $\text{mld}(P, X, C) = \text{codim } P - \mu_P [C] - \sum c_i > x$ where $\mu_P [C] \in \mathbb{N}$ is the multiplicity of the reduced part $[C]$ at P . Thus, $\text{mld}(P, X, C + \sum \varepsilon_i D_i) = \text{codim } P - \mu_P [C] - \sum (c_i + \varepsilon_i) > x$ where $|\varepsilon_i|$ are small enough.

Now by taking $\beta = s$ and $\alpha = 1 - \beta$, we get

$$x = \text{mld}(P, X, (\alpha + \beta)B + \beta(B' - B)) = \text{mld}(P, X, \alpha B + \beta B').$$

□

Proof of Proposition 2.1

Suppose that the proposition does not hold. Then, there exists a sequence of positive real numbers $\tau_1 > \tau_2 > \dots$ with $\lim_{i \rightarrow +\infty} \tau_i = 0$, and a sequence of d -dimensional log pairs $(X_i, B_i), i = 1, 2, \dots$, such that the proposition does not hold for τ_i on (X_i, B_i) in a point $P_i \in X_i$. In other words, there exists $B'_i \in \Gamma$ on X_i under (2-4) with $\tau = \tau_i$, and

$$\text{mld}(P_i, X_i, B_i) \geq a \quad \text{but} \quad \text{mld}(P_i, X_i, B'_i) < a$$

We now construct a new sequence of d -dimensional log pairs (T_i, A_i) and points $Q_i \in T_i$ such that $a_i = \text{ldis}(T_i, A_i)$ is strictly increasing with i and such that Ω , the set of multiplicities of all boundaries A_i , satisfies the dcc.

By ACC for mld's the set $\{a'_i = \text{mld}(P_i, X_i, B'_i)\}$ has a maximum which is less than a . We can assume that this maximum is equal to $\text{mld}(P_1, X_1, B'_1)$.

Put $(T_1, A_1) := (X_1, B'_1)$ and $Q_1 = P_1$ and let $a_1 = \text{mld}(Q_1, T_1, A_1)$. Note that a_1 is a real number ≥ 0 and $a_1 = -\infty$ is impossible by (4).

Suppose that we have already constructed (T_j, A_j) for $1 \leq j \leq i-1$. Since Γ satisfies the dcc, we can choose τ_k such that there are no multiplicities of A_j , for $1 \leq j \leq i-1$, in $(r-\tau_k, r)$ for any $r \in \Gamma$. Take $(T_i, A_i) := (X_k, \alpha B_k + \beta B'_k)$, for some $\alpha, \beta > 0$ with $\alpha + \beta = 1$, and $Q_i = P_k$ such that

$$\frac{a_{i-1} + a}{2} < a_i = \text{mld}(Q_i, T_i, \Delta_i) < a.$$

The existence follows from Lemma 2.2 with $X = X_k, B = B_k, B' = B'_k, a' = a'_k$, and any x in the interval (a', a) (here the a in the proposition!) but $a = a_k$ in the lemma (not the a in the proposition). Such x exists because, by construction and assumptions (2-4), $a_{i-1}, a'_k < a$ (for both a).

Also by construction for every real number $\varepsilon > 0$, almost all (expect for finitely many) multiplicities of Ω belong to intervals $(r - \varepsilon, r]$ where $r \in \Gamma$. This implies that Ω satisfies the dcc because Γ does so. On the other hand, the set of mld's $\{a_i\}$ does not satisfy the acc which contradicts the ACC for mld's. \square

Proposition 2.3 *Under the assumptions of Proposition 2.1, let $Y \rightarrow X$ be an extremal divisorial extraction such that*

(1) $K_Y + B'_Y + (1 - a)E$ is \mathbb{R} -Cartier,

where E is the exceptional reduced divisor (but not necessarily irreducible), and B'_Y is the birational transform of B' on X . Then $K_Y + B'_Y + (1 - a)E$ is seminegative/ X .

Example 2.4 The typical situation where we apply the proposition is as follows. Let $(X_i, B_i), i = 1, 2, \dots$, be a sequence of d -dimensional plt log pairs such that

- a) $a_1 = \text{ldis}(X_1, B_1) \geq \dots \geq a_i = \text{ldis}(X_i, B_i) \geq \dots > 0$ with
- b) $a = \lim_{i \rightarrow \infty} a_i > 0$; and
- c) $B_1 \leq \dots \leq B_i \leq \dots$ with
- d) $B = \lim_{i \rightarrow \infty} B_i \in R$.

The c)-d) means that there exist prime divisors $D_{i,k}, k = 1, \dots, n$, on each X_i such that every

$$B_i = \sum_{k=1}^n b_{i,k} D_{i,k};$$

and, for every $k = 1, \dots, n$,

c') $b_{1,k} \leq \dots \leq b_{i,k} \leq \dots$ with

d') $b_k = \lim_{i \rightarrow \infty} b_{i,k} \in R$

(cf. types in Definition 2.6 below).

In particular, $B = \sum b_k D_{i,k}$ approximates B_i on X_i , that is, for any $\tau > 0$, $\|B_i - B\| < \tau$ for all $i \gg 0$ (divisors B on X_i have the same type (b_1, \dots, b_n) in the sense of Definition 2.6 below; this is why we use this ambiguous notation). Thus if we take $R = \{b_k \mid k = 1, \dots, n\}$, for all $i \gg 0$ for given $\tau > 0$, we satisfy all assumptions of Proposition 2.1 for any (X_i, B_i) and (X_i, B) with X_i, B_i, B , and P_i instead of X, B, B' , and P respectively, except for the \mathbb{R} -Cartier property of $K_{X_i} + B$ in (3), and the lc property in (4). The \mathbb{R} -Cartier property will hold if for example X is \mathbb{Q} -factorial. Moreover, if $a < 1$, then each $a_i < 1$ for $i \gg 0$, and there exists a crepant extremal divisorial extraction $Y_i \rightarrow X_i$ of an exceptional prime b-divisor E_i with centre P_i on X_i and $a_i = \text{mld}(P_i, X_i, B_i) = a(E_i, X_i, B_i)$ [23, Theorem 3.1]. The extraction Y_i is \mathbb{Q} -factorial too [23, Theorem 3.1], and $K_{Y_i} + B_{Y_i} + (1 - a)E_i$ is \mathbb{R} -Cartier where B_{Y_i} denotes the birational transform of B_i on Y_i .

Thus in the \mathbb{Q} -factorial case and under (4), for all $i \gg 0$, (X_i, B) is a -lc, and $K_{Y_i} + B_{Y_i} + (1 - a)E_i$ is seminegative/ X_i . Cf. the proof of Proposition 2.3 below, and Step 8 in the proof of Proposition 4.1 where we either assume (4) or, we assume ACC for lc thresholds and derive (4) from that assumption.

Finally, note that we can derive (4) from ACC for lc thresholds in dimension d when X is \mathbb{Q} -factorial. Indeed, if (4) does not hold, then possibly after passing to a subsequence, we can construct a strictly increasing sequence of boundaries B'_i such that $B_i < B'_i < B$, and $\text{ldis}(X_i, B'_i) = 0$. Essentially, B'_i is constructed by taking an appropriate lc threshold. Moreover, the multiplicities of B'_i will satisfy the dcc with finitely many accumulation points in R . This contradicts ACC for lc thresholds in dimension d .

Proof (of Proposition 2.3)

Suppose that $K_Y + B'_Y + (1 - a)E$ is not seminegative. Then by property (1) of the proposition and the extremal property, it is numerically positive/ X .

On the other hand, by (3) of Proposition 2.1,

$$K_Y + B'_Y + \sum (1 - a(E_i, X, B'))E_i \equiv 0/X,$$

where by Proposition 2.1 the discrepancy $a(E_i, X, B') \geq a$ for each prime

component E_i of $E = \sum E_i$. Thus the \mathbb{R} -Cartier divisor

$$\sum (a - a(E_i, X, B'))E_i = (K_Y + B'_Y + \sum (1 - a(E_i, X, B'))E_i) - (K_Y + B'_Y + (1 - a)E)$$

is numerically negative/ X . According to Negativity [25, 1.1], the divisor is effective and $\neq 0$, that is, each $a - a(E_i, X, B') > 0$, a contradiction. \square

The following result is the big chunk of (i) in our Main Theorem 1.8, and it gives another application of Main Proposition when the support of B is not universally bounded.

Proposition 2.5 *ACC for mld's and lc thresholds in dimension d implies ACC for a -lc thresholds in the same dimension for all $a > 0$, in particular, for canonical thresholds.*

Proof Suppose that we have a monotonic increasing sequence t_i of d -dimensional a -lc thresholds, that is, there exists a sequence (X_i, B_i) of d -dimensional log pairs with boundaries $B_i \in \Gamma$, and \mathbb{R} -divisors $M_i \in S$ on X_i such that

(1) (X_i, B_i) is a -lc;

and

(2) $t_i = \text{th}_a(M_i, X_i, B_i)$;

in particular, on each X_i there exists a point $P_i \in \text{Supp } M_i \subset X_i$ of codimension ≥ 2 , at which

(3) $\text{mld}(P_i, X_i, B_i + t_i M_i) = a$; or

(4) $B_i + t_i M_i$ has a reduced component (the payment for ldis in codimension ≥ 2 ; see Remark 1.5).

We need to verify the acc for the sequence t_i , that is, the sequence stabilizes.

If for infinitely many i , $B_i + t_i M_i$ has a reduced component $D_{i,j}$ as in (4), that is, $b_{i,j} + t_i m_{i,j} = 1$ for multiplicities in $D_{i,j}$, then $t_i = (1 - b_{i,j})/m_{i,j}$ and stabilizes by the dcc for multiplicities $b_{i,j}$ and $m_{i,j}$. This gives the acc in the case (4).

Thus after taking a subsequence, we can assume (3) for all i . Note that by the lc property of $(X, B_i + t_i M_i)$ the limit $t = \lim_{i \rightarrow \infty} t_i$ exists because t_i are bounded from above: $t_i \leq \frac{1}{m_0}$ where $m_0 = \min\{m \in S\}$; $t \geq 0$. We can apply Proposition 2.1 for each $X = X_i, B = B_i, B' = B_i + t M_i$, and $P = P_i$. Indeed, (1) of the proposition holds because $\dim X = d$. The assumption (2) of the proposition follows from construction, in particular, the multiplicities

of B' are as $b_{i,j} + tm_{i,j}$ and satisfy the dcc as their components $b_{i,j}$ and $m_{i,j}$ do.

