Boundedness of ϵ-lc Complements on Surfaces II: via Surface Geometry

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Contents

1 Introduction

In [Bir1] we proved the boundedness of ϵ-lc complements on surfaces, using the theory of complements. In this paper, we give another proof of this boundedness (except in the local over curve case) using properties of algebraic surfaces rather than complement theory and LMMP. This new proof is interesting on its own for people who work on surface geometry. It is also interesting to see how different this proof and our proof in [Bir1] are.

For an introduction to ϵ-lc complements and the terminology used in this paper, we refer you to [Bir1].

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Definition 1.1 Let (X, B) be a log pair of dimension (dim) d. A log divisor $K_X + B^+$ is an $(\epsilon, \mathbb{R})$-*complement*$/P \in Z$ for $K_X + B$ if $(X, K_X + B^+)$ is ϵ-lc$/P \in Z$, $K_X + B^+ \sim_{\mathbb{R}} 0/P \in Z$ and $B^+ \geq B$. An $(\epsilon, \mathbb{Q})$-*complement*$/P \in Z$ can be similarly defined where $\sim_{\mathbb{R}}$ is replaced by $\sim_{\mathbb{Q}}$.

Definition 1.2 (ϵ-lc complements) Let $\big(X/Z, B = \sum_i b_i B_i\big)$ be a pair of dim d. Then, $K_X + B^+$ is called an (ϵ, n)-*complement*$/P \in Z$ for $K_X + B$, where $B^+ = \sum_i b_i^+ B_i$, if the following properties hold:

◇ $(X, K_X + B^+)$ is an ϵ-lc pair$/P \in Z$ and $n(K_X + B^+) \sim 0/P \in Z$.

◇ $\llcorner (n+1) b_i \lrcorner \leq n b_i^+$.

We say $(X/P \in Z, B)$ is (ϵ, n)-*complementary*$/P$ if there exists an (ϵ, n)-complement$/P$ for $K_X + B$.

Remark 1.3 In Definitions 1.1 and 1.2, if we take $\epsilon = \delta = 0$ then we have the usual notion of complement as defined in [Sh2].

Definition 1.4 Let $\Gamma \subseteq [0, 1]$. For a divisor $B = \sum_i b_i B_i$, we write $B \in \Gamma$ if all $b_i \in \Gamma$. The set $\Phi_{\text{sm}} = \{\frac{k-1}{k} | k \in \mathbb{N}\} \cup \{1\}$ is called the set of *standard* boundary multiplicities. Γ_f denotes a finite subset of $[0, 1]$.

We now state Shokurov's conjectures on the boundedness of complements.

Conjecture 1.5 (Weak ϵ-lc complements) *Let $\Gamma \subset [0, 1]$ be a set of real numbers which satisfies the descending chain condition (DCC). Then, for any $\delta > 0$ and d there exist a finite set $\mathcal{N}_{\delta,d,\Gamma}$ of positive integers and $\epsilon > 0$ such that any d-dimensional δ-lc weak log Fano pair $(X/P \in Z, B)$, where $B \in \Gamma$, is (ϵ, n)-complementary$/P \in Z$ for some $n \in \mathcal{N}_{\delta,d,\Gamma}$.*

We refer to this conjecture as $\mathrm{WC}_{\delta,d,\Gamma}$. The following conjecture due to Alexeev and Borisov brothers is closely related to the above conjecture.

Conjecture 1.6 (BAB) *Let $\delta > 0$ be a real number, $d > 0$ and $\Gamma \subset [0, 1]$. Then, varieties X for which $(X/\mathrm{pt.}, B)$ is a d-dimensional δ-lc WLF pair for a boundary $B \in \Gamma$ form a bounded family.*

We divide the surface case of conjecture 1.5 into the following cases:

local isomorphic X/Z is an isomorphism.

local birational X/Z is birational but may not be an isomorphism.

local over curve Z is a curve.

global Z is a point.

We prove $\mathrm{WC}_{\delta,2,\Gamma_f}$ in the global case (Corollary 4.6) and $\mathrm{WC}_{\delta,2,\{0\}}$ in the local isomorphic case (Theorem 2.1) and the local birational case (Theorem 3.1).

2 Local isomorphic case

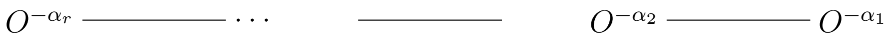
The main theorem in this section is Theorem 2.1. We use classification of surface singularities.

Theorem 2.1 *Conjecture* $\mathrm{WC}_{\delta,2,\{0\}}$ *(1.5) holds in the local isomorphic case, that is, when* $X \to Z$ *is the identity and* $\Gamma = \{0\}$.

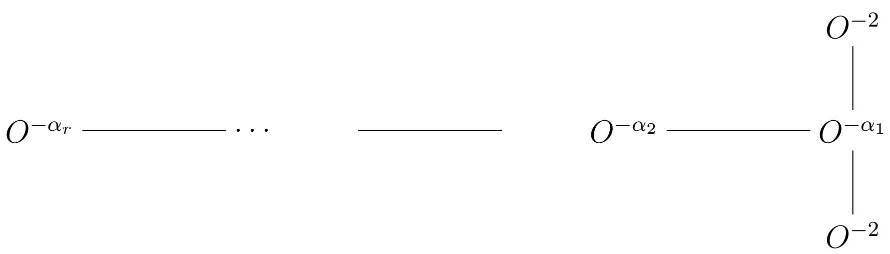
Proof Note that $(X, 0)$ is Klt$/P \in Z$ by assumptions of Conjecture 1.5 ($\delta > 0$). If $\delta > 1$, then X is smooth at P so we are already done. If $\delta = 1$ then X is canonical at P so K_X is Cartier. In this case K_X is a $(1,1)$-complement$/P$ for K_X. From now on we assume that $\delta < 1$.

If the singularity at P is of type E_6, E_7 or E_8, then there are only a finite number of possibilities for such singularities up to analytic isomorphism because of the δ-lc assumption [Pr, 6.1.2].

If the singularity at P is of type A_r, then the graph of the resolution is as



where $\alpha_i \geq 2$. If the singularity at P is of type D_r, then the graph is as



where $\alpha_i \geq 2$.

$\mathbf{A}_r$ case: Let $K_W - \sum_i e_i E_i = {}^*K_Z$ where e_i are the discrepancies for a log resolution $W \to Z$ near P. The following lemma is well known and a proof can be found in [AM, 1.2].

Lemma 2.2 *The numbers $(-E_i^2)$ are bounded from above in terms of δ.*

□

By computing the intersection numbers $(K_W - \sum_i e_i E_i) \cdot E_j$ we get the following system:

$$\begin{cases} a_1(-E_1^2) - a_2 - 1 = 0 \\ a_2(-E_2^2) - a_1 - a_3 = 0 \\ a_3(-E_3^2) - a_2 - a_4 = 0 \\ \vdots \\ a_{r-1}(-E_{r-1}^2) - a_{r-2} - a_r = 0 \\ a_r(-E_r^2) - a_{r-1} - 1 = 0 \end{cases}$$

where a_i is the log discrepancy of E_i with respect to K_Z.

From the equation $a_i(-E_i^2) - a_{i-1} - a_{i+1} \leq 0$ we get the inequality $a_i(-E_i^2 - 2) + a_i - a_{i-1} \leq a_{i+1} - a_i$ which shows that if $a_{i-1} \leq a_i$, then $a_i \leq a_{i+1}$ and moreover if $a_{i-1} < a_i$ then $a_i < a_{i+1}$. So the solution for the system above must satisfy the following:

$$a_1 \geq \cdots \geq a_i \leq \cdots \leq a_r \tag{2.2.1}$$

for some $i \geq 1$. If $r \leq 2$ (or any fixed number), then the theorem is trivial. So we may assume that $r > 3$ and also can assume $i \neq r$ unless $a_1 = a_2 = \cdots = a_r$. Now, for any $i \leq j < r$, if $-E_j^2 > 2$, then $a_{j+1} - a_j \geq a_j(-E_j^2 - 2) \geq \delta$. Hence if $l := \#\{j \mid -E_j^2 > 2 \text{ and } i \leq j < r\}$, then $a_r \geq l\delta$. Thus $a_r(-E_r^2 - 1) + a_r - a_{r-1} \geq l\delta$, which contradicts the last equation in the system for l large enough. In any case, $l\delta \leq 1$ and $l \leq \frac{1}{\delta}$, so l is bounded . Similarly, we deduce that $l' := \#\{j \mid -E_j^2 > 2 \text{ and } 1 \leq j \leq i\}$ is bounded. Then, $l + l' \leq \frac{2}{\delta}$.

Now suppose that $a_{i_2} = \cdots = a_i = \cdots = a_{i_1}$, $a_{i_1-1} \neq a_{i_1}$ (unless $i_1 = 1$) and $a_{i_2} \neq a_{i_2+1}$ (unless $i_2 = r$) where $i_2 \leq i \leq i_1$. Assume that $i_1 \neq i$ or $i_2 \neq i$. If $i_1 \neq i$ and all a_j are not equal (to 1), then we have

$$\begin{aligned}1 = (-E_r^2 - 1)a_r + a_r - a_{r-1} & \\ & \geq (r - i_1)(a_{i_1+1} - a_{i_1}) \\ & = (r - i_1)[(-E_{i_1}^2 - 2)a_{i_1} + a_{i_1} - a_{i_1-1}] \\ & = (r - i_1)(-E_{i_1}^2 - 2)a_{i_1} \geq (r - i_1)\delta\end{aligned}$$

because $-E_{i_1}^2$ cannot be equal to 2.

