To think about over the term and Christmas vacation.

1. In which integer dimension is the volume of a unit ball the largest?

2. Does there exist an uncountable family $\mathcal{A}$ of subsets of $\mathbb{N}$ such that for every $A, B \in \mathcal{A}$ either $A \subseteq B$ or $B \subseteq A$?

3. Does there exist a function $f : \mathbb{R} \to \mathbb{R}$ which takes every value on every open interval?

4. Suppose that two points x and y are chosen independently at random from the unit cube $[0,1]^d$ in d dimensions. Show that there is some absolute constant $c > 0$ (independent of d) such that $||x| - |y|| \leqslant 1$ with probability at least c. Here, $|\cdot|$ denotes the Euclidean distance to the origin.

5. Let K be a convex shape in the plane. Show that congruent copies of K may be used to tile the plane with efficiency at least 51%.

6. Let G be a finite group which is not abelian. Show that at most 5/8 of the pairs of elements in G commute, and that 5/8 cannot be replaced by any smaller constant.

7. A sequence $(a_i)_{i=1}^{\infty} \subseteq [0,1]$ is such that $a_{i+1} - a_i$ is either $\sqrt{2}$ or $\sqrt{3} (\bmod 1)$. Can the set $A = \{a_1, a_2, \dots\}$ be closed? (That means that any point not in A is contained in an open interval disjoint from A.)

8. Do there exist two 2×2 matrices A and B which generate a free group? (That means that no word $A^{n_1} B^{m_1} A^{n_2} B^{m_2} \dots A^{n_k} B^{m_k}$ is equal to the identity, where n_i, m_i are nonzero integers.)

9. A function $f : \{1, \dots, n\} \to \mathbb{R}$ has the property that $f(x+y) = f(x) + f(y)$ for at least 99.9% of pairs x, y with $x + y \leqslant n$. Show that there is some λ such that $f(x) = \lambda x$ for 90% of $x \in \{1, \dots, n\}$.

10. Show that for any choice of integers $a_1, \dots, a_n$ such that $a_1 + \dots + a_n \equiv 0 \pmod{n}$ there are two permutations $\pi, \sigma : \{1, \dots, n\} \to \{1, \dots, n\}$ such that $\pi(i) + \sigma(i) \equiv a_i (\bmod n)$ for each $i = 1, \dots, n$.