The assumption (3) of the proposition $\text{mld}(P_i, X_i, B_i) \geq a$ holds by (1) above; $K_X + B'$ is \mathbb{R} -Cartier because each M_i is \mathbb{R} -Cartier.

Finally, the assumption (4) of the proposition, that is, (X, B') is lc in a neighborhood of P , follows from ACC for lc thresholds. Indeed, if the lc property does not hold for $i \gg 0$, then we get an increasing set of $t'_i = \text{lct}(M_i, X_i, B_i)$ for infinitely many i , such that $t_i \leq t'_i < t$ (The lc property in codimension 1 holds by construction.). This contradicts ACC for lc thresholds.

Therefore, by Proposition 2.1 and (1-3) $t = t_i$ for all $i \gg 0$ and t_i stabilizes.

□

Proposition 2.1 also gives some relations between different ACC versions besides the ones for mld's and thresholds in the Introduction. Now we recall some of them.

Definition 2.6 (cf. [15, Section 18].) (i) The *type order* is a direct sum of \mathbb{R} countably many times, that is, the set of sequences (b_1, \dots, b_n) with $b_i \in \mathbb{R}, n \geq 0$, and the following order: $(b_1, \dots, b_m) \leq (b'_1, \dots, b'_n)$ if either $n < m$ or $n = m$ and each $b_i \leq b'_i$. The *maximal element* is the empty sequence with $n = 0$.

(ii) A *type* of an \mathbb{R} -divisor $D = \sum d_i D_i$ on X , where D_i are distinct prime divisors on X , is the sequence (d_1, \dots, d_n) of its nonzero multiplicities (in any possible ordering). We usually do not think of D with a specific ordering of the prime components in mind, so D can have several types. Even one can add finitely many zeros.

(iii) (Cf. [15, Definition 18.3].) A log pair $(X, H + D)$ with \mathbb{R} -divisors H, D and prime divisors D_1, \dots, D_n on X has *maximal a -lc type* (d_1, \dots, d_n) near $Z \subset X$ and respectively at $P \in X$ if $D = \sum d_i D_i$, in particular, has type (d_1, \dots, d_n) , if (X, D) is a -lc near Z and respectively at P , and (X, D') is not a -lc near Z and respectively at P for any \mathbb{R} -divisor $D' = \sum d'_i D_i$ of type (d'_1, \dots, d'_n) such that $K + H + D'$ is \mathbb{R} -Cartier and $D' > D$ in any neighborhood of Z and respectively of P . For $H = 0$, the pair (X, D) has that maximal property.

(iv) (Cf. [15, 18.15.1].) $\mathcal{S}_d(\text{Fano})$ is the set of types (b_1, \dots, b_n) such that there is a nonsingular Fano variety X of dimension at most d and a boundary B of type (b_1, \dots, b_n) such that $K + B \equiv 0$. (See Example 2.9, (1) below.)

(v) (Cf. [15, 18.15.1].) $\mathcal{S}_d(\text{global})$ is the set of types (b_1, \dots, b_n) such that there is a proper normal variety X of dimension at most d and a boundary B of type (b_1, \dots, b_n) such that (X, B) is lc, and $K + B \equiv 0$. (See Example 2.9, (2) below.) We denote by $\mathcal{S}_d^0(\text{global})$ its subset with $\text{nonklt}(X, B)$.

(vi) (Cf. [15, 18.15.2].) $\mathcal{S}_{a,d}(\text{local})$ is the set of types (b_1, \dots, b_n) such that there is a pointed \mathbb{Q} -factorial variety $P \in X$ of dimension at most d , and prime divisors D_1, \dots, D_n on X such that $B = \sum b_i D_i$ is a boundary,

$$P \in \cap D_i \text{ and } \text{mld}(P, X, B) \geq a,$$

actually $= a$ or $n \geq 1$ and of maximal lc type in some point of any neighborhood of P (see Example 2.9, (4) below), or equivalently, $B = \sum b_i D_i$ is a boundary, and (X, B) locally has maximal a -lc type (b_1, \dots, b_n) at P with given divisors D_i . (See Example 2.9, (3) below.)

(vii) (Cf. [15, 18.15.2].) $\overline{\mathcal{S}}_{a,d}(\text{local})$ is the set of types (b_1, \dots, b_n) such that there is a \mathbb{Q} -factorial variety of dimension at most d , prime divisors D_1, \dots, D_n on X , and a closed subvariety $Z \subset X$ such that $B = \sum b_i D_i$ is a boundary, (X, B) is a -lc near Z , every D_i intersects Z , and (X, B) has maximal a -lc type $(b_{i_1}, \dots, b_{i_l}), 1 \leq i_1 < \dots < i_l \leq n$, at some of the intersection points of D_i with the divisors D_{i_1}, \dots, D_{i_l} passing through such a point, or equivalently, $B = \sum b_i D_i$ is a boundary, and (X, B) locally has maximal a -lc type (b_1, \dots, b_n) near Z with the divisors D_i .

(viii) (Cf. [15, 18.15.3].) $\mathcal{S}_{a,d}^0(\text{local})$ is the set of types (b_1, \dots, b_n) such that there is a pointed \mathbb{Q} -factorial variety $P \in X$ of dimension at most d , and prime divisors D_1, \dots, D_n on X such that $B = \sum b_i D_i$ has a reduced component,

$$P \in \cap D_i \text{ and } \text{mld}(P, X, B) \geq a,$$

actually $= a$ or $n \geq 1$ and of maximal lc type in some point in any neighborhood of P (see Example 2.9, (4) below), or equivalently, $B = \sum b_i D_i$ has a reduced component, and (X, B) locally has maximal a -lc type (b_1, \dots, b_n) at P with given divisors D_i .

(ix) (Cf. [15, 18.15.3].) $\overline{\mathcal{S}}_{a,d}^0(\text{local})$ is the set of types (b_1, \dots, b_n) such that there is a \mathbb{Q} -factorial variety X of dimension at most d , a subset $Z \subset X$, and prime divisors D_1, \dots, D_n on X such that $B = \sum b_i D_i$ has a reduced component, (X, B) is a -lc near Z , Z is in the reduced part of B , every D_i intersects Z , and (X, B) has maximal a -lc type $(b_{i_1}, \dots, b_{i_l}), 1 \leq i_1 < \dots < i_l \leq n$, at some of the intersection points of D_i with the divisors D_{i_1}, \dots, D_{i_l} passing through such a point, or equivalently, B has a reduced component, Z

is in the reduced part of B , (X, B) locally has maximal a -lc type (b_1, \dots, b_n) near Z with given divisors D_i .

(x) (Cf. [15, 18.15.1].) $\mathcal{S}_d(\text{Mori-Fano})$ is the set of types (b_1, \dots, b_n) such that there is a Mori-Fano variety X of dimension at most d , and a boundary B of type (b_1, \dots, b_n) such that (X, B) is lc, and $K + B \equiv 0$. We denote by $\mathcal{S}_d^0(\text{Mori-Fano})$ its subset with nonklt (X, B) .

(xi) (Cf. [15, 18.15.1].) $\mathcal{S}_d(\text{Mori-Fano cn})$ is the set of types (b_1, \dots, b_n) such that there is a cn Mori-Fano variety X of dimension at most d and a boundary B of type (b_1, \dots, b_n) , and $K + B \equiv 0$.

We consider each of the above sets $\mathcal{S}_d(\text{Fano}), \dots, \mathcal{S}_d(\text{Mori-Fano cn})$ as a subset of the order \mathcal{B} . Thus it has ordering induced from \mathcal{B} .

For $a = 0$, we set $\mathcal{S}_d = \mathcal{S}_{a,d}$, e.g., $\overline{\mathcal{S}}_d^0(\text{local}) = \overline{\mathcal{S}}_{0,d}^0(\text{local})$. Some of them are slightly more general than in [15, 18.15].

Nonetheless we expect the same.

Conjecture 2.7 (cf. [15, Conjecture 18.16]) *Each set $\mathcal{S}_d(\text{global}), \mathcal{S}_d^0(\text{global}), \mathcal{S}_{a,d}(\text{local}), \overline{\mathcal{S}}_{a,d}(\text{local}), \mathcal{S}_{a,d}^0(\text{local}), \overline{\mathcal{S}}_{a,d}^0(\text{local}), \mathcal{S}_d(\text{Mori-Fano}), \mathcal{S}_d(\text{Mori-Fano cn})$ satisfies the acc.*

Remark 2.8 (1) Equivalently, in Definition 2.6 (iii) for effective D and $D' > D$, the assumption $D' = \sum d'_i D_i$ has type (d'_1, \dots, d'_n) can be replaced by the strict inequality of types: D' has type $(d'_1, \dots, d'_m) > (d_1, \dots, d_n)$ when each $d_i \neq 0$, that is, $m = n$ in this situation.

Of course, the maximal a -lc property depends on a, H and divisors D_i . Actually, at P , it depends only on D itself; for $n = 0$, $D = 0$ and $D' > 0$ with $\text{Supp } D' = 0$ does not exist (cf. Example 2.9, (4) below). However if we consider a type (d_1, \dots, d_n) of D with all $d_i \neq 0$, maximal a -lc near Z also depends only on D itself. This condition can be stated as the maximal type (d_1, \dots, d_n) for D . It is unique up to permutation.

In general, we can add some $b_i = 0$ (see Example 2.9 for $n = 2$, or proof of ??^{2.1} below)

(2) In Definition 2.6, (vi-ix) the \mathbb{Q} -factorial assumption can be replaced by the \mathbb{Q} -Cartier property of prime divisors D_i (cf. Example 2.9, (3) below).

(3) In Definition 2.6, (vii) and (ix) for $a > 0$, (X, B) is plt near Z . Thus the reduced part $B_0 = \sum_{b_i=1} D_i$ of B is locally irreducible near each point of Z . Actually, in this situation $a \leq 1$ in dimension $d \geq 2$.