So $(r-i_1)\delta \leq 1$ which in turn implies that $r-i_1 \leq \frac{1}{\delta}$ is bounded. Similarly, we deduce that i_2 is bounded.

These observations show that, given that all $-E_k^2$ are bounded, the denominators of a_k are bounded. Therefore, the index of K_Z at P is bounded and so we are done in this case.

But if $i_1 = i = i_2$, then the situation is different. Note that in this case $\delta \leq (-E_i^2 - 2)a_i = a_{i-1} - a_i + a_{i+1} - a_i$. Hence $\frac{\delta}{2} \leq a_{i-1} - a_i$ or $\frac{\delta}{2} \leq a_{i+1} - a_i$. If $\frac{\delta}{2} \leq a_{i+1} - a_i$, then similar to the calculations we just carried out above, $r - i$ is bounded. But it can happen that $a_{i-1} - a_i$ is very small so we will not be able to bound i. The same argument applies to the case $\frac{\delta}{2} \leq a_{i-1} - a_i$.

Actually, we try to find a solution with bounded denominators for the following system:

$$\begin{cases} u_1(-E_1^2) - u_2 - 1 \leq 0 \\ u_2(-E_2^2) - u_1 - u_3 \leq 0 \\ u_3(-E_3^2) - u_2 - u_4 \leq 0 \\ \vdots \\ u_{r-1}(-E_{r-1}^2) - u_{r-2} - u_r \leq 0 \\ u_r(-E_r^2) - u_{r-1} - 1 \leq 0 \end{cases}$$

To find such a solution, note that if $-E_{i-1}^2 > 2$, then $\delta \leq (-E_{i-1}^2 - 2)a_{i-1} = a_{i-2} - a_{i-1} + a_i - a_{i-1} \leq a_{i-2} - a_{i-1}$. Hence similar computations to the above show that i is bounded. Now let j be the smallest number such that $-E_j^2 = \cdots = -E_{i-1}^2 = 2$ (remember that we have assumed $\frac{\delta}{2} \leq a_{i+1} - a_i$). Hence j is bounded. Now take $u_j = \cdots = u_i = \frac{1}{I}$ for a natural number I. Then, the following equations are satisfied if $i - j > 2$:

$$\begin{cases} u_{j+1}(-E_{j+1}^2) - u_j - u_{j+2} = 2u_j - u_j - u_j = 0 \\ \vdots \\ u_{i-1}(-E_{i-1}^2) - u_{i-2} - u_i = 2u_{i-1} - u_{i-2} - u_i = 0 \end{cases}$$

Since $r-i$ and j are bounded the number of remaining equations is bounded. Therefore, there is a bounded I such that there is a solution $(u_1,\dots,u_r)$ where $u_j=\dots=u_i=\frac{1}{I}$. This completes the proof of A_r case.

Form the solution $(u_1,\dots,u_r)$ we construct a Klt log divisor K_W+D with bounded index such that $-(K_W+D)$ is nef and big$/P\in Z$. Now we may use [Bir1, Remark 2.16]. This completes the proof of A_r case.

Remark 2.3 In Shokurov's case [Sh1, §5], where $\delta=\epsilon=0$, we just take $u_1=\dots=u_r=0$.

$\mathbf{D}_r$ case: We have a chain $E_1,\dots,E_r$ of exceptional divisors together with E and E', where E and E' intersect only E_1. In this case we have the following system:

$$\begin{cases} a(-E^2)-a_1-1=0\\ a'(-E'^2)-a_1-1=0\\ a_1(-E_1^2)-a-a'-a_2+1=0\\ a_2(-E_2^2)-a_1-a_3=0\\ a_3(-E_3^2)-a_2-a_4=0\\ \vdots\\ a_{r-1}(-E_{r-1}^2)-a_{r-2}-a_r=0\\ a_r(-E_r^2)-a_{r-1}-1=0 \end{cases}$$

Note that $-E^2=-E'^2=2$, so $2a-a_1-1=0$ and $2a'-a_1-1=0$. Hence $a+a'=a_1+1$ and the third equation becomes $a_1(-E_1^2-1)-a_2=0$. We now consider the system obtained from the last system after ignoring the first two equations:

$$\begin{cases} a_1(-E_1^2-1)-a_2=0\\ a_2(-E_2^2)-a_1-a_3=0\\ a_3(-E_3^2)-a_2-a_4=0\\ \vdots\\ a_{r-1}(-E_{r-1}^2)-a_{r-2}-a_r=0\\ a_r(-E_r^2)-a_{r-1}-1=0 \end{cases}$$

Any solution of this system satisfies the following:

$$a_1=\dots=a_i<a_{i+1}<\dots<a_r$$

If $i = r$, then $a = a' = a_1 = \cdots = a_r = 1$. So we may assume $i < r$. We show that $r - i$ is bounded. In this case, if $i > 1$, then $-E_1^2 = \cdots = -E_{i-1}^2 = 2$ but $-E_i^2 > 2$. Now $\delta(-E_i^2 - 2) \leq a_i(-E_i^2 - 2) + a_i - a_{i-1} = a_{i+1} - a_i$ (if $i = 1$ then $\delta(-E_1^2 - 2) \leq a_1(-E_1^2 - 2) = a_2 - a_1$). We also have $a_{k+1} - a_k \leq a_{k+2} - a_{k+1}$ for $i \leq k < r - 1$. On the other hand, $\sum_{i \leq k < r} a_{k+1} - a_k \leq a_r < a_r + a_r - a_{r-1} < 1$. So we conclude that $r - i$ is bounded.

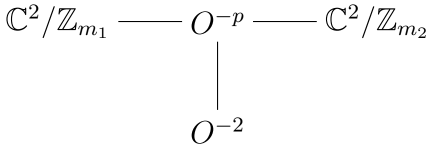
Moreover since $-E_k^2$ is bounded, this proves that the denominators of all a_k in the D_r case are bounded hence the index of K_Z at P is bounded. In this case $B^+ = 0$ and we complete the proof of Theorem 2.1.

$\square$

Remark 2.4 All the bounds occurring in the proof are effective and can be calculated in terms of δ.

Remark 2.5 Essentially, the boundedness properties that we proved and used in the proof of Theorem 2.1 have been more or less discovered by other mathematicians independently. In particular, Shokurov has used these ideas in an unpublished preprint on mlds [Sh8].

Remark 2.6 Here we recall the diagrams for the E_6, E_7 and E_8 types of singularities [Pr, 6.1.2]. The following is a general case of such singularities:



where the only possibilities for (m_1, m_2) are $(3, 3)$, $(3, 4)$ and $(3, 5)$. So the only possible diagrams are as follows: For $(m_1, m_2) = (3, 3)$ we have

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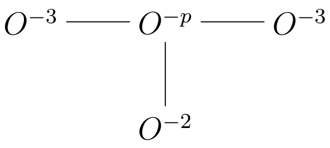

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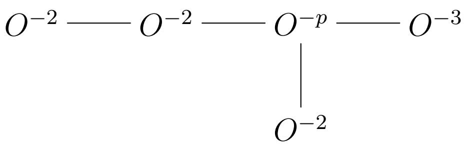

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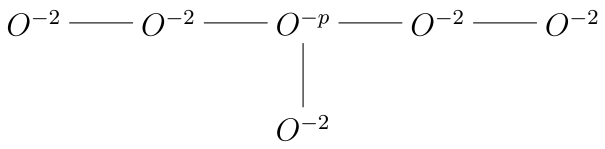

For $(m_1, m_2) = (3, 4)$ we have

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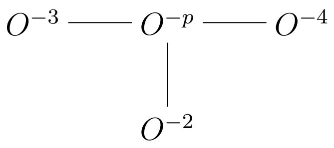

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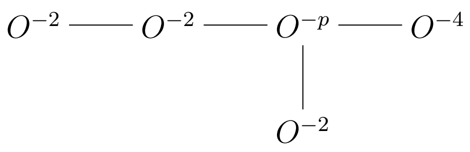

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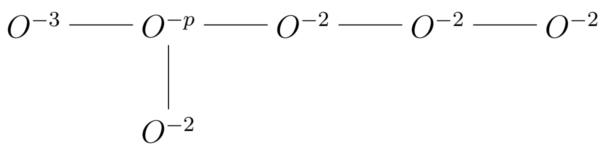

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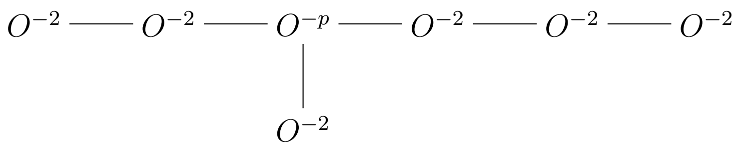

Finally for $(m_1, m_2) = (3, 5)$ we have

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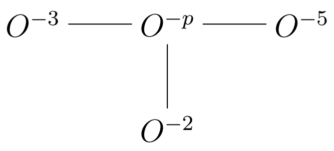

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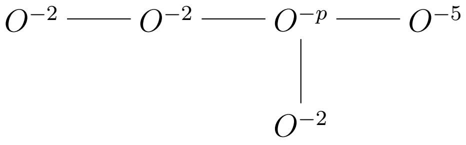

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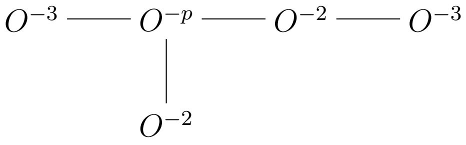

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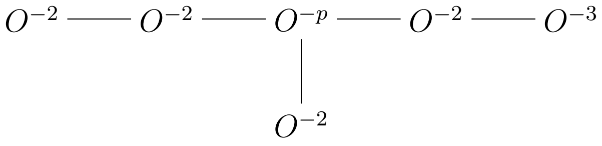

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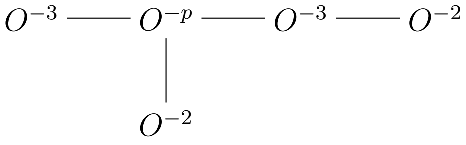

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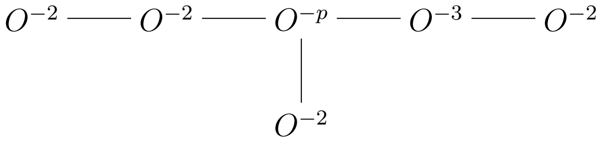

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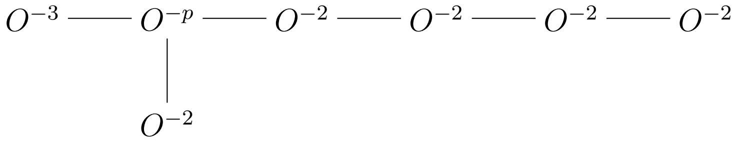

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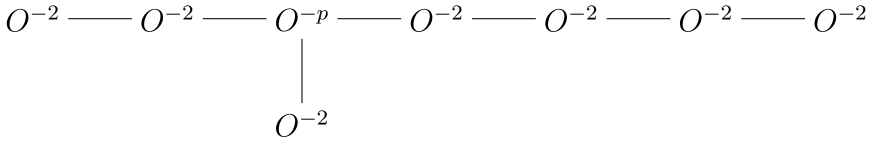

3 Local birational case

In this section whenever we write $/Z$ we mean $/P \in Z$ for a fixed point P on Z.

Theorem 3.1 *Conjecture* $\mathrm{WC}_{\delta,2,\{0\}}$ *(1.5) holds in the birational case, that is, when* $X \to Z$ *is birational and* $\Gamma = \{0\}$*.*

Strategy of the proof: Let W be a minimal resolution of X and let $\{E_i\}$, $\{F_j\}$ be the exceptional divisors $/Z$ on W where the E_i are exceptional$/X$ but F_j are not. We use the notation E for a typical E_i and similarly F for F_j or its birational transform). We construct an antinef$/Z$ and Klt log divisor $K_W + \Omega = K_W + \sum_i u_i E_i + \sum_j u_j F_j$ where $u_i, u_j < 1$ are rational numbers with bounded denominators. Then, we use [Bir1, Remark 2.16].

Proof By contracting those curves where $-K_X$ is numerically zero, we can assume that $-K_X$ is ample$/Z$ (we can pull back the complement). Let W be the minimal resolution of X. Then, since K_W is nef$/X$ by the negativity lemma we have $K_W - \sum_i e_i E_i = K_W + \sum_i (1 - a_i) E_i \equiv {}^*K_X$ where $e_i \leq 0$.

Definition 3.2 For any smooth model Y where $W/Y/Z$ we define $\overline{\mathrm{exc}}(Y/Z)$ to be the graph of the exceptional curves ignoring the birational transform of exceptional divisors of type F. For an exceptional$/Z$ divisor G on Y not of type F, $\overline{\mathrm{exc}}(Y/Z)_G$ means the connected component of $\overline{\mathrm{exc}}(Y/Z)$ where G belongs to.

Lemma 3.3 *We have the following on* W*:*

◇ *The exceptional divisors$/Z$ on W are with simple normal crossings.*

◇ *Each F (that is, each exceptional divisor of type F) is a -1-curve.*

◇ *The model $\overline{W}$ obtained by blowing down -1-curves$/Z$ is the minimal resolution of Z.*

◇ *Each F cuts at most two exceptional divisors of type E.*

Proof Let F be an exceptional divisor$/Z$ on W which is not exceptional$/X$. Then, $(K_W - \sum_i e_iE_i)\cdot F = K_W \cdot F + \sum_i(-e_i)E_i \cdot F = 2p_a(F) - 2 - F^2 + \sum_i(-e_i)E_i \cdot F < 0$ where $p_a(F)$ stands for the arithmetic genus of the curve F. Then, $2p_a(F) - 2 - F^2 < 0$ and so $p_a(F) = 0$ and $-F^2 = 1$. In other words F is a -1-curve.
On the other hand by contracting -1-curves$/Z$ (i.e. running the classical minimal model theory for smooth surfaces on W/Z) we get a model $\overline{W}/Z$ where $K_{\overline{W}}$ is nef$/Z$. Actually $\overline{W}$ is the minimal resolution of $P \in Z$.

The exceptional divisors$/Z$ on $\overline{W}$ are with simple normal crossings and since W is obtained from $\overline{W}$ by a sequence of blow ups, then the exceptional divisors$/Z$ on W are also with simple normal crossings. Further, since all the F, exceptional$/Z$ but not$/X$, are contracted$/\overline{W}$ then they can intersect at most two of E_i because exc$(\overline{W}/Z)$ is with simple normal crossings and F is exceptional$/\overline{W}$.

$$\begin{tikzcd} & W \arrow[dl] \arrow[dr] & \\ X \arrow[dr] & & \overline{W} \arrow[dl] \\ & Z & \end{tikzcd}$$

□

Moreover no two exceptional divisors of type F can intersect on W because they are both -1-curves. This means that the intersection points of any two exceptional divisor$/Z$ on X is a singular point of X. Also any exceptional divisor$/Z$ on X contains at most two singular points of X.

Let $\{Q_k\}_k$ be the singular points of X. If none of the points $\{Q_k\}$ is of type A_r, then the proof of Theorem 2.1 shows that the index of K_X is bounded so we are done. But if there is one point of type A_r, then the proof is more complicated. Surprisingly, the A_r type is the most simple case in the sense of Shokurov, that is, when $\delta = 0$ (see Remark 2.3). Similar to the proof of Theorem 2.1 we try to understand the structure of exc(W/Z) and the blow ups $W \to \overline{W}$.

Definition 3.4 A smooth model $\ddot{W}$ where $W/\ddot{W}$ and $\ddot{W}/\overline{W}$ are series of smooth blow ups, is called a *blow up model* of $\overline{W}$. Such a model is *perfect* if there is X' such that $K_{\ddot{W}}$ is nef$/X'$ and $X/X'/Z$. In other words, it is the minimal resolution of X'. The connected components of $\overline{\text{exc}}(\ddot{W}/Z)$ are either of type A_r, D_r, E_6, E_7 or E_8 for a perfect blow up model.

Definition 3.5 We call the divisor $K_W + \omega = K_W + \sum_i (1 - a_i)E_i = {}^*K_X$ the *primary log divisor*. The pair (X, B) has a $(0, n)$-complement $K_X + B^+$ over Z by Shokurov [Sh2] ($n \in \{1, 2, 3, 4, 6\}$). From now on we call it a *Shokurov complement.* So $K_W + \omega_{Sh} + C = K_W + \sum_i (1 - a_i^{Sh})E_i + \sum_j (1 - a_j^{Sh})F_j + C = {}^*(K_X + B^+)$ where C is the birational transform of the nonexceptional part of B^+. We call $K_W + \omega_{Sh}$ a *Shokurov log divisor* and the numbers a_i^{Sh} and a_j^{Sh} *Shokurov log discrepancies.*

Definition 3.6 Consider the graph $\text{exc}(W/Z)$. If we ignore those F that appear with zero coefficient in ω_{Sh} (that is, $a^{Sh} = 1$), then we get a graph $\text{exc}(W/Z)_{>0}$ with some connected components. The connected graph $\mathcal{C}$ consisting of exceptional$/Z$ curves with $a^{Sh} = 0$, belong to one of the components of the graph $\text{exc}(W/Z)_{>0}$ which we show by $\mathcal{G}$ ($\mathcal{C}$ is connected because of the connectedness of the locus of log canonical centres$/P \in Z$). Now contracting all -1-curves$/Z$ in $\mathcal{G}$ and continuing the contractions of subsequent -1-curves$/Z$ which appear in $\mathcal{G}$, we finally get a model which we denote by $W_{\mathcal{G}}$. The transform of $\mathcal{G}$ on $W_{\mathcal{G}}$ is denoted by $\mathcal{G}_1$ and similarly the transform of $\mathcal{C}$ is $\mathcal{C}_1$.

Definition 3.7 A chain of exceptional curves consisting of $G_{\beta_1}, \dots, G_{\beta_r}$ is called *strictly monotonic* if $r = 1$ or if $a_{\beta_1} < a_{\beta_2} < \cdots < a_{\beta_r}$ (these are log discrepancies with respect to K_X). G_{β_1} is called the *base curve.*

Definition 3.8 Let $G \in \text{exc}(\ddot{W}/Z)$ for a smooth blow up model $\ddot{W}$. Then, we define the *negativity* of G on this model as $N_{\ddot{W}}(G) = (K_{\ddot{W}} + {}_*\omega) \cdot G \leq 0$. We also define the *total negativity* by $N_{\ddot{W}} = \sum_\alpha N_{\ddot{W}}(G_\alpha)$ where G_α runs over all exceptional divisors$/Z$ on $\ddot{W}$. For $G \in \overline{\text{exc}}(\ddot{W}/Z)$ we define $N_{\ddot{W},G} = \sum_\alpha N_{\ddot{W}}(G_\alpha)$ where the sum runs over all members of $\overline{\text{exc}}(\ddot{W}/Z)_G$. Similarly, we define the negativity functions N^{Sh} and N^+ replacing ω with ω_{Sh} and $\omega_{Sh} + C$ respectively. Note that the latter is always zero, because $K_W + \omega_{Sh} + C \equiv 0/Z$.