^{2.1?}

(4) Since we omit the lc assumption in Definition 2.6, (iv) and (xi), the sets $S_d(\text{Fano})$ and $S_d(\text{Mori-Fano cn})$ are not subsets of $S_d(\text{Mori-Fano})$, and their intersections with $S_d(\text{Mori-Fano})$ are determined by the lc property of (X, B) . Nonetheless acc is known for $S_d(\text{Fano})$ (see Example 2.9, (1) below) and expected for $S_d(\text{Mori-Fano cn})$. ACC for $S_d(\text{Mori-Fano cn})$ follows from Conjecture 1.2 in dimension d as so does acc for $S_d(\text{Fano})$ from the boundedness of nonsingular Fano varieties in Example 2.9, (1). Moreover, we can omit in both cases the assumption $b_i \leq 1$, and in the last case the assumption $\rho(X) = 1$ because the boundedness of cn Fano varieties in any dimension d is expected (cf. Example 2.9, (5) below).

Example 2.9 (1) $\mathcal{S}_d(\text{Fano})$ satisfies the acc in any dimension d by the boundedness of nonsingular Fano varieties X of dimension $\leq d$ [16]. Indeed, there exists a generic curve $C \subset X$ which positively intersects each prime divisor D_i on X and with bounded $(-K \cdot C)$: C is a generic curve section for an embedding $X \subset \mathbb{P}^N$ of bounded degree in a fixed projective space \mathbb{P}^N . Then for positive integers $m_i = (D_i \cdot C)$, $\sum b_i m_i = (B \cdot C) = (-K \cdot C)$. On the other hand, for any increasing sequence of types $(b_1^l, \dots, b_{n_l}^l)$, $l = 1, 2, \dots$, we can suppose that their sizes stabilize: $n_l = n$ for all $l \gg 0$, and each $b_i^l > 0$. Therefore, after taking a subsequence, the multiplicities $m_i^l = (D_i^l \cdot C)$ stabilize too: for each $i = 1, \dots, n$, $m_i^l = m_i > 0$ for all $l \gg 0$. Hence the types stabilize: for each $i = 1, \dots, n$, $b_i^l = b_i > 0$ for all $l \gg 0$ (cf. the proof of [25, Second Termination 4.9]).

Equivalently, for a finite set R of real (nonnegative) numbers, we can consider the set of types (d_1, \dots, d_n) such that each $d_i \geq 0$ and $\sum d_i = d \in R$ (cf. Example 4.2 below). For example, the case $R = \{2\}$ includes $\mathcal{S}_1(\text{Fano})$ with an extra condition, namely, each $d_i = b_i \leq 1$.

According to our arguments for the acc, the assumption that (X, B) is lc, and other ones on singularities of the log pair are not necessary, in particular, we can omit the assumption $b_i \leq 1$. Hence acc holds for $\mathcal{S}_d(\text{fano})$ as in [15, 18.15.1].

Let $X = \mathbb{P}^d$ be the projective space of dimension d , and D a generic hypersurface in \mathbb{P}^d of degree $d + 2$. Then

$$K_{\mathbb{P}^d} + \frac{d+1}{d+2}D \equiv 0.$$

Thus $((d+1)/(d+2)) \in \mathcal{S}_d(\text{Fano})$, and the dimension condition for sets in Definition 2.6, (iv-v), and (x-xi) is necessary to satisfy the acc in Conjecture 2.7. Similarly, for all other sets in the conjecture.

(2) However, for $\mathcal{S}_d(\text{global})$ and $\mathcal{S}_d(\text{Mori-Fano})$, the assumption that (X, B) is lc is very important. Let $Q_n \subset \mathbb{P}^{(n+1)}$ be the cone over a rational normal curve of degree n with a line generator L . Then for a generic hyperplane section H ,

$$-K = (n+2)L \equiv 3L + \frac{n-1}{n}H.$$

If we replace $3L$ by $L_1 + L_2 + L_3$ or $L_1/2 + \dots + L_6/2$ with distinct generators L_i , we construct strictly increasing sequences of types $(1, 1, 1, (n-1)/n)$ and $(1/2, 1/2, 1/2, 1/2, 1/2, 1/2, (n-1)/n)$ respectively. However, they are not in $\mathcal{S}_d(\text{global})$ because $(Q_n, L_1 + L_2 + L_3 + (n-1)H/n)$ and $(Q_n, L_1/2 + \dots + L_6/2 + (n-1)H/n)$ are not lc (at the vertex of Q_n).

(3) The \mathbb{Q} -factorial property in Definition 2.6, (vi-ix) is very important, too (cf. Remark 2.8, (2) above). Let $f: Y \rightarrow X$ be a contraction of a nonsingular rational curve C on a nonsingular 3-fold Y , and D_1, D_2 two nonsingular prime divisors on Y with intersection only along C with normal crossings. Set $-n = C^2$ on D_1 . For any $n \geq 2$ there exists such a contraction, e.g., toric one. Then $K + B \equiv 0/X$ for $B = D_1 + (n-2)D_2/n$. Thus we have a strictly increasing sequence of types $(1, (n-2)/n)$ which does not belong (entirely) to any set in Conjecture 2.7 if it satisfies the acc. That is in Definition 2.6, (v) the *proper* assumption is very important. The same types correspond to the image (X, f_*B) . However it does not belong to the sets in Definition 2.6, (vi-ix) because X is not \mathbb{Q} -factorial (cf. Remark 2.8, (2) above).

Let D_3 be a divisor which transversally intersects D_1, D_2 in a single point. (Again such a divisor exists in a toric case.) Then, for $B' = D_1 + D_3 + (n-1)/nD_2$, $K + B' \equiv 0/X$, and (X, f_*B') is exactly lc near $P = f(C)$: $\text{ldis}(X, f_*B') = 0$, but $(n-1)/n$ does not satisfies the acc and is not a counter example to ACC for thresholds since f_*D_2 is not a \mathbb{Q} -Cartier divisor. Similar examples can be constructed for any a instead of 0.

However it is expected that (the existence of \mathbb{Q} -factorialization implies that), for any strictly increasing types, $\text{ldis } 0$ is never attained at P .

(4) If in Definition 2.6, (vi) and (viii) $n = 0$, that is, the type itself is *maximal*, then $B = 0$, and $\text{mld}(P, X, 0) \geq a$ only but can be $\neq a$. More generally, $(X, H + 0)$ has maximal a -lc type \emptyset near Z and respectively at P if locally $K + H$ is \mathbb{R} -Cartier and (X, H) is a -lc near Z and respectively $\text{mld}(P, X, H) \geq a$ but not necessary $= a$.

(5) Acc of $\mathcal{S}_d(\text{Mori-Fano cn})$ holds for $d \leq 3$ by [17]. Moreover, one of the boundary properties of B , namely that each $b_i \leq 1$, and the condition $\rho(X) = 1$ are not necessary (cf. Remark 2.8, (4) above).

The main result of this section is

Theorem 2.10 ACC for mld's and lc thresholds in dimension $\leq d$ imply

(i) Acc for $\mathcal{S}_{a,d}(\text{local})$, $\overline{\mathcal{S}}_{a,d}(\text{local})$, $\mathcal{S}_{a,d}^0(\text{local})$, and $\overline{\mathcal{S}}_{a,d}^0(\text{local})$ with any $a > 0$;

Acc for $\mathcal{S}_d(\text{Mori-Fano cn})$, LMMP and ACC for mld's in dimension $\leq d$ imply

(ii) acc for $\mathcal{S}_d(\text{global})$, and $\mathcal{S}_d(\text{Mori-Fano})$;

(iii) acc for $\mathcal{S}_{d+1}^0(\text{Mori-Fano})$;

(iv) acc for $\mathcal{S}_{d+1}(\text{local})$, $\overline{\mathcal{S}}_{d+1}(\text{local})$, $\mathcal{S}_{d+1}^0(\text{local})$, and $\overline{\mathcal{S}}_{d+1}^0(\text{local})$; and

(v) ACC for lc thresholds in dimension $\leq d + 1$;

If in addition, LMMP holds in dimension $d + 1$, then

(vi) acc for $\mathcal{S}_{d+1}^0(\text{global})$ holds.

Addendum 2.11 Acc for $\mathcal{S}_d(\text{Mori-Fano cn})$ can be replaced by Conjecture 1.2 in dimension $\leq d$ (everywhere!) because the latter implies the former. (Cf. Example 2.9, (1) and Remark 2.8, (4) above.)

Theorem 2.12 Let $\Gamma \subset [0, 1]$ be a set satisfying the dcc. Then Theorem 2.10 holds when each $B \in \Gamma$.

Notation: We denote the corresponding subsets by $\mathcal{S}_d(\Gamma, \text{local})$, etc. Of course, we can apply it in other cases too: e.g., $\mathcal{S}_d(\Gamma, \text{Fano})$, and similar results hold in these cases.

Addendum 2.13 Moreover, then Γ can be assumed to be finite, that is, there exists its finite subset Γ_f such that $\mathcal{S}_{a,d}(\Gamma, \text{local}) = \mathcal{S}_{a,d}(\Gamma_f, \text{local})$, $\mathcal{S}_d(\Gamma, \text{global}) = \mathcal{S}_d(\Gamma_f, \text{global})$, etc.

We also have $\mathcal{S}_d(\Gamma, \text{Fano}) = \mathcal{S}_d(\Gamma_f, \text{Fano})$.

Corollary 2.14 (ACC for anticanonical (ac) thresholds) Assume acc for $\mathcal{S}_d(\text{Mori-Fano cn})$, LMMP and ACC for mld's in dimension $\leq d$. Then,

(i) acc for $\mathcal{S}_{a,d}(\text{global FT})$ where the $\mathcal{S}_{a,d}(\text{global FT})$ is the set of types (b_1, \dots, b_n) such that there is a FT variety X of dimension at most d , an ample Cartier divisor H on X , and a boundary B of type (b_1, \dots, b_n) such

that (X, B) is lc, and $K + B + aH \equiv 0$; in particular, $\mathcal{S}_{0,d}(\text{global FT}) = \mathcal{S}_d(\text{global FT})$ the subset in $\mathcal{S}_d(\text{global})$ corresponding to FT X ;

(ii) for subsets $\{0\} \subseteq \Gamma \subseteq [0, 1]$ and $\{0\} \subsetneq S \subseteq \mathbb{R}$ of nonnegative numbers satisfying the dcc,

(ACC) the following subset of $\mathbb{R}^+ \cup \{+\infty\} = \{r \in \mathbb{R} \mid r \geq 0\} \cup \{+\infty\}$

$\overline{\mathcal{A}}_{a,d}(\Gamma, S) = \{\text{act}_a(M, X, B) \mid X \text{ is complete, } \dim X = d, B \in \Gamma, (X, B) \text{ is lc,}$

$X \text{ is FT, } K + B \text{ is seminegative and } M \text{ is an } S\text{-Cartier divisor on } X\}$.

satisfies the acc; $+\infty$ corresponds to the case $M = 0$, where $t = \text{act}_a(M, X, B)$ means that $K + B + tM + aH \equiv 0$ on X for some ample Cartier divisor H on X , and the S -Cartier property means that M is a linear combination of ample Cartier divisors with multiplicities in S .