Definition 3.9 Let $\ddot{W}/Z$ be a smooth blow up model and $\xi \in \ddot{W}$. If ξ belongs to two exceptional divisors$/Z$, then the blow up at ξ is a *double blow up.* If ξ belongs to just one exceptional divisor$/Z$, then the blow up at ξ is a *single blow up.* If ξ belongs to two components of ${}_*(\omega_{Sh} + C)$, then the blow up at ξ is a *double*$^+$ *blow up.* If ξ belongs to just one component of $_*(\omega_{Sh} + C)$, then the blow up at ξ is a *single*$^+$ *blow up.*

Lemma 3.10 *For any exceptional* $G_\beta \in \overline{\text{exc}}(\ddot{W}/Z)$ *on a blow up model* $\ddot{W}$ *we have:*

- ◇ $-1+\delta \le N_{\ddot{W},G_\beta}$ *if* $\overline{\text{exc}}(\ddot{W}/Z)_{G_\beta}$ *is of type* D_r, E_6, E_7 *or* E_8. *In particular, in these cases* $-1+\delta \le N_{\ddot{W}}(G_\beta)$ *holds.*
- ◇ $2(-1+\delta) \le N_{\ddot{W},G_\beta}$ *and* $-1+\delta \le N_{\ddot{W}}(G_\beta)$ *if* $\overline{\text{exc}}(\ddot{W}/Z)_{G_\beta}$ *is of type* A_r *unless it is strictly monotonic.*

Proof D$_r$ case: Similar to the notation in the proof of Theorem 2.1 let $G_\beta, G_{\beta'}, G_{\beta_1}, \dots, G_{\beta_r}$ be the exceptional divisors in $\overline{\text{exc}}(\ddot{W}/Z)_{G_\beta}$. Then, from the equations in the proof of Theorem 2.1 for the D$_r$ case we get the following system for the log discrepancies:

$$\begin{cases} 2a_\beta - a_{\beta_1} - 1 \le 0 \\ 2a_{\beta'} - a_{\beta_1} - 1 \le 0 \\ 2a_{\beta_1} - a_\beta - a_{\beta'} - a_{\beta_2} + 1 \le 0 \\ 2a_{\beta_2} - a_{\beta_1} - a_{\beta_3} \le 0 \\ \vdots \\ 2a_{\beta_{r-1}} - a_{\beta_{r-2}} - a_{\beta_r} \le 0 \\ 2a_{\beta_r} - a_{\beta_{r-1}} - 1 \le 0 \end{cases}$$

Adding the first and the second inequalities gives $2a_\beta + 2a_{\beta'} - 2a_{\beta_1} - 2 \le 0$. Accordingly, the third inequality becomes $a_{\beta_1} \le a_{\beta_2}$ and so $a_{\beta_1} \le a_{\beta_2} \le \cdots \le a_{\beta_r}$. Therefore

$$\begin{aligned} N_{\ddot{W},G_\beta} & \\ &\ge a_\beta + a_{\beta'} + a_{\beta_r} - a_{\beta_1} - 2 \\ &\ge a_\beta + a_{\beta'} + a_{\beta_2} - a_{\beta_1} - 2 \\ &\ge 2a_{\beta_1} + 1 - a_{\beta_1} - 2 \\ &\ge a_{\beta_1} - 1 \ge \delta - 1 \end{aligned}$$

because $2a_{\beta_1} + 1 \le a_\beta + a_{\beta'} + a_{\beta_2}$ and X is δ-lc.

A$_r$ case (nonstrictly monotonic): In this case assume that the exceptional divisors in $\overline{\text{exc}}(\ddot{W}/Z)_{G_\beta}$ are $G_{\beta_1}, \dots, G_{\beta_r}$. We get the system:

$$\begin{cases} 2a_{\beta_1} - a_{\beta_2} - 1 \leq 0 \\ 2a_{\beta_2} - a_{\beta_1} - a_{\beta_3} \leq 0 \\ \vdots \\ 2a_{\beta_{r-1}} - a_{\beta_{r-2}} - a_{\beta_r} \leq 0 \\ 2a_{\beta_r} - a_{\beta_{r-1}} - 1 \leq 0 \end{cases}$$

So there will be k such that $a_{\beta_1} \geq a_{\beta_2} \geq \cdots \geq a_{\beta_k} \leq a_{\beta_r}$. Thus $N_{\ddot{W}}(G_{\beta_1}) \geq a_{\beta_1} + a_{\beta_1} - a_{\beta_2} - 1 \geq a_{\beta_1} - 1 \geq \delta - 1$. In this way we get the similar inequalities for all other inequalities except for $N_{\ddot{W}}(G_{\beta_k})$. Suppose $N_{\ddot{W}}(G_{\beta_k}) < \delta - 1$. So we get $2a_{\beta_k} - a_{\beta_{k-1}} - a_{\beta_{k+1}} < \delta - 1$ and so $1 - \delta < a_{\beta_{k-1}} + a_{\beta_{k+1}} - 2a_{\beta_k} \leq a_{\beta_1} + a_{\beta_r} - 2a_{\beta_k}$. On the other hand by adding all the inequalities in the system we get $N_{\ddot{W},G_\beta} \geq a_{\beta_1} + a_{\beta_r} - 2 > 1 - \delta + 2a_{\beta_k} - 2 \geq \delta - 1$. This contradicts the fact that $N_{\ddot{W}}(G_{\beta_k}) \geq N_{\ddot{W},G_{\beta_k}}$.

To get the inequality for $N_{\ddot{W},G_{\beta_k}}$ add all the equations in the system above. Note that if $r = 2$, then $a_{\beta_1} = a_{\beta_2}$ and lemma is immediate.

E_6, E_7, E_8 cases:[3.10.1] In these cases the graph $\overline{\mathrm{exc}}(\ddot{W}/Z)_{G_\beta}$ is as in Remark 2.6. It is enough to substitute 2 for all the self-intersection numbers because the negativity becomes smaller. We start from the smallest possible graph, that is, case 1 in 2.6.

$$\begin{cases} 2a_{\beta} - a_{\beta_2} - 1 \leq 0 \\ 2a_{\beta_2} - a_{\beta} - a_{\beta_1} - a_{\beta_3} + 1 \leq 0 \\ 2a_{\beta_1} - a_{\beta_2} - 1 \leq 0 \\ 2a_{\beta_3} - a_{\beta_2} - 1 \leq 0 \end{cases}$$

Adding all inequalities we get $N_{\ddot{W},G_\beta} = a_\beta + a_{\beta_1} + a_{\beta_3} - a_{\beta_2} - 2$. By the second inequality we have $a_\beta + a_{\beta_1} + a_{\beta_3} - a_{\beta_2} \geq a_{\beta_2} + 1$, so $N_{\ddot{W},G_\beta} \geq a_{\beta_2} + 1 - 2 \geq \delta - 1$. In fact, this was a special case of the D_r type inequalities (the similarity of the system not necessarily the graph $\overline{\mathrm{exc}}(\ddot{W}/Z)_{G_\beta}$). Note that the inequality for the total negativity implies the inequality for the negativity of each exceptional curve.

Now we prove the other cases by induction on the number of the exceptional curves. The minimum is four exceptional curves and we have just proved this case. Suppose we have proved the lemma for graphs with $\leq k-1$

[3.10.1] I only prove that $-1 + \delta \leq N_{\ddot{W}}(G)$ for any exceptional G. We will not need the inequality for total negativity.

exceptional curves and that our graph has k members. Let the exceptional curves be $G_\beta, G_{\beta_1}, \ldots, G_{\beta_{k-1}}$ and such that G_{β_l} cuts $G_\beta, G_{\beta_{l-1}}$ and $G_{\beta_{l+1}}$. If $l = 2$ or $l = k-2$, then we obtain again a system of type D_r. Otherwise, since $-a_\beta + 1 \geq 0$ we get a system as follows

$$\begin{cases} 2a_{\beta_1} - a_{\beta_2} - 1 \leq 0 \\ 2a_{\beta_2} - a_{\beta_3} - a_{\beta_1} \leq 0 \\ \vdots \\ 2a_{\beta_{k-1}} - a_{\beta_{k-2}} - 1 \leq 0 \end{cases}$$

This is a system of type A_{k-1}, so we have either $a_{\beta_1} \geq a_{\beta_2}$ or $a_{\beta_{k-1}} \geq a_{\beta_{k-2}}$. We study the first case (the other case being similar). Now note that $N_{\ddot{W}}(G_{\beta_1}) \geq 2a_{\beta_1} - a_{\beta_2} - 1 = a_{\beta_1} - a_{\beta_2} + a_{\beta_1} - 1 \geq \delta - 1$. By ignoring G_{β_1} we get a system for a graph with a smaller number of elements:

$$\begin{cases} 2a_{\beta_2} - a_{\beta_3} - 1 \leq 2a_{\beta_2} - a_{\beta_3} - a_{\beta_1} \leq 0 \\ \vdots \\ 2a_{\beta_l} - a_{\beta_{l-1}} - a_{\beta_{l+1}} - a_\beta + 1 \leq 0 \\ \vdots \\ 2a_{\beta_{k-1}} - a_{\beta_{k-2}} - 1 \leq 0 \end{cases}$$

and the lemma is proved by induction. □

Lemma 3.11 *Suppose $\xi \in \ddot{W}/\overline{W}$ ($\ddot{W}$ is a blow up model). Let $\tilde{W}$ be the blow up of $\ddot{W}$ at ξ and G_α the exceptional divisor of the blow up. Then,*

If G_α is the double blow up of G_β and G_γ (that is, $\xi \in G_\beta \cap G_\gamma$), then:

◇ *$N_{\tilde{W}}(G_\alpha) = a_\alpha - a_\beta - a_\gamma$ where a_α is the log discrepancy of G_α for K_X and similarly a_β and a_γ.*

◇ *$N_{\tilde{W}}(G_\beta) = N_{\ddot{W}}(G_\beta) - N_{\tilde{W}}(G_\alpha)$ and $N_{\tilde{W}}(G_\gamma) = N_{\ddot{W}}(G_\gamma) - N_{\tilde{W}}(G_\alpha)$.*

◇ *$N_{\tilde{W}} = N_{\ddot{W}} - N_{\tilde{W}}(G_\alpha)$.*

If G_α is the single blow up of G_β, then:

◇ *$N_{\tilde{W}}(G_\beta) = N_{\ddot{W}}(G_\beta) - N_{\tilde{W}}(G_\alpha)$, $N_{\tilde{W}}(G_\alpha) = a_\alpha - a_\beta - 1 \leq -\delta$ and $N_{\ddot{W}}(G_\beta) + \delta \leq 0$.*

◇ *$N_{\tilde{W}} = N_{\ddot{W}}$.*

Proof Standard computations. □

Corollary 3.12 *Let $\ddot{W}$ be a blow up model/$\overline{W}$. If G_α is a single blow up of G_β on $\ddot{W}$ and $N_{\ddot{W}}(G_\beta) \geq \delta - 1$, then $a_\alpha \geq a_\beta + \delta$.*

Proof Since G_α is a single blow up of G_β, $1 + a_\beta - a_\alpha + N_{\ddot{W}}(G_\beta) \leq 0$ and so $1 + a_\beta - a_\alpha + \delta - 1 \leq 0$. Therefore, $a_\beta + \delta \leq a_\alpha$.

Definition 3.13 Let ξ be a point on a blow up model $\ddot{W}$. Define the *multiplicity of double blow ups* as

$$\mu_{db}(\xi) = \max\left\{\#\{\text{double blow ups}/\xi \text{ before having a single blow up}/\xi\}\right\}$$

where the maximum is taken over all sequences of blow ups from $\ddot{W}$ to W. The next lemma shows the boundedness of this number.

Lemma 3.14 *$\mu_{db}(\xi)$ is bounded.*

Proof By Lemma 3.11 each double blow up adds a non-negative number to the total negativity of the system. Moreover, the total negativity is bounded because the total negativity on $\overline{W}$ is bounded.[3.14.1] Therefore, except for a bounded number of double blow ups, we have

$$\frac{-\delta}{2} \leq N_{\tilde{W}}(G_\alpha) = a_\alpha - a_\beta - a_\gamma \leq 0$$

where G_α is the double blow up of some G_β and G_γ and $G_\beta \cap G_\gamma/\xi$. The inequality shows that $a_\beta + \frac{\delta}{2} \leq a_\beta + a_\gamma - \frac{\delta}{2} \leq a_\alpha$ and similarly $a_\gamma + \frac{\delta}{2} \leq a_\alpha$. In other words the log discrepancy is increasing at least by $\frac{\delta}{2}$. Since log discrepancies are in $[\delta, 1]$, the number of these double blow ups has to be bounded. □

Definition 3.15 Let $\xi \in \ddot{W}/W$ a blow up model. Define the *single blow up multiplicity* of ξ as:

$$\mu_{sb}(\xi) = \max\left\{\#\{G\colon G \text{ is the exceptional divisor of a single blow up}/\xi\}\right\}$$

The maximum is taken over all sequences of blow ups from $\ddot{W}$ to W. Moreover, define $\mu_{sb}(G_\beta) = \sum_{\xi \in G_\beta} \mu_{sb}(\xi)$ and $\mu_{sb}(\ddot{W}) = \sum_{\xi \in \ddot{W}} \mu_{sb}(\xi)$.

[3.14.1] This boundedness for the A_r and D_r cases is shown in Lemma 3.10 and in the other cases it is obvious.

So, if ξ_2/ξ_1 (these points may be on different models), then $\mu_{sb}(\xi_1) \geq \mu_{sb}(\xi_2)$.

Remark 3.16 Usually there is not a unique sequence of blow ups from $\ddot{W}$ to W. In fact, if ξ_1, ξ_2 are distinct points on $\ddot{W}$ and they are centres of some exceptional divisors on W, then it does not matter which one we blow up first in order to get to W.

Definition 3.17 Let $\xi \in \overline{\text{exc}}(\ddot{W}/Z)$ be a point on a blow up model $\ddot{W}$. We call such a point a *generating point* if there is an exceptional divisor G_α/ξ on a blow up model $\tilde{W}$ such that $N_{\tilde{W}}(G_\alpha) < \delta - 1$.

Remark 3.18 By Lemma 3.10 and Lemma 3.11 if $\xi \in \overline{\text{exc}}(\ddot{W}/Z)_{G_\beta}$ and $\overline{\text{exc}}(\ddot{W}/Z)_{G_\beta}$ is of type A_r (non-strictly monotonic) , D_r, E_6, E_7 or E_8, then ξ can not be a generating point. Moreover, again by Lemma 3.10, if $\overline{\text{exc}}(\ddot{W}/Z)_{G_\beta}$ is strictly monotonic, then there can be at most one generating point in $\overline{\text{exc}}(\ddot{W}/Z)_{G_\beta}$ and it can only belong to the base curve.

Lemma 3.19 *$\mu_{sb}(\xi)$ is bounded if $\xi \in \ddot{W}$ is not a generating point.*

Proof If G_α/ξ is a single blown up exceptional divisor, then $\delta - 1 \leq N_{\tilde{W}}(G_\alpha)$ since ξ is not a generating point. So if G_α is a single blow up$/\xi$ of G_β, then $a_\alpha \geq a_\beta + \delta$, that is, it increases the log discrepancy at least by δ. Moreover, as in the proof of Lemma 3.14, except for a bounded number of double blow ups, any double blow up$/\xi$ increases the log discrepancy at least by $\frac{\delta}{2}$. Hence, there can be only a bounded number of blow ups$/\xi$ from $\ddot{W}$ to W. $\square$

Corollary 3.20 *The number of exceptional curves$/\xi$ on W is bounded for any nongenerating point $\xi \in \ddot{W}$.*

We now continue the proof of Theorem 3.1. If no divisor in ω_{Sh} has coefficient 1, then this is what we are looking for. Since in this case $K_W + \omega_{Sh}$ will be a $\frac{1}{6}$-lc log divisor. If the opposite happens,that is, some divisors appear with coefficient 1 in ω_{Sh}, then these divisors will form a connected chain $\mathcal{C}$ which does not intersect with any other exceptional divisor$/Z$ with positive coefficient in ω_{Sh}, except the edges of this chain. Some of the exceptional divisors of type F may appear with positive coefficients and some with zero coefficients in ω_{Sh}.

The image of the graph $\mathcal{G}$ on $W_{\mathcal{G}}$, that is $\mathcal{G}_1$ (see Definition 3.6), is either of type A_r, D_r, E_6, E_7 or E_8 because similar to what we proved above for $\overline{W}$ the model $W_{\mathcal{G}}$ is the minimal resolution of some surface, namely, the minimal resolution of the surface $X_{\mathcal{G}}$ obtained from X by contracting the exceptional$/Z$ curves on X whose birational transform belong to $\mathcal{G}$. In fact, there is no -1-curve$/X_{\mathcal{G}}$ on $W_{\mathcal{G}}$.

Now, suppose $\mathcal{G}_1$ is of type A_r and not strictly monotonic. Let the push-down on $W_{\mathcal{G}}$ of the chain $\mathcal{C}$ be $\mathcal{C}_1$. Let the exceptional divisors of $\mathcal{G}_1$ be $G_{\beta_1}, \dots, G_{\beta_r}$ and assume that the chain $\mathcal{C}_1$ consists of $G_{\beta_k}, \dots, G_{\beta_l}$. Hence $N^{Sh}_{W_{\mathcal{G}}}(G_{\beta_k}) \leq -\frac{1}{6}$, $N^{Sh}_{W_{\mathcal{G}}}(G_{\beta_{k+1}}) = \cdots = N^{Sh}_{W_{\mathcal{G}}}(G_{\beta_{l-1}}) = 0$ and $N^{Sh}_{W_{\mathcal{G}}}(G_{\beta_l}) \leq -\frac{1}{6}$. Here the superscript Sh means that we compute the negativity according to the Shokurov log divisor not the primary log divisor. Note that if $a^{Sh}_{\beta} > 0$ for some β, then $a^{Sh}_{\beta} \geq \frac{1}{6}$ because the denominator of a^{Sh}_{β} is in $\{1, 2, 3, 4, 6\}$. The chain $\mathcal{C}_1$ is of type A_{l-k+1}. From the constructions in the local isomorphic section we can replace the Shokurov log numbers $a^{Sh}_{\beta_k} = 0, \dots, a^{Sh}_{\beta_l} = 0$ with new log numbers with bounded denominators and preserve all other Shokurov log numbers in the graph $\mathrm{exc}(W_{\mathcal{G}}/Z)$ so that we obtain a new log divisor $K_{W_{\mathcal{G}}} + \Omega_1$ on $W_{\mathcal{G}}$ which is antinef$/Z$ and Klt. Now put $K_W + \Omega =^* (K_{W_{\mathcal{G}}} + \Omega_1)$. The only problem with Ω is that it may have negative coefficients (it is a sub-boundary). Remark 3.18 and Corollary 3.20 guarantee that the negativity of these coefficients is bounded from below. Moreover if an exceptional divisor has negative coefficient in Ω, then it must belong to the graph $\mathcal{G}$. But any exceptional divisor in $\mathcal{G}$ appears with positive coefficient in ω_{Sh}. Since $\omega_{Sh} \geq \omega$ and by the definition of $\mathcal{G}$, any exceptional divisor of type F in $\mathcal{G}$ has positive coefficient at least $\frac{1}{6}$. If E is not of type F but belongs to $\mathcal{G}$, then since B^+ is not zero $P \in Z$ we get positive coefficients in ω_{Sh} for all exceptional$/Z$ curves which are not of type F. Thus all members of $\mathcal{G} = \mathrm{exc}(W/Q)$ appear with positive coefficient in ω_{Sh}.