In particular, the ACC holds for $a = 0$ and $S = \{1\}$ which gives the anticanonical threshold (see [12, p. 47]).

(iii) the log Fano indices, that is, a maximal real positive number a such that $K + B + aH \equiv 0$ for ample Cartier divisors H , satisfies the acc for the lc pairs (X, B) , with FT variety X of dimension $\leq d$ with $B \in \Gamma$ as in (ii).

Addendum 2.15 Acc for $\mathcal{S}_d(\text{Mori-Fano cn})$ can be replaced by Conjecture 1.2 in dimension $\leq d$.

Remark 2.16 We expect that Corollary 2.14 holds when FT is omitted, that is, for $a > 0$, (X, B) is just a lc Fano variety as in Theorem 2.10 (ii).

Corollary 2.17 Let $\Gamma \subset [0, 1]$ be a set satisfying the dcc. Then Corollary 2.14 (i) holds when each $B \in \Gamma$.

Addendum 2.18 Moreover, then Γ can be assumed to be finite in Corollary 2.17 (i), that is, there exists its finite subset Γ_f such that $\mathcal{S}_{a,d}(\Gamma, \text{global}) = \mathcal{S}_{a,d}(\Gamma_f, \text{global})$.

Remark 2.19 If $\rho(X) = 1$, then a -anticanonical (a -ac) threshold is well defined for any ample Cartier divisor H , e.g., for such a generator in $\text{Pic}(X)$: there exists a (unique) real number t such that

$$K + B + tM + aH \equiv 0.$$

If $K + B + aH$ is seminegative then $t \geq 0$. The condition $\rho(X) = 1$ replaces the \mathbb{Q} -factorial property of the local case. For $a = 0$, we get the ac threshold [12, p. 47].

In Corollaries 2.14 and 2.17 we can suppose that a is varying in a dcc set. Then it is expected that the corresponding thresholds t in dimension $\leq d$ satisfies the acc (ACC conjecture). This is clear from the proof of Corollaries 2.14 and 2.17 (below).

Proof (of Corollary 2.14)

The case (i) follows from its counterpart in Theorem 2.10 (ii) for $a = 0$. To apply the theorem we replace the boundary B with $B + aH$ with an appropriate choice of H (see proof of (ii) below). The type of B will be extended by that of aH . Since the latter has finitely many possible multiplicities, acc for B is equivalent to acc of extended types to which we apply Theorem 1.10.

(ii) Suppose that such thresholds do not satisfy the acc. Let Ω be an infinite set of such thresholds, which satisfies the dcc. Now take a $t \in \Omega$. Also take X, B, H and $M \neq 0$ corresponding to t . Since M is S -Cartier, there are $s_j \in S$ and ample Cartier divisors H_j such that $M = \sum_j s_j H_j$. By anticanonical boundedness, ts_j is bounded. By effective base point freeness [14], there is h , a natural number (not depending on X, H, H_j but depending only on the dimension d), such that hH_j and hH are free divisors and $(a/h), (ts_j/h) \in [0, 1]$.

Write $M \equiv \sum_j (ts_j/h)H'_j$ where $H'_j \in |hH_j|$ is a general member. For $d \geq 2$, general H'_j is irreducible. For $d = 1$, the number of components in H'_j is bounded. Thus we can assume that

$$(X, B + \sum_j (ts_j/h)H'_j + (a/h)H')$$

is lc where $H' \in |hH|$ is general. In particular, the possible multiplicities of $B + \sum_j (ts_j/h)H'_j + (a/h)H'$ satisfies the dcc. Now use Addendum 2.13 for $S_d(\Gamma_f, global)$.

(iii) If $S = 1$ then any M is an ample Cartier divisor H and any ac threshold satisfies $K + B + tH \equiv 0$. Thus possible t satisfy the acc by (ii). This implies acc for the Fano indices.

□

Proof (of Addendum 2.15)

Same as Addendum 2.11 below. \square

Proof (of Corollary 2.17)

There is a natural number h such that hH is free where h does not depend on X, H . Choose h big enough such that $a/h \in [0, 1]$. Now for a general $H' \in |hH|$, $K + B + (a/h)H' \equiv 0$ and $(X, B + (a/h)H')$ is klt. Since $B \in \Gamma$, possible multiplicities of $B + (a/h)H'$ satisfy the dcc because Γ satisfies the dcc, a, h are fixed and H' is a reduced Cartier divisor. Therefore, the result follows from acc for $\mathcal{S}_d(\text{global})$ as in Theorem 2.12 (ii).

\square

Proof (of Addendum 2.18)

We can use the extension of boundaries B as in the proof of Corollary 2.14, and then use Addendum 2.13. \square

Proof (of Theorem 2.10) Each statement follows from the same statement under the assumption $B \in \Gamma$ for some Γ under the dcc. Thus it follows from the corresponding statement in Theorem 2.12.

For the statements (iii) and (vii) such a set Γ is given by assumptions.

In the other cases, we need to verify that each increasing sequence of types $(b_1^l, \dots, b_{n_l}^l), l = 1, 2, \dots$, stabilizes. By definition of the ordering, their sizes stabilize: $n_l = n$ for all $l \gg 0$. Then for the corresponding pairs (X_l, B_l) , $B_l = \sum_{i=1}^n b_i^l D_i^l \in \Gamma$ where

$$\Gamma = \{b_i^l \mid i = 1, \dots, n, \text{ and } l = 1, 2, \dots\} = \cup_{i=1}^n \Gamma_i, \text{ and } \Gamma_i = \{b_i^l \mid l = 1, 2, \dots\}.$$

Since each sequence $b_i^l, l = 1, 2, \dots$, increases, the sets Γ_i and Γ satisfy the dcc.

Proof (of Addendum 2.11) Immediate by Remark 2.8 (4). \square

Proof (of Theorem 2.12) (i) By inclusions

$$\mathcal{S}_{a,d}^0(\Gamma, \text{local}) \subset \mathcal{S}_{a,d}(\Gamma, \text{local}) \text{ and } \overline{\mathcal{S}}_{a,d}^0(\Gamma, \text{local}) \subset \overline{\mathcal{S}}_{a,d}(\Gamma, \text{local})$$

It is enough to prove acc for the ambient sets. On the other hand, it is known (almost by definition) that, for any type (b_1, \dots, b_n) in $\overline{\mathcal{S}}_{a,d}(\Gamma, \text{local})$ and any of its component b_i there exists a type (b'_1, \dots, b'_m) in $\mathcal{S}_{a,d}(\Gamma, \text{local})$ with a component $b'_i = b_i$: take a type corresponding to an appropriate intersection point of $D_i \cap Z$ (cf. [15, 18.19.1]). Thus it is enough to prove acc for $\mathcal{S}_{a,d}(\Gamma, \text{local})$.

Let $(b_1^l, \dots, b_{n_l}^l), l = 1, 2, \dots$, be an increasing sequence of types in the set $\mathcal{S}_{a,d}(\Gamma, \text{local})$. The fact that this sequence of types is increasing, implies that n_l is bounded and it stabilizes: $n_l = n$ for $l \gg 0$. So we can assume that $n_l = n$ for any l . Since the sequence stabilizes for $n = 0$, we can suppose that $n \geq 1$. According to construction we have a sequence of pointed \mathbb{Q} -factorial varieties $P_l \in X_l$ of dimension d and prime divisors D_1^l, \dots, D_n^l on X_l such that $B_l = \sum b_i^l D_i^l$ is a boundary,

$$P_l \in \cap D_i^l \text{ and } \text{mld}(P_l, X_l, B_l) = a.$$

and P_l has codimension ≥ 2 , or $(b_1^l, \dots, b_{n_l}^l)$ is maximal lc at some point Q_l near P_l (possibly of codimension 1 but not a closed point since $a > 0$). Note that in the last case, all D_i^l with $b_i^l < 1$ pass through Q_l by the maximal lc property. Thus a subvector of $(b_1^l, \dots, b_{n_l}^l)$ and so the set of all $(b_1^l, \dots, b_{n_l}^l)$ satisfies the acc by ACC for lc thresholds in dimension $\leq d - 1$ (see the arguments below). Thus taking a subsequence we can suppose that the first case: $\text{mld}(P_l, X_l, B_l) = a$.

We can choose a subsequence such that the limits below exist (e.g., unique) by monotonic increasing and boundedness (≤ 1 , see Example 2.4 above)

$$b_i = \lim_{i \rightarrow \infty} b_i^l \text{ for } i = 1, \dots, n, \quad B' = \sum_{i=1}^n b_i D_i^l, \quad \text{and} \quad R = \{b_i \mid i = 1, \dots, n\}.$$

Then for any $\tau > 0$, $\|B_l - B'\| < \tau$ for all $l \gg 0$. Note that $K_{X_l} + B'$ is \mathbb{R} -Cartier because X_l is \mathbb{Q} -factorial. By ACC for lc thresholds and Proposition 2.1 for $X = X_l, B = B_l, P = P_l$, and every $l \gg 0$, we can assume that (X_l, B') is lc near P_l , and a -lc at P_l ; $a > 0$ by assumptions. Therefore, $\text{mld}(P_l, X_l, B') \geq a = \text{mld}(P_l, X_l, B_l)$.