Now, consider the sum

$$K_W + \Omega + I[K_W + \omega_{Sh}] = (1 + I)K_W + [\Omega + I\omega_{Sh}]$$

where I is an integer. Given that the negative coefficients appearing in Ω are bounded from below, this implies that there is a large bounded I such that the sum $\Omega + I\omega_{Sh}$ is an effective divisor. So by construction the log divisor

$K_W + \frac{[\Omega + I\omega_{Sh}]}{1+I}$ is ϵ-lc and antinef/Z for some fixed rational number $0 < \epsilon$ and the denominators of the coefficients in the log divisor are bounded.

Now assume that $\mathcal{G}_1$ is strictly monotonic and the generating curve is G_{β_1}. By Corollary 3.20 and Remark 3.18 the only place where we may have difficulties is a generating point ξ on the generating curve if there is any such point.

We blow up ξ and get the exceptional divisor G_{α_1}. The chain $G_{\alpha_1}, G_{\beta_1}, \dots, G_{\beta_r}$ is not exactly of type A_{r+1} because G_{α_1} is a -1-curve. But still we can claim that there is at most a base on this chain and it can only be on G_{α_1}. Obviously a generating point cannot be on $G_{\beta_2}, \dots, G_{\beta_r}$. Now suppose that the intersection point of G_{α_1} and G_{β_r} is a generating point. Then, the sum of negativities of all $G_{\alpha_1}, G_{\beta_1}, \dots, G_{\beta_r}$ must be less than $2\delta - 2$. This is impossible because the sum of negativities of all $G_{\beta_1}, \dots, G_{\beta_r}$ on $W_{\mathcal{G}}$ is at least $2\delta - 2$ (remember that blowing up reduces negativity).

Now if on G_{α_1} there is a generating point ξ_1, then again we blow up this point to get G_{α_2} and so on. This process has to stop after finitely many steps (not after bounded steps!). Let the final model be W_ξ and let $G_{\alpha_1}, \dots, G_{\alpha_s}$ be the new exceptional divisors. In fact, we have constructed a chain (because there was at most one generating point on each curve) and by adding the new exceptional divisors to $\mathcal{G}_1$ we get a new graph $\mathcal{G}_2$. Now there is no base point on $\mathcal{G}_2$. All the divisors G_{α_i} have self-intersection equal to -2 except G_{α_s} which is a -1-curve.

Next let $\mathcal{C}_2$ be the pushdown of $\mathcal{C}$, that is, the connected chain of curves with coefficient one in ω_{Sh} on W_ξ. If G_{α_s} is not in $\mathcal{C}_2$, then we proceed exactly as in the non-monotonic case above; that is we assign appropriate coefficients to the members of $\mathcal{C}_2$ and keep all other coefficients in ω_{Sh} on W_ξ. If G_{α_s} is in $\mathcal{C}_2$, then let $\mathcal{C}'$ be the chain $\mathcal{C}_2$ except the member G_{α_s}. This new chain (i.e. $\mathcal{C}'$) if of type A_x and so we can assign appropriate coefficients to its members and put the coefficient of G_{α_s} simply equal to zero and retain all other coefficients in ω_{Sh} on W_ξ. In any case, we construct a Klt log divisor $K + \Omega$ on W_ξ which is antinef/Z and the boundary coefficients are with bounded denominators. The rest is as in the non-monotonic case above.

Suppose the graph $\mathcal{G}_1$ is of type D_r and $\mathcal{C}_\infty \neq \emptyset$ (if it is empty, then we already have Ω_1). Assume that the members of $\mathcal{G}_1$ are $G_\beta, G_{\beta'}, G_{\beta_1}, \dots, G_{\beta_r}$ and the members of $\mathcal{C}_1$ are $G_{\beta_k}, \dots, G_{\beta_l}$. As in the proof of Lemma 3.10 for the D_r case, we have $a^{Sh}_{\beta_1} \leq a^{Sh}_{\beta_2} \leq \dots$. So $k = 1$, therefore $2a^{Sh}_\beta - 0 - 1 \leq 0$ and so $a^{Sh}_\beta \leq \frac{1}{2}$. Similarly $a^{Sh}_{\beta'} \leq \frac{1}{2}$. The chain $\mathcal{C}_1$ is of type A_l and so we can change the coefficients of its members in ω_{Sh} on $W_{\mathcal{G}}$. The rest of the argument is very similar to the above cases. Just note that there is no base point in this case.

The cases E_6, E_7 and E_8 are settled by Remark 3.18 and Corollary 3.20. In these cases, the graph $\mathcal{G}$ is bounded, so assigning the primary log numbers to the members of $\mathcal{G}_1$ and Shokurov log numbers to the rest of the graph $\mathrm{exc}(W_{\mathcal{G}}/Z)$ gives a log divisor which can be used as $K_{W_{\mathcal{G}}}+\Omega_1$. This completes the proof of Theorem 3.1.

$\square$

4 Global case

Remember that all the varieties are algebraic surfaces unless otherwise stated. We first prove the boundedness of varieties and then prove the boundedness of complements. This is somehow the opposite of what we did in the last section. However our proof was inspired by the theory of complements. The following proof makes heavy use of properties of algebraic surfaces. That means that it is not expected to generalise to higher dimension. The method also has some similarity with the proof of Alexeev and Mori [AM] in the sense that both analyse a series of blow ups, but in different ways.

Theorem 4.1 *The* $\mathrm{BAB}_{\delta,2,[0,1]}$ *(1.6) holds.*

Proof Now we reduce to the case $B=0$. Run the anti LMMP on the pair $(X,0)$ i.e. if $-K_X$ is not nef, then contract an extremal ray R where $K_X\cdot R>0$. This obviously contracts a curve in B. Repeating this process gives us a model $(X',0)$ where $-K_{X'}$ is nef and big. Otherwise X' must be with Picard number one and $K_{X'}$ nef. But this is impossible by our assumptions. We prove the boundedness of $\{X'\}$ which in turn implies the boundedness of $\{X\}$. Now we replace (X,B) with $(X',0)$ but we denote it by $(X,0)$. We also assume that $\delta<1$ otherwise X will be smooth and so with bounded index.

Let $W\to X$ be a minimal resolution. The main idea is to prove that there is only a bounded number of possibilities for the coefficients in B_W where $K_W+B_W={}^*K_X$, that is, the index of K_X is bounded.

Strategy: We apply the familiar division into nonexceptional and exceptional cases.

First assume that $(X, 0)$ is nonexceptional. So there will be a $(0, n)$-complement $K_X + B^+$ for $n < 58$. If we run the classical MMP on the pair $(W, 0)$, then we end up with S which is either $\mathbb{P}^2$ or a ruled surface. Since $-(K_S + B_S) = -_*(K_W + B_W)$ is nef and big, K_S cannot be nef. Let $K_W + B_W^+ = {}^*(K_X + B_X^+)$

Lemma 4.2 *Let G be a component of the boundary B_S^+ where $K_S + B_S^+ = {}_*(K_W + B_W^+)$. Then, G^2 is bounded from below and above. Moreover there is only a bounded number of components in B_S^+.*

Proof The boundedness of G^2 follows from the next lemma and the fact that X is δ-lc. The boundedness of number of components in B_S^+ is left to the reader. $\square$

The more general lemma below will also be needed later.

Lemma 4.3 *Let $(T/Z, B_T)$ be an δ-lc WLF pair where T is either $\mathbb{P}^2/\mathrm{pt.}$ or a smooth ruled surface (with no -1-curves) over a curve and suppose $K_T + \overline{B}$ is antinef and lc for a boundary $\overline{B}$. Let M, B'_T be effective divisors with no common component such that $\overline{B} = B'_T + M$. Then, M^2 is bounded from above.*

Proof First assume that $T = \mathbb{P}^2$. In this case the lemma is obvious because if M^2 is too big, then so is $\deg M$ and so it contradicts the fact that $\deg M \leq 3$.

Now assume that T is a ruled surface where F is a general fibre other than those curves in the boundary and let C be a section. The Mori cone of T is generated by its two edges. F generates one of the edges. If all the components of M are fibres, then $M^2 = 0$ and we are done. So, assume otherwise and let $M \equiv aC + bF$, then $0 < M \cdot F = (aC + bF) \cdot F = a$ so a is positive. Let $C^2 = -e$ and consider the following two cases:

1. $e \geq 0$: We know that $K_T \equiv -2C + (2g - 2 - e)F$ where g is a non-negative number [H, V, 2.11]. So we have

$$0 \geq (K_T + M + tC) \cdot F = -2 + a + t$$

for some $t \geq 0$ where $B'_T \equiv tC + uF$ ($u \geq 0$ since $e \geq 0$). Hence $a + t \leq 2$. Calculations give $M^2 = a(2b - ae)$. Since a and e are both nonnegative, M^2

big implies that b is big. But on the other hand we have:

$$0 \geq (K_T + M + tC) \cdot C = (-2 + a + t)(-e) + 2g - 2 - e + b$$

This gives a contradiction if b is too big because e is also bounded. The boundedness of e follows from the fact that T is δ-lc. In the local isomorphic section, we proved that exceptional divisors have bounded selfintersection numbers.

2. $e < 0$: in this case, by [H, V, 2.12] we have $e + 2g \geq 0$ and so:

$$\begin{aligned} 0 \geq (K_T + M) \cdot C &= (-2 + a)(-e) + 2g - 2 - e + b \\ &= 2g + e - 2 - (ae/2) + (2b - ae)/2 \end{aligned}$$

Now since $2g + e - (ae/2) \geq 0$, $(2b - ae)/2 \leq 2$. So, M^2 is bounded because a is also bounded. $\square$

Let $P \in X$ be a singular point. If P is not in the support of B^+, then the index of K_X at P is at most 57 and so bounded. Now suppose that P is in the support of B^+. If the singularity of P is of type E_6, E_7, E_8 or D_r, then again the index of K_X at P is bounded. So assume that the singularity at P is of type A_r. The goal is to prove that the number of curves in $\mathrm{exc}(W/P)$ is bounded. We must prove that the number of -2-curves is bounded because the number of other curves is bounded by the proof of local isomorphic case. Note that the coefficient of any $E \in \mathrm{exc}(W/P)$ in B_W^+ is positive and there is only a bounded number of possibilities for these coefficients. Let $\mathcal{C}$ be the longest connected subchain of -2-curves in $\mathrm{exc}(W/P)$. Run the classical MMP on W to get a model W' such that there is a -1-curve F on W' s.t. it is the first -1-curve that intersects the chain $\mathcal{C}$ (if there is no such W' and F, then $\mathcal{C}$ must consist of a single curve). We have two cases:

1. F intersects, transversally and in one point, only one curve in $\mathcal{C}$, say E. First suppose that E is a middle curve, that is, there are E' and E'' in the chain which both intersect E. Now contract F so E becomes a -1-curve. Then, contract E and then E' and then all those which are on the side of E'. In this case by contracting each curve we increase E''^2 by one. Hence E'' will be a divisor on S in B_S^+ with high self-intersection. By Lemma 4.3 there can be only a bounded number of curves in $\mathcal{C}$ on the side of E'. Similarly there is only a bounded number of curves on the side of E''. So we are done in this case.

Now suppose that E is on the edge of the chain and intersects E'. Let t_E and t_F be the coefficients of E and F in B_W^+ and similarly for other curves.

Let h be the intersection number of F with the curves in $B^+_{W'}$ except those in $\mathcal{C}$ and F itself. Now we have

$$0 = (K_{W'} + B^+_{W'}) \cdot F = t_E + h - 1 - t_F$$

and hence $h = 1 + t_F - t_E$. If $h \neq 0$, then F intersects some other curve not in the chain $\mathcal{C}$. By contracting F then E and then other curves in the chain we get a contradiction again. Now suppose $h = 0$, that is, $t_E = 1$ and $t_F = 0$. In this case let x be the intersection of E with the curves in $B^+_{W'}$ except those in $\mathcal{C}$. So we have

$$0 = (K_{W'} + B^+_{W'}) \cdot E = -2t_E + t_{E'} + x$$

therefore $x = 2t_E - t_{E'} > 0$ and similarly we again get a contradiction.

2. Now assume that F intersects the chain in more than one curve or intersects a curve with intersection number more than one. Suppose the chain $\mathcal{C}$ consists of $E_1, \dots, E_s$ and F intersects $E_{j_1}, \dots, E_{j_l}$. Note that l is bounded. If $F \cdot E_{j_k} > 1$ for all $0 \leq k \leq l$, then contract F. So $E^2_{j_k} \geq 0$ after contraction of F. In addition, they will not be contracted later and so they appear in the boundary B^+_S. Now replace $\mathcal{C}$ with longest connected subchain when we disregard all E_{j_k}. Now go to step one again. If it does not hold return to step two and so on.

Now suppose $F \cdot E_{j_k} = 1$ for some k. So F must intersect at least another E_{j_t} where $t = k+1$ or $t = k-1$. Now contract F so E_{j_k} becomes a -1-curve and it will intersect E_{j_t}. Contracting E_{j_k} and possible subsequent -1-curves will prove that there is a bounded number of curves between E_{j_t} and E_{j_k}. Now after contracting E_{j_k} and all other curves between E_{j_t} and E_{j_k} we have $E^2_{j_m} \geq 0$ for each $m \neq k$. So we again take the longest connected subchain excluding all E_{j_t}. Repeat the procedure. It must stop after a bounded number of steps because the number of curves in B^+_S is bounded. This boundedness implies that there is only a bounded number of possibilities for the coefficients in B_W where $K_W + B_W = {}^*K_X$. By Borisov-M$^{\text{c}}$Kernan W belongs to a bounded family and so complements will be bounded.

Here the proof of the nonexceptional case finishes and from now on we assume that $(X, 0)$ is exceptional.

Let $W \to X$ be a minimal resolution. Let $\tau \in (0, \frac{1}{2})$ be a rational number. If $(X, 0)$ is $\frac{1}{2} + \tau$-lc, then we know that X belongs to a bounded family according to [Bir1, proof of Theorem 5.1, step1]. So we may assume that $(X, 0)$ is not $\frac{1}{2} + \tau$-lc. Blow up all exceptional curves E with log discrepancy

$a_E = a(E, X, 0) \leq \frac{1}{2} + \tau$ to get $Y \to X$ and put $K_Y + B_Y = {}^*K_X$. Fix E_1, one of these exceptional divisors. Let $t \geq 0$ be such that there is an extremal ray R such that $(K_Y + B_Y + tE_1) \cdot R = 0$ and $E_1 \cdot R > 0$ (and s.t. $K_Y + B_Y + tE_1$ is Klt and antinef). Such R exists otherwise there is a $t > 0$ such that $K_Y + B_Y + tE_1$ is lc (and not Klt) and antinef. This is a contradiction by [Sh2, 2.3.1]. Now contract $R\colon Y \to Y_1$ if it is of birational type.

Again by increasing t there will be an extremal ray R_1 on Y_1 such that $(K_{Y_1} + B_{Y_1} + tE_1) \cdot R_1 = 0$ and $E_1 \cdot R_1 > 0$ (preserving the nefness of $-(K_{Y_1} + B_{Y_1} + tE_1)$). If it is of birational type, then contract it and so on. After finitely many steps we get a model $(V_1, B_{V_1} + t_1E_1)$ and a number $t_1 > 0$ with the following possible outcomes:

(4.3.1)

- ◇ $(V_1, B_{V_1} + t_1E_1)$ is Klt, $\rho(V_1) = 1$ and $K_{V_1} + B_{V_1} + t_1E_1 \equiv 0$.
- ◇ $(V_1, B_{V_1} + t_1E_1)$ is Klt and $\rho(V_1) = 2$ and there is a non-birational extremal ray R on V_1 such that $(K_{V_1} + B_{V_1} + t_1E_1) \cdot R = 0$. Moreover $K_{V_1} + B_{V_1} + t_1E_1$ is antinef.

Note that for each element $E \in \operatorname{exc}(Y/X)$, either E is a divisor on V_1 or it is contracted to a point in the support of E_1.

Lemma 4.4 *For any $h > 0$ there is a $\mu > 0$ such that if (T, B) is a δ-lc pair (δ is already fixed) with a component C of B passing through $P \in T$, with a coefficient $t \geq h$, then either K_T is $\delta+\mu$-lc at P or $1-a_E > \mu$ for each exceptional divisor E/P on a minimal resolution of T (a_E = log discrepancy of (T, B) at E).*

Proof If P is smooth or has E_6, E_7, E_8 or D_r type of singularity, then the lemma is clear since the index of K_T at P is bounded in all these cases (see the local isomorphic section). In all these cases there will be an $\mu > 0$ such that K_T is $\delta + \mu$-lc at P.

Now suppose that the singularity at P is of type A_r. Take a minimal resolution $W_T \to T$ with $\operatorname{exc}(W_T/P) = \{E_1, \ldots, E_r\}$ (notation as in the local isomorphic section) and suppose that j is the maximal number such that $\operatorname{mld}(P, T, 0) = a'_j$ (a'_* is the log discrepancy of $(T, 0)$ at E_*) for an exceptional divisor E_j/P. Actually we may assume that $r - j$ is bounded.

By the local isomorphic section , the distance of E_j from one of the edges of $\operatorname{exc}(W_T/P)$ is bounded. We denote the birational transform of C on W_T again by C. Suppose C intersects E_k in $\operatorname{exc}(W_T/P)$. If $k \neq 1$ or r, then $(-E_k^2)a_k - a_{k-1} - a_{k+1} + x = 0$ where a_* shows the log discrepancy of the pair (T, B) at E_* and $x \geq h$. So either $a_{k-1} - a_k \geq \frac{h}{2}$ or $a_{k+1} - a_k \geq \frac{h}{2}$. In either case the distance of E_k is bounded from one of the edges of $\operatorname{exc}(W_T/P)$. If this edge is the same edge as for Ej, then again the lemma is clear since the coefficients of E_k and E_j in ${}^*C$ (now C is on T and ${}^*C$ on W_T) are bounded from below (in other words they are not too small).

Now assume that E_k and E_j are close to different edges. In this case we claim that the coefficients of the members of $\operatorname{exc}(W_T/P)$ in $\overline{B}_{W_T}$ are bounded from below where $K_W + \overline{B}_{W_T} = {}^*(K_T + tC)$. Suppose that the smallest coefficient occurs at E_m. A simple calculation shows that we can assume that E_m is one of the edges of $\operatorname{exc}(W_T/P)$. Hence E_m is within a bounded distance from E_j or from E_k.