We can derive the lc property, (4) of Main Proposition 2.1, of (X_l, B') from the assumptions as follows (cf. proof of Proposition 2.5). If (X_l, B') is not lc near P_l for $l \gg 0$, then (since X_l is \mathbb{Q} -factorial), for infinitely many l there is $G_l = \sum_{i=1}^n g_i^l D_i^l$ such that $B_l \leq G_l \leq B'$ and such that (X_l, G_l) is precisely lc (i.e., lc but not klt) near P_l . The set of multiplicities of those G_l satisfies the dcc and is not finite. We can assume that $\{g_1^l\}$ is not finite but increasing and that D_1^l contains a lc centre. So, g_1^l is the lc threshold of D_1^l with respect to $(X_l, G_l - g_1^l D_1^l)$. This contradicts ACC for lc thresholds.

Now by Monotonicity of mld's (see [Sh. 3-fold log flips, 1.3.3]) and since $B' \geq B_l$, the sequence stabilizes: $B' = B_l$ for every $l \gg 0$. This proves the acc.

(ii) This will be established in the weak finiteness section modulo (v) in dimension d which can be assumed by induction.

(iii) Let (X_l, B_l) be a pair of dimension $d + 1$ for each l such that $B_l = \sum_{i=1}^{n_l} b_i^l D_i^l$ has a type $(b_1^l, \dots, b_{n_l}^l) \in \mathcal{S}_{d+1}^0(\Gamma, \text{Mori-Fano})$ such that these types are strictly increasing with respect to l . We can assume that $\{b_1^l\}$ is a strictly increasing sequence. By assumptions, (X_l, B_l) is lc but not klt. We can take a strictly lt model (Y_l, B_{Y_l}) for (X_l, B_l) ; this needs special termination and existence of flips in dimension $d + 1$ which follow from LMMP in dimension d [11].

Suppose that D_1^l intersects $\text{LCS}(X_l, B_l)$ for infinitely many l . Thus, the birational transform of D_1^l intersects the reduced part of B_{Y_l} . Then using adjunction, restrict to an appropriate component in the reduced part of the boundary which intersects D_1^l . The multiplicities that we get are of the following type

$$1 > b' = \frac{m-1}{m} + \sum \frac{c_i}{m} b_i^l$$

with natural numbers m, c_i [25, 3.10].

Lemma 2.20 *Any set of such b' satisfies the dcc where $b_i^l \in \Gamma$ (cf. [25, Second termination 4.9] and [15, 18.21.4]). Moreover, if it is finite, then the set of b_i^l is finite.*

Here the finiteness of the set of b' comes from induction on d (part (ii)). So we get a contradiction.

Then, we can assume that D_1^l does not intersect $\text{LCS}(X_l, B_l)$. There is an extremal ray R_l on Y_l such that the birational transform of D_1^l intersects R_l positively. If R_l is of fibre type, then by restricting to the general fibre

and using induction on d we get a contradiction. So assume otherwise. The reduced part of B_{Y_l} intersects R_l , otherwise R_l corresponds to a flipping or divisorial type extremal ray R'_l on X_l . This is not possible since $\rho(X_l) = 1$.

Let $(Y_l^+, B_{Y_l}^+)$ be the model after operating on R_l (i.e., after a flip or divisorial contraction). Thus, the birational transform of D_1^l on Y_l^+ intersects the reduced part of $B_{Y_l}^+$. We get a contradiction as above by restricting to a component of the reduced part of $B_{Y_l}^+$.

(iv) As above it is enough to verify acc for $\mathcal{S}_{d+1}(\Gamma, \text{local})$. Suppose that there are (X_l, B_l) such that $B_l = \sum_{i=1}^{n_l} b_i^l D_i^l$ has a type $(b_1^l, \dots, b_{n_l}^l)$ in $\mathcal{S}_{d+1}(\Gamma, \text{local})$ such that these types are strictly increasing with respect to l . We can assume that the set $\{b_i^l\}$ is strictly increasing. If for infinitely many l , D_1^l passes through a lc centre of (X_l, B_l) of dimension ≥ 1 , then by taking hyperplane sections, we reduce the problem to dimension $\leq d$ for which we may assume that the theorem is already proved.

So, we assume that none of D_1^l passes through a lc centre of (X_l, B_l) of dimension ≥ 1 . Now, take a strictly lt model of each (X_l, B_l) . Then using adjunction, restrict to an appropriate exceptional divisor in the reduced part of the boundary which intersects the birational transform of D_1^l . The multiplicities that we get are as in Lemma 2.20. We get a contradiction by (ii).

(v) This is proved exactly as in (iv) using induction on d (part (ii)).

(vi) Let (X_l, B_l) be a $d + 1$ -dimensional pair for each l such that $B_l = \sum_{i=1}^{n_l} b_i^l D_i^l$ has a type $(b_1^l, \dots, b_{n_l}^l) \in \mathcal{S}_{d+1}^0(\Gamma, \text{global})$ such that these types are strictly increasing with respect to l . We can assume that $\{b_1^l\}$ is a strictly increasing sequence. By assumptions, (X_l, B_l) is lc but not klt. As in the proof of Proposition 4.1, run the anti-LMMP on D_1^l . After finitely many steps, either we get a fibration or the Mori-Fano case. For the former case we use induction on d and for the latter case use (iii).

□

To prove Addendum 2.13 we use the following

Lemma 2.21 *Any suborder $\mathcal{S} \subset \mathcal{B}$ satisfies the acc if each (b_1, \dots, b_n) in \mathcal{S} is in R , that is, each $b_i \in R$, for some fixed finite set of real numbers R . The converse holds, that is, there exists finite R such that each (b_1, \dots, b_n) in \mathcal{S} is in R , when \mathcal{S} satisfies the acc, (b_1, \dots, b_n) in \mathcal{S} is in Γ for some Γ under the dcc, and with each (b_1, \dots, b_n) in \mathcal{S} some abridged type $(b'_1, \dots, b'_{n'})$ with bounded n' is in \mathcal{S} . Abridged means that both types have the same components*

$b_i \neq 0$.

Proof First suppose that each (b_1, \dots, b_n) in \mathcal{S} is $\in R$ for some fixed finite set of real numbers R . If \mathcal{S} does not satisfy the acc, then we can find a strictly increasing set of elements β_1, β_2, \dots in \mathcal{S} . We can assume that they all have the same size, that is, there is n such that $\beta_l = (b_1^l, \dots, b_n^l)$. Since R is finite, there are only finitely many such types, a contradiction.

Now suppose that we have \mathcal{S} satisfying the acc and other assumptions of the lemma. Let $R \subset \Gamma$ be the set of all real numbers appearing as a component in some type in \mathcal{S} . It is enough to prove that R is finite. If R is not finite, then there is a strictly increasing sequence $\{r_l\}_{l \in \mathbb{N}} \subset R$ and an infinite set of types β_1, β_2, \dots in \mathcal{S} such that r_l is a component of β_l . Replacing each β_l by an abridged one $\beta_l = (b_1^l, \dots, b_{n'}^l)$, we can assume that $b_1^l = r_l$. If $n' = 1$, then we get a contradiction. Otherwise, consider types $\lambda_l = (b_2^l, \dots, b_{n'}^l)$ and use induction on size and the dcc property of Γ to get an infinite increasing subsequence of β_i . By construction it is strictly increasing. This is a contradiction, because the set $\{\beta_l\}_{l \in \mathbb{N}}$ does not satisfy acc.

□

Proof (of Addendum 2.13)

By Theorem 2.12, each set satisfies the acc. Now Lemma 2.21 guarantees the existence of $\Gamma_f \subseteq \Gamma$. □

We also proved the following.

Corollary 2.22 *Under the assumptions of Theorem 2.12, (ii) implies (iii), (iv) and (v).*

Proof See the proof of Theorem 2.12 above.

Corollary 2.23 *In Theorems 2.10 and 2.12 and Addenda for $d \leq 4$, we can drop the assumption on LMMP in dimension ≤ 4 .*

Proof The LMMP in dimension 4 follows from ACC of mld's in dimension 4 [22, Cor 5].

Corollary 2.24 *In (ii-v) of Theorems 2.10 and 2.12 and Addenda for $d = 3$, we can drop the assumption on LMMP in dimension ≤ 3 and acc for $\mathcal{S}_3(\text{Mori-Fano cn})$.*

Proof LMMP and boundedness of Mori-Fano cn varieties are known in dimension 3 [23][17].

Corollary 2.25 *In (i-v) of Theorems 2.10 and 2.12 and Addenda for $d = 2$, we can drop the assumption on LMMP in dimension ≤ 2 , acc for $\mathcal{S}_2(\text{Mori-Fano cn})$ and ACC for mld's in dimension 2.*

Proof The same plus the fact that ACC for mld's is known in dimension 2 [26][2].

3 Log twist

In this section, we introduce a construction which is crucial for us and which generalizes [reminds] Sarkisov links of Type I and II [18, Theorem 13-1-1], and we establish its basic properties.

Construction 3.1 (Log Twist) Let X be a d -dimensional Mori-Fano variety, and B be a boundary such that (X, B) is klt and noncanonical in codimension ≥ 2 (noncn for short), and $K + B \equiv 0$. Fix a prime b-divisor (exceptional divisor) E such that $a := 1 - e := \text{ldis}(X, B) = a(E, X, B)$. Assume the LMMP in dimension d . Then there exists (and is unique for the fixed E) the following transformation of X which we call a *log twist*:

$$\begin{array}{ccccccc}
 Y = Y_1 & \dashrightarrow & Y_2 & \dashrightarrow & \cdots & \dashrightarrow & Y' = Y_n \\
 \downarrow f & & & & & & \downarrow f' \\
 X & & & & & & X'
 \end{array}$$

where $f: Y = Y_1 \rightarrow X$ is an extremal divisorial extraction of E , all horizontal modifications $Y_i \dashrightarrow Y_{i+1}, i = 1, \dots, n - 1$, are extremal $-E$ -flips, and $f': Y' = Y_n \rightarrow X'$ is either a Mori-Fano fibration with $\dim X' \geq 1$ and the crepant boundary $B_{Y'}$ such that

- (1) $(Y', B_{Y'})$ is klt, and $K_{Y'} + B_{Y'} \equiv 0$, or
 an extremal divisorial contraction of a divisor E' onto a Mori-Fano variety X' with the crepant boundary $B_{X'}$ such that
- (2) $(X', B_{X'})$ is klt, and $K_{X'} + B_{X'} \equiv 0$.
- In addition, the following two facts hold:
- (3) If D is an effective divisor on Y which is (numerically) seminegative over X then its birational transform D' on Y' is semipositive over X' , and strictly positive when $D \neq 0$.
- (4) Thus, if in (3) D' is also seminagative over X' , then $D = D' = 0$.

Caution 3.2 Since we consider log discrepancies of the pair (X, B) , the first blowup $Y \rightarrow X$ can be in a terminal and even nonsingular point of X .

Definition 3.3 We say that the twist has *Type I* if $Y' \rightarrow X$ is a fibration. Otherwise the twist has *Type II* (cf. [18, Theorem 13-1-1]).

Lemma 3.4 (cf. [23, Theorem 3.1]) *Assume the LMMP in dimension d . Let (X, B) be a log pair, and E be an exceptional prime divisor of X such that*

- a) (X, B) is klt;
- b) $\dim X = d$; and
- c) $a = a(E, X, B) < 1$.

Then there exists an extraction of E : that is, a contraction

$$f: Y \rightarrow X$$

with the only exceptional divisor E such that $-E$ is ample over X .

Moreover, Y and f are unique, and if X is \mathbb{Q} -factorial, then so does Y and f is extremal, that is $\rho(Y/X) = 1$.

Proof Let $g: W \rightarrow X$ be a log resolution of (X, B) such that E is a divisor on W . Let $B_W = B^\sim + \sum_{E_i \neq E} E_i$ where B^\sim is the birational transform of B and E_i are the exceptional/ X divisors on W . Run the LMMP/ X on $K_W + B_W$. At the end, we get a model \overline{W} where $K_{\overline{W}} + B_{\overline{W}}$, the pushdown of $K_W + B_W$, is nef (and big)/ X . By construction, \overline{W} is \mathbb{Q} -factorial and $K_{\overline{W}} + B_{\overline{W}}$ is dlt. All the E_i are contracted/ \overline{W} except E , by the negativity lemma [25] which in turn implies that $K_{\overline{W}} + B_{\overline{W}}$ is klt.

In fact, $K_{\overline{W}} + B_{\overline{W}} \equiv -eE/X$ where $e = 1 - a > 0$, and so $-E$ is nef and big/ X by construction. Moreover, $K_{\overline{W}} + B_{\overline{W}}$ is semiample/ X [11,

Theorem 7.1], so $\overline{W} \rightarrow X$ can be factored through contractions $h: \overline{W} \rightarrow Y$ and $f: Y \rightarrow X$ such that $K_{\overline{W}} + B_{\overline{W}} \equiv 0/Y$ and $K_Y + B_Y$, the pushdown of $K_{\overline{W}} + B_{\overline{W}}$, is ample/ X . Since $E \not\equiv 0/X$, E is a divisor on Y and it is numerically negative/ X which implies that E is the only exceptional/ X divisor on Y . Note that, Y is the log canonical model of $K_W + B_W$ [23, Definition 2.1]. This implies the uniqueness of f and Y .

Now suppose that X is \mathbb{Q} -factorial. Let D be a Weil divisor on Y . If $D = E$, then D is \mathbb{Q} -Cartier by construction. If $D \neq E$, let $D' = f_*D$ and $D'' = f^*D'$. Since D' is \mathbb{Q} -Cartier, so is D'' . On the other hand $D'' = D + \alpha E$ for some rational number α . Since E is \mathbb{Q} -Cartier, so is D . This implies that Y is also \mathbb{Q} -factorial. Moreover, this observation also shows that $\rho(Y/X) = 1$.

□

Remark 3.5 We expect that Lemma 3.4 holds in the lc case as well. One then needs not only the LMMP but also the semiampleness for dlt pairs in the birational case.

Proof (of Construction 3.1) There is an extremal extraction $f: Y = Y_1 \rightarrow X$ of E by Lemma 3.4. Let B_Y be the crepant pullback of B on Y . According to our assumptions, B_Y is a boundary. Moreover,

- (5) $a < 1$ and $0 < e \leq 1$; and
- (6) $\text{ldis}(Y, B_Y) \geq \text{ldis}(X, B) = a$.

By construction $K_Y + B_Y \equiv 0$, and Y is \mathbb{Q} -factorial by Lemma 3.4.

Now we run the LMMP starting from Y with respect to $K_Y + B_Y - eE \equiv -eE$. By (5) this is the same as with respect to $-E$. Since E is always positive on the generic member of some covering family of curves, after finitely many flips $Y_i \dashrightarrow Y_{i+1}$, we get an extremal contraction $f': Y' = Y_n \rightarrow X'$ which is not a flipping, that is, f' is a Mori-Fano fibration or a divisorial contraction, contracting E' . The first case gives a twist of Type I, and the second one gives a Type II twist.

In both cases, E is positive with respect to f' , and also so does E with respect to the flipping contraction of each flip $Y_i \dashrightarrow Y_{i+1}$. In particular, E is a divisor on X' if f' has Type II. In both cases, the flips are log flops with respect to $K_{Y_i} + B_{Y_i}$, and all B_{Y_i} are (crepant) boundaries. Thus, both Type I and Type II twists satisfy property (1), and in addition, the Type II also satisfies (2). By (6) in both cases,

(6') $\text{ldis}(Y', B_{Y'}) \geq \text{ldis}(X, B) = a$.

However, after the contraction in Type II, this may fail (see Definition 3.6).

By construction, $\rho(X') = \rho(Y') - 1 = \rho(Y_i) - 1 = \rho(Y) - 1 = \rho(X) = 1$, and, for Type II, X' is \mathbb{Q} -factorial. Hence, for this case, since E is not exceptional on X' and by (5), $-K_{X'}$ is ample which means that X' is a Mori-Fano variety.

Now let D be an effective divisor on Y which is seminegative/ X . According to the previous paragraph, each $\rho(Y_i) = 2$. Let R_1 be an extremal ray corresponding to the contraction $Y \rightarrow X$, and R_2 be the other extremal ray. By our assumption, D is seminegative on R_1 . Since $D \geq 0$, D is semipositive on R_2 . Thus the first flip $Y_1 \dashrightarrow Y_2$ is a $-D$ -flip or $-D$ -flop. Similarly, each next flip $Y_i \dashrightarrow Y_{i+1}$ is a $-D_i$ -flip or $-D_i$ -flop where D_i denotes the birational transform of D . This flip corresponds to the second extremal ray R_2 , whereas R_1 is flipped on Y_{i+1} . So, D_i is always seminegative on R_1 and semipositive on R_2 . For $Y' = Y_n$ and $D' := D_n$, this gives the semipositivity in (3) because R_2 determines the last contraction $Y' \rightarrow X'$. (This proves also the uniqueness of the twist for the fixed E .)

Moreover, if D' is also seminegative over X' , then D' is seminegative on Y' and thus it is 0 because $D' \geq 0$. Thus $D = 0$ too. This proves (4) and the strict positivity in (3). \square

Definition 3.6 A log twist is called *final*, if

- (a) $Y' \rightarrow X'$ is a fibration, that is, it is of Type I; or
- (b) $Y' \rightarrow X'$ is of Type II, X' is noncn, and $\text{ldis}(X', B_{X'}) = 1 - e'$ where $e' = \text{mult}_{E'} B_{Y'}$; or
- (c) $Y' \rightarrow X'$ is of Type II, and X' is canonical (cn for short).

Indeed, if a log twist is not final, it is of Type II with noncn X' . Thus we can take a log twist of $(X', B_{X'})$. In case (b) of Definition 3.6, an inverse log twist can be constructed. Otherwise we expect that a sequence of log twists:

$$(7) \quad (X, B) \dashrightarrow (X', B_{X'}) \dashrightarrow \cdots \dashrightarrow (X^{(i)}, B_{X^{(i)}}) \dashrightarrow \cdots$$

terminates, where each log twist is nonfinal, except possibly for the last one.

Proposition 3.7 (Termination of log twists) *Suppose that for a sequence as in (7), there exists a real number $a_0 < 1$ such that*

$$(UBD) \quad \text{each } a^{(i)} = \text{mld}(X^{(i)}, B_{X^{(i)}}) \leq a_0$$

Then, assuming LMMP in dimension d , the sequence terminates and universally with respect to a_0 , that is, the sequence is finite and the number of twists in it is bounded whereas the bound depends only on a_0 and the dimension of X .

By ACC for mld's in dimension d near 1, we mean that 1 is not an upper limit in the mld spectrum (1.3) in dimension d . This is a special case of Conjecture 1.3.

Lemma 3.8 ACC for mld's in dimension d for $\Gamma = \{0\}$ and only near 1 implies UBD.

More precisely, there exists $a_0 < 1$ depending only on d such that, for any log pair (X, B) , with \mathbb{Q} -factorial noncn X of dimension d ,

$$a = \text{ldis}(X, B) \leq a_0.$$

Proof Put

$$a_0 = \max\{\text{ldis}(X, 0) \mid \dim X = d\} \cap [0, 1).$$

Then by ACC for mld's, $a_0 < 1$, and for any log pair in the lemma, $a = \text{ldis}(X, B) \leq \text{ldis}(X, 0) \leq a_0$. \square

Corollary 3.9 Assume that LMMP and ACC for mld's near 1 hold in dimension d . Let $\Gamma \subset [0, 1]$ be a set of real numbers satisfying dcc. Then, universal termination of Proposition 3.7 holds in dimension d for $B \in \Gamma$ without the UBD assumption, that is, the length of a sequence of nonfinal twists for (X, B) is bounded by a natural number N , where $\dim X = d$ and $B \in \Gamma$; and N depends only on d .

Proof Immediate by Proposition 3.7 and Lemma 3.8 because each X is \mathbb{Q} -factorial. \square

Addendum 3.10 Let

$$\Gamma' = \Gamma \cup \{1 - \text{ldis}(X, B) \mid (X, B) \text{ as in } (\gamma), B \in \Gamma, X \text{ noncn and } \dim X = d\}$$

and let $\Gamma^{(i)} = (\Gamma^{(i-1)})'$. Then, the increasing sequence

$$\Gamma \subseteq \Gamma' \subseteq \dots \subseteq \Gamma^{(i)} \subseteq \dots$$

stabilizes, and satisfies the dcc, that is, the union

$$\Gamma^\infty = \bigcup_1^N \Gamma^{(i)} = \Gamma^{(N)}$$

and it satisfies the dcc.