Suppose that E_m is within a bounded distance from E_j. If $a'_j \geq \frac{1+\delta}{2}$, then K_T is $\frac{1+\delta}{2}$-lc at P. So we can assume that $a'_j < \frac{1+\delta}{2}$. We prove that all the numbers $1 - a'_j, \ldots, 1 - a'_r$ are bounded from below. In fact, if $1 < j < r$, then $(-E_j^2)a'_j - a'_{j-1} - a'_{j+1} = 0$ (note that $-E_j^2 > 2$ in this case). Now if $a'_{j-1} - a'_j \geq \frac{\delta}{2}$, then the chain will be bounded and thus the index of K_T at P. But if $a'_{j+1} - a'_j \geq \frac{\delta}{2}$ then $a'_r - a'_{r-1} \geq \frac{\delta}{2}$ and so $(-E_r^2 - 1)a'_r = 1 - (a'_r - a'_{r-1}) \leq 1 - \frac{\delta}{2}$. Hence if $m = r$, then we are done. But if $m = 1$, then again the whole chain is bounded and so the index of K_T at P. Now if $j = r$, then again the chain is bounded if $m = 1$ and $a'_m = a'_j = a'_r < \frac{1+\delta}{2}$ if $m = r$.

In the second case, that is, if E_m is within a bounded distance from E_k, then the coefficient of E_m in ${}^*C$ on W is bounded from below. □

Lemma 4.5 *For any $h > 0$ there is a $\gamma > 0$ such that if $(T/\mathrm{pt.}, B)$ is a δ-lc WLF pair (δ is already fixed) with a component C of B passing through $P \in T$ and $t \geq h$ where t is the coefficient of C in B, then K_T is $\delta + \gamma$-lc at P.*

Proof As discussed in Lemma 4.4 we may assume that the singularity at P is of type A_r. Moreover, we assume that $1 - a_k > \mu$ for some fixed number $\mu > 0$ where a_k is the log discrepancy of the pair (T, B) at an exceptional divisor E_k/P on W_T. Here $W_T \to T$ is a minimal resolution and $\operatorname{exc}(W_T/P) = \{E_1, \ldots, E_r\}$. Let $\mathcal{C}$ be the longest connected sub-chain of -2-

curves in $\mathrm{exc}(W_T/P)$ and W_1 a model where $\mathcal{C}$ is intersected by a -1-curve F for the first time, that is, we blow down -1-curves on W_T until we get a model W_1 and a morphism $W_T \to W_1$ such that W_1 is the first model where there is a -1-curve F intersecting $\mathcal{C}$ (on W_1). Let $K_{W_T} + B^+ \equiv 0$ be a (lc) $(0, \mathbb{Q})$-complement of $K_{W_T} + B_{W_T}$. Assume that F intersects E_j in $\mathcal{C}$ and let t_{E_j} and t_F be the coefficients of E_j and F in B^+ on W_T (similar notation for the coefficients of other exceptional divisors). Then, an argument as in the proof of the nonexceptional case gives a contradiction:

1. Suppose F intersects, transversally and in one point, only one curve in $\mathcal{C}$, say E_j. First suppose that E_j is a middle curve, that is, there are E_{j-1} and E_{j+1} in $\mathcal{C}$ which both intersect E_j. Now contract F so E_j becomes a -1-curve. Then, contract E_j and then E_{j-1} and then all those which are on the of E_{j-1}. By contracting each curve we increase E_{j+1}^2 by one. If we continue contracting -1-curves we get S ($S = \mathbb{P}^2$ or a ruled surface with no -1-curve) where E_{j+1} is a component of B_S. By Lemma 4.3 there can be only a bounded number of curves in $\mathcal{C}$ on the side of E_{j-1}. Similarly there is only a bounded number of curves in $\mathcal{C}$ on the side of E_{j+1}. So we are done in this case.

Now suppose that E_j is on the edge of the chain $\mathcal{C}$ and that it intersects E_{j-1}. Let ${B^+}_{W_1} = \dot{B}^+ + M$ (M and $\dot{B}^+$ with no common component) where each component of $\dot{B}^+$ is either F or an element of $\mathcal{C}$. Now we have

$$0 = (K_{W_1} + {B^+}_{W_1}) \cdot F = t_{E_j} - 1 - t_F + (M \cdot F)$$

and thus $M \cdot F = 1 + t_F - t_{E_j}$. Similarly let ${B^+}_{W_1} = \ddot{B}^+ + N$ (N and $\ddot{B}^+$ with no common component) where each component of $\ddot{B}^+$ is either F or an element of $\mathcal{C}$. Then, we have

$$0 = (K_{W_1} + {B^+}_{W_1}) \cdot E_j = -2t_{E_j} + t_{E_{j-1}} + t_F + (N \cdot E_j)$$

and so $t_{E_j} = t_{E_{j-1}} - t_{E_j} + t_F + (N \cdot E_j) > \mu$. Hence $t_{E_{j-1}} - t_{E_j} > \frac{\mu}{3}$ or $t_F > \frac{\mu}{3}$ or $(N \cdot E_j) > \frac{\mu}{3}$.

If $t_F > \frac{\mu}{3}$, then by contracting F we increase M^2 by at least $(M \cdot F)^2 \geq t_F^2 > (\frac{\mu}{3})^2$. We have the same increase when we contract E_j and then E_{j-1} and so on. So Lemma 4.3 shows the boundedness of $\mathcal{C}$.

If $(N \cdot E_j) > \frac{\mu}{3}$, then proceed similar to the last paragraph.

If $t_{E_{j-1}} - t_{E_j} > \frac{\mu}{3}$, then $t_{E_{j-1}} > t_{E_j} + \frac{\mu}{3}$. This implies that $t_{E_j} \leq 1 - \frac{\mu}{3}$, hence $M \cdot F \geq \frac{\mu}{3}$ and so we continue as above.

2. Now assume that F intersects $\mathcal{C}$ in more than one curve or intersects a curve in $\mathcal{C}$ with intersection number more than one. Suppose the chain $\mathcal{C}$ consists of $E_s, \ldots, E_u$ and F intersects $E_{j_1}, \ldots, E_{j_l}$. Note that l is bounded.

If $F \cdot E_{j_k} > 1$ for all $1 \leq k \leq l$, then contract F. So $E_{j_k}^2 \geq 0$ after contraction of F hence E_{j_k} can not be contracted. Therefore, it appears in the boundary on a "minimal" model S (namely, S is the projective plane or a smooth ruled surface with no -1-curve). Replace $\mathcal{C}$ with its longest connected subchain when we disregard all E_{j_k}. From here we can return to step one and repeat the argument.

Now suppose $F \cdot E_{j_k} = 1$ for some k. So F must intersect at least another E_{j_q} where $q = k+1$ or $q = k-1$. Now contract F so E_{j_k} becomes a -1-curve and would intersect E_{j_q}. Contracting E_{j_k} and possible subsequent -1-curves will prove that there are only a bounded number of curves between E_{j_q} and E_{j_k} in $\mathcal{C}$. Now after contracting E_{j_k} and all other curves between E_{j_q} and E_{j_k} we will have $E_{j_m}^2 \geq 0$ for each $m \neq k$. So again we take the longest connected subchain excluding $E_{j_1}, \ldots, E_{j_l}$ and return to step one.

This process must stop after a bounded number of steps because the number of curves in B_S^+ with coefficient $> \mu$ is bounded (S is again a "minimal" model). To prove this latter boundedness note that $(K_S + B^+{}_S) \cdot F = 0$, where we assume that S is a ruled surface and F a fibre. This implies that there is only a bounded number of non-fibre components in B_S^+ with coefficient $> \mu$. Let L be a section and t_L be its coefficient in B_S^+ and F_i fibre components of B_S^+ with $t_{F_i} > \mu$. Then,

$$\begin{aligned} 0 \geq\ & (K_S + t_L L + \textstyle\sum_i t_{F_i} F_i) \cdot L \\ =\ & (-2L + (2g-2-e)F + t_L L + \textstyle\sum_i t_{F_i} F_i) \cdot L \\ =\ & -t_L e + e + 2g - 2 + \textstyle\sum_i t_{F_i} \end{aligned}$$

which proves that there is a bounded number of F_i ($L^2 = -e$ and $e + 2g \geq 0$ if $e < 0$). So the chain $\mathcal{C}$ must have a bounded length. This implies that if we throw C away in the boundary B, then the mld at P will increase by at least a fix number $\gamma > 0$ (γ does not depend on P or T). This proves the lemma. $\square$

Lemma 4.5 settles the first case in 4.3.1 by deleting the boundary B_{V_1}.

Now assume the second case in 4.3.1. Let F be a general fibre of the contraction defined by the extremal ray R. If the other extremal ray of V_1

defines a birational map $V_1 \to Z$, then let H be the exceptional divisor of this contraction (otherwise delete the boundary and use 4.5).

If K_{V_1} is antinef, then use again 4.5. If K_{V_1} is not antinef and if $E_1 \neq H$ then apply Lemma 4.5 to (Z, B_Z). Boundedness of Z implies the boundedness of V_1 and so we can apply [Bir1, Lemma 5.6]. But if K_{V_1} is not antinef and $E_1 = H$, then perform a hat of the third type, as defined in the proof of Theorem 5.1 in [Bir1], with $(U, G_U) := (V_1, B_{V_1} + t_1E_1)$ and $V_2 := U'$. We can use Lemma 4.5 on V_2 or after contracting a curve on V_2 to get the boundedness of V_2. Boundedness of V_2 implies the boundedness of V_1. $\square$

Corollary 4.6 *Conjecture* $\mathrm{WC}_{\delta,2,\Gamma_f}$ *(1.5) holds in the global case where* Γ_f *is a finite subset of rational numbers in* $[0,1]$.

Proof Obvious by Theorem 4.1.

5 References

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