Proof More generally, in the addendum, we can take lc pairs (X, B) with noncn (even non terminal) X , $\dim X = d$, and $B \in \Gamma$ but we need to assume that $i \leq N$, that is, we state that $\Gamma^{(N)}$ satisfies the dcc. The assumption holds for the pairs in Proposition 3.7 by Corollary 3.9.

Then the new Γ' satisfies the dcc as a union of two sets under the dcc. The second set

$$\{1 - \text{ldis}(X, B) | (X, B) \text{ is lc, but noncn, } \Gamma \in B \text{ and } \dim X = d\},$$

satisfies the dcc by ACC for mld's. Etc. \square

Theorem 3.11 *Let $(X/Z, B = \sum b_i D_i)$ be a log pair of dimension d such that:*

- a) $X \rightarrow Z$ is a proper contraction;*
- b) (X, B) is lc; and*
- c) $K + B$ is seminegative/ Z .*

Then LMMP in dimension d implies:

$$\rho^W(X/Z) \geq -\dim X + \sum b_i$$

where ρ^W is the Weil number, that is, the rank of Weil divisors on X modulo numerical equivalence.

Proof See [19, Theorem 2.3].

Corollary 3.12 *Under the assumptions of Theorem 3.11, $\sum b_i \leq \dim X + 1$ when $\rho^W(X/Z) = 1$.*

Note that $\rho^W = \rho$ for \mathbb{Q} -factorial X .

Proof Obvious by Theorem 3.11.

Proof (of Proposition 3.7) Note that if a log twist is not final then (6') implies

$$(6'') \quad a' = \text{ldis}(X', B_{X'}) = \text{ldis}(Y', B_{Y'}) \geq \text{ldis}(X, B) = a$$

and

(8) for the prime divisor E' on X , $1 \geq a(E', X', B_{X'}) = a(E', X, B) > a'$.

Similarly, for any nonfinal twist $X^{(i)} \dashrightarrow X^{(i+1)}$,

(8') the prime divisor $E^{(i+1)}$
contracted by $X^{(i)} \dashrightarrow X^{(i+1)}$ satisfies

$$1 \geq a(E^{(i+1)}, X^{(i)}, B_{X^{(i)}}) = a(E^{(i+1)}, X^{(i+1)}, B_{X^{(i+1)}}) > a^{(i+1)}.$$

Thus the sequence of mld's $a, a', \dots, a^{(i)}, \dots$ increasing:

$$a \leq a' \leq \dots \leq a^{(i)} \leq \dots,$$

or equivalently,

$$(9) \quad e = 1 - a \geq e' = 1 - a' \geq \dots \geq e^{(i)} = 1 - a^{(i)} \geq \dots$$

Since the set of boundary multiplicities can be expanded by twists, it might not be expected that the sequence $a^{(i)}$ stabilizes or that it is finite.

To establish this we introduce the *difficulty* $d^{(i)}$ of the $(X^{(i)}, B_{X^{(i)}})$ to be: the number of prime components D_i of $B_{X^{(i)}}$ with $b_i = \text{mult}_{D_i} B_{X^{(i)}} \geq e^{(i)}$.

The difficulty increases: for any nonfinal twist $X^{(i)} \dashrightarrow X^{(i+1)}$,

$$(10) \quad d^{(i+1)} \geq d^{(i)} + 1.$$

Indeed, $e^{(i+1)} \leq e^{(i)}$ by (9), and by (6') none of the prime boundary components D_i of $B_{X^{(i)}}$ with $b_i = 1 - a(D_i, X^{(i)}, B_{X^{(i)}}) \geq e^{(i)} \geq e^{(i+1)} = 1 - a^{(i+1)}$ is contracted. On the other hand, the nonfinal twist of $X^{(i)}$ blows up a new prime component with the multiplicity $e^{(i)} \geq e^{(i+1)}$ which adds 1 in the inequality (10).

Note now, that by Theorem 3.11 there exists a natural number N such that, on each Mori-Fano variety of dimension d , a boundary B has at most N boundary components with multiplicities $\geq e_0 = 1 - a_0 > 0$ if $K + B$

is seminegative, and (X, B) is lc. More precisely, we can take any $N \geq (d+1)/e_0$.

By UBD each $a^{(i)} \leq a_0$, except for the last one (if such exists), or equivalently, each $e^{(i)} \geq e_0 = 1 - a_0$, except for the last one. Thus we have at most N nonfinal twists. \square

Lemma 3.13 *Let $\Gamma \subset [0, 1]$ be a set of real numbers satisfying the dcc, and $X_i \dashrightarrow X'_i$ be a family of birational, (e.g., nonfinal) log twists in dimension d such that*

- a) $B_i \in \Gamma$, and
- b) *the set of multiplicities of all boundaries B_i is infinite.*

Then ACC for mld's and lc thresholds in dimension d imply that the set of multiplicities of all boundaries $B_{X'_i}$ is infinite too.

Proof By our assumptions each X'_i is a birational modification of X_i with a divisorial contraction $Y'_i \rightarrow X'_i$. In particular, all crepant boundaries $B_{Y'_i}$ and $B_{X'_i}$ are well defined.

By the dcc of Γ and after taking a subsequence, we can suppose that there exists a sequence of prime divisors D_i on X_i such that the corresponding sequence of boundary multiplicities $b_i = \text{mult}_{D_i} B_i$ is strictly increasing, and the increasing holds for other multiplicities. If infinitely many members of the sequence D_i are nonexceptional on X'_i , the required statement holds.

Otherwise, after taking a subsequence, we can suppose that each D_i is contracted on X'_i , that is, $D_i = E'_i$ on Y'_i , and it is numerically negative on Y'_i over X'_i . Thus by the property (3) of twists, D_i is numerically positive on Y_i over X_i . Hence each D_i passes through $P_i = C_{X_i} E_i$, the center of E_i on X_i . According to ACC for mld's and lc thresholds, Proposition 2.5 and the monotonicity of multiplicities, the set of new boundary multiplicities $e_i = 1 - \text{mld}(P_i, X_i, B_i)$ is not finite. But each component E_i is nonexceptional on X'_i which gives the required infinity in this case too. \square

Addendum 3.14 *We can omit ACC for lc thresholds in Lemma 3.13 if we assume the LMMP and Conjecture 1.2 in dimension $d - 1$.*

Proof Will be given in section 5.

Corollary 3.15 *Assume ACC for mld's and lc thresholds in dimension d . Under the assumptions of Corollary 3.9 let (X_i, B_i) be a family of pairs in dimension d such that*

- a) *each has at least one twist,*
- b) *$B_i \in \Gamma$, and*
- c) *the set of multiplicities of all boundaries B_i is infinite.*

Then, for their final twists $X_i^{(j)} \dashrightarrow X_i^{(j+1)}$, the set of multiplicities of all boundaries $B_{X_i^{(j)}}$ is infinite too. (Taking a subsequence we can suppose also that j is the same for all final twists.)

Note that the final twists exist by Corollary 3.9.

Proof Immediate by Lemma 3.13 and induction on j . (The statement about a subsequence follows from the universal termination in Corollary 3.9.)

Addendum 3.16 *We can omit ACC for lc thresholds in Corollary 3.15 if we assume the LMMP and Conjecture 1.2 in dimension $d - 1$.*

Proof Will be given in section 5.

4 Weak finiteness

Let $\mathcal{F}_d(\Gamma)$ be the set of log pairs (X, B) where X is a d -dimensional projective variety, and B is a boundary such that $B \in \Gamma$, (X, B) is lc, and $K + B \equiv 0$.

Proposition 4.1 (Weak finiteness) *We assume LMMP, ACC for mld's and Boundedness Conjecture 1.2 in dimension $\leq d$. Let $\Gamma \subset [0, 1]$ be a set satisfying the dcc. Then there exists a finite subset $\Gamma_f \subset \Gamma$ such that*

$$\mathcal{F}_d(\Gamma) = \mathcal{F}_d(\Gamma_f),$$

that is, for each pair $(X, B) \in \mathcal{F}_d(\Gamma)$, actually $B \in \Gamma_f$.

Example 4.2 Let Γ be a set satisfying the dcc, and $(X, B) \in \mathcal{F}_1(\Gamma)$. So, X is a nonsingular curve, and since $K + B \equiv 0$, either $X \cong \mathbb{P}^1$, or X is an elliptic curve. In the latter case, $B = 0$ and $\Gamma_f = \{0\}$ is enough.

In the former case, $\deg K + B = 0$, and $\sum_1^k b_j = -2$ where $B = \sum b_j P_j$ and each $b_j \in \Gamma$. By the dcc of Γ , we may assume that k is bounded, that is, it only depends on Γ . Since (X, B) is lc, $k \geq 2$. Note that if $\{s_i\}$ and $\{s'_i\}$ are two sequences satisfying the dcc, then $\{s_i + s'_i\}$ is also a sequence satisfying the dcc. Similarly, the sum of n dcc sequences, satisfies the dcc.

Now, if there is no finite Γ_f as in Proposition 4.1, then there is a sequence $(X_i, B_i) \in \mathcal{F}_1(\Gamma)$ such that the set of multiplicities of all $B_i = \sum_1^k b_{i,j} P_{i,j}$ is not finite. In particular, we may assume that all X_i are the projective line and that $\{b_{i,1}\}$ is an infinite dcc sequence. On the hand, $2 - b_{i,1} = \sum_2^k b_{i,j}$. Since each $\{b_{i,j}\}$ is a dcc sequence, $\sum_2^k b_{i,j}$ is also a dcc sequence. This contradicts the fact that $\{2 - b_{i,1}\}$ is an infinite acc sequence.

Proof (of Proposition 4.1)

We use induction on d .

Step 1. For $d = 1$, see Example 4.2. Now we assume that the theorem holds in dimension $\leq d - 1$, and we establish it in dimension d . Suppose that there exists a sequence of log pairs $(X_i, B_i) \in F_d(\Gamma)$, $i = 1, 2, \dots$, such that the set of boundary multiplicities $M = \{b_{i,k}\}$, for boundaries $B_i = \sum b_{i,k} D_{i,k}$ is infinite. Since M satisfies the dcc we can assume that the sequence $b_{i,1}$, $i = 1, 2, \dots$ is strictly increasing, and has only positive real numbers. Below we derive a contradiction (see Step 8).

Step 2. *We can suppose that each X_i is \mathbb{Q} -factorial, and, in particular, each divisor $D_{i,1}$ is \mathbb{Q} -Cartier.* Indeed a \mathbb{Q} -factorialization $Y_i \rightarrow X_i$ exists by LMMP [12, Corollary 6.7], if (X_i, B_i) is klt, or the same construction gives a crepant strictly lt model, and we can replace each pair (X_i, B_i) by its crepant \mathbb{Q} -factorialization (Y_i, B_i) or respectively by a crepant strictly log terminal model (Y_i, B_i^{\log}) where the boundary B_i on Y_i is the birational transform of B_i on X_i or respectively B_i^{\log} is the log birational transform of B_i . In the last case we need to extend Γ by 1.

Step 3. *We can suppose that each X_i is a Mori-Fano variety*, that is, X_i is a projective \mathbb{Q} -factorial variety with Picard number $\rho(X_i) = 1$ having only lc singularities and ample $-K_{X_i}$ (cf. [12, Definition 1.6 (v)]).

Indeed, by our assumptions we can apply LMMP to $(X_i, B'_i = B_i - b_{i,1} D_{i,1})$ or, equivalently, with respect to the log canonical divisor $K_{X_i} + B_i - b_{i,1} D_{i,1}$. Note that $K_{X_i} + B_i - b_{i,1} D_{i,1} \equiv -b_{i,1} D_{i,1}$ where $b_{i,1} > 0$. Thus $D_{i,1}$ is positive on each extremal contraction $X_i \rightarrow Z_i$. In particular, the divisor $D_{i,1}$ will never be contracted. Moreover, if the extremal contraction is birational, its

birational modification $(X_i^+, (B_i')^+)$ (a divisorial contraction or a log flip; both are log flips in the sense of [23]) belongs again to $F_d(\Gamma)$ and we can replace pair (X_i, B_i) by its log flop (X_i^+, B_i^+) . Of course, the entire set of multiplicities can decrease but its monotonic infinite subset $\{b_{i,1}\}$ will remain. On the other hand, we always have an extremal contraction: $-D_{i,1}$ is not nef, for any i . Therefore, after finitely many steps, the extremal contraction will be a Mori-Fano fibration $X_i \rightarrow Z_i$. By construction each X_i has such a contraction.

If $\dim Z_i = d_i \geq 1$, then the generic fibre of $(X_i/Z_i, B_i)$ with induced (intersection) boundary belongs to $F_{d-d_i}(\Gamma)$, and $D_{i,1}$ gives a boundary component with the same multiplicity $b_{i,1}$. Hence, by induction we do not have infinite subsequence (X_i, B_i) with $d - d_i < d$. Thus, replacing with a subsequence, we can assume that each $d_i = 0$, $Z_i = \text{pt.}$, and X_i is a Mori-Fano variety.

Step 4. *We can suppose that only finitely many varieties X_i are cn, and thus, replacing by a subsequence, we can suppose that all varieties X_i are noncn.* Otherwise, we can suppose that each X_i is cn. Then by Conjecture 1.2, varieties X_i belong to a bounded family. Hence, by [20, Lemma 8.1] the set of boundary multiplicities, in particular, $\{b_{i,1}\}$ is finite. This is a contradiction.

So, replacing by a subsequence, we can suppose that each X_i has a noncn point P_i (it may be nonclosed) and its codimension ≥ 2 . We can assume also that each (X_i, B_i) is klt by Theorem 2.12 (iii). Fix a prime b-divisor E_i with center $C_{X_i} E_i = P_i$, and $\text{ldis}(X_i, B_i) = \text{mld}(P_i, X_i, B_i) = a(E_i, X, B_i)$. Thus, we can apply to each X_i a log twist as in Construction 3.1.

Step 5. *We can suppose that each log twist $X_i \dashrightarrow X'_i$ is final.* Indeed, if it is not final, put $(X_i, B_i) = (X'_i, B_{X'_i})$, and take another twist, etc. According to our assumptions (ACC of mld's) and Corollary 3.9, after a bounded number of twists, we can suppose that each twist $X_i \dashrightarrow X'_i$ is final. By Addendum 3.10, the extended set of boundary multiplicities $\Gamma^\infty = \Gamma^{(N)}$ again satisfies the dcc.

For the final twist, we denote the resulting contraction by $f': Y'_i \rightarrow X'_i$. By Corollary 3.15 there exists an infinite set of distinct boundary multiplicities for pairs (X_i, B_i) . As in Step 1 and after taking a subsequence, we can suppose that there exists a sequence of prime divisors $D_{i,1}$ on X_i with strictly increasing boundary multiplicities $b_{i,1}$.

Put $a_i = \text{mld}(P_i, X_i, B_i) = a(E_i, X_i, B_i)$ and codiscrepancy $e_i = 1 - a_i$. Note that by ACC for mld's and the dcc for Γ , the set of mld's $\{a_i | i =$

$1, 2, \dots \}$ satisfies the acc. Hence the numbers e_i satisfy the dcc, and we can suppose (after taking a subsequence) that numbers e_i form a monotonically increasing sequence. Thus the crepant boundaries $B_{Y_i} = B_i + e_i E_i$ on Y_i , where divisors B_i on Y_i denote the birational transform of B_i , and their modifications $B_{Y'_i}$ belong to the set $\Gamma \cup \{e_i\}$ which again satisfies the dcc (cf. Addendum 3.10), and the divisors itself have two monotonically increasing multiplicities $b_{i,1}$ and e_i . The former is strictly monotonic.

Step 6. *Infinitely many twisted contractions are divisorial*, that is, of Type II (see 3.3). Otherwise infinitely many of twisted contractions $Y'_i \rightarrow X'_i$ are Mori-Fano fibrations. Then, replacing by a subsequence, we can assume that all of them are fibrations with varieties X'_i of the same dimension ≥ 1 . Thus, generic fibres will be of the same dimension $\leq d - 1$.

On the other hand, it is impossible by induction that infinitely many birational transforms of $D_{i,1}$ on Y'_i are strictly positive over X'_i , because then they intersect the generic fibre (cf. Step 3).

Therefore, by (3) in Construction 3.1, we can assume that each $D_{i,1}$ is positive over X_i . Then, by Proposition 2.5 the set $\{e_i\}$ is not finite, and by ACC for mld's we can suppose that e_i is strictly increasing (cf. Step 8 below and the proof of Lemma 3.13). This again gives a contradiction because, by (3) in Construction 3.1, the birational transform of each E_i on Y'_i is positive over X'_i .

Thus we can assume that each twisted contraction $Y'_i \rightarrow X'_i$ is divisorial with X'_i a Mori-Fano variety, and some divisor E'_i is contracted to a point P'_i in X'_i of codimension ≥ 2 . Put $a'_i = a(E'_i, X'_i, B_{X'_i})$ and the codiscrepancy $e'_i = 1 - a'_i$.

Step 7: *We can assume that $a'_i = \text{mld}(P'_i, X'_i, B_{X'_i})$ for infinitely many i , so we can assume that for all i .* Otherwise, since the twists are final, X'_i is canonical for infinitely many i . This is a contradiction, because such varieties are bounded.

Step 8. *Contradiction: M is finite.* By the dcc of Γ and since the support of B_i has a bounded number of components (Corollary 3.12), we can assume that each sequence $b_{i,k}, i = 1, 2, \dots$, is increasing with respect to i . The crepant divisors

$$B_{Y_i} = \sum_{k=1}^{n+1} b_{i,k} D_{i,k} = e_i E_i + \sum_{k=1}^n b_{i,k} D_{i,k}$$

satisfy the same property where $b_{i,n+1} := e_i$. Now we define the set $R =$

$\{r_k | k = 1, 2, \dots, n + 1\}$ as the set of limits (not necessarily distinct)

$$r_k = \lim_{i \rightarrow \infty} b_{i,k}$$

First suppose that $a = \lim_{i \rightarrow \infty} a_i > 0$ and $a' = \lim_{i \rightarrow \infty} a'_i > 0$. Let τ be a positive real number constructed in Main Proposition 2.1. We can assume that each $b_{i,k} \in [r_k - \tau, r_k]$. Hence by Proposition 2.3 each $K_{Y_i} + B_{Y_i}^\tau$ is seminegative over X_i , and so does its birational transform $K_{Y'_i} + B_{Y'_i}^\tau$ over X'_i . Here and for the rest of the proof, the superscript τ stands for the limit, that is, for example $B_{Y_i}^\tau = \lim_{j \rightarrow \infty} B_{Y_j}$ on Y_i in the sense of Example 2.4. On the other hand, by construction

$$B_{Y_i}^\tau \geq B_{Y_i}.$$

Hence

$$D := K_{Y_i} + B_{Y_i}^\tau = K_{Y_i} + B_{Y_i} + (B_{Y_i}^\tau - B_{Y_i}) \equiv B_{Y_i}^\tau - B_{Y_i} \geq 0$$

is an effective divisor which is seminegative over X_i . Since flips preserve numerical equivalence, D' the birational transform of D is seminegative over X'_i . Thus, by (3) in Construction 3.1, $D = D' = 0$. This means that all limits are stabilized. This contradicts infinity of M , in particular strict monotonicity of $b_{i,1}$.

Now if $a = 0$, then to get the seminegativity of $K_{Y_i} + B_{Y_i}^\tau$ over X_i , we can use Theorem 2.10 (vi). In fact, if $K_{Y_i} + B_{Y_i}^\tau$ is not seminegative over X_i , then $K_{X_i} + A_i$ is maximally 0-lc at some point of X_i for some $B_i \preceq A_i \preceq B_i^\tau$ which contradicts Theorem 2.12 (vi). We have a similar argument for $a' = 0$. The rest of the proof is exactly as in the $a, a' > 0$ case.

□

5 Proof of Main Theorem

Proof (of Main Theorem 1.8)

(i) By induction, we can assume ACC for lc thresholds in dimension $\leq d$. Now we can use Proposition 2.5.

(ii) This follows from Addendum 2.11 (v).

(iii) We can use (ii) and the main result of [5].

(iv) This follows from [11] (see [6] or [27] for more information).

□

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