## Baire's category theorem and the axiom of dependent choice

The following theorem from set theory does not need any form of the axiom of choice; it is a consequence of the Zermelo-Fraenkel axioms.

**Theorem 0.1 (The recursion theorem)** *Suppose that $A$ is a non-empty set, that $f$ is a mapping of $A$ to itself and that $\bar{a} \in A$. Then there is a unique sequence $(a_n)_{n \in P}$ such that $a_0 = \bar{a}$ and $a_{s(n)} = f(a_n)$ for $n \in P$.*

(Here $s(n)$ is the successor of $n$, more commonly known as $n+1$: but all this happens before we worry about arithmetic.)

Although recursion enables us to construct sequences, it requires the use of a given function $f$. Let us consider a more general situation. Suppose that $A$ is a nonempty set, and that $\phi$ is a mapping from $A$ into the set $P(A) \setminus \{\emptyset\}$ of non-empty sets of $A$. Suppose that $\bar{a} \in A$. Does there exist a sequence $(a_n)_{n \in P}$ such that $a_0 = \bar{a}$ and $a_{s(n)} \in \phi(a_n)$, for $n \in P$? At stage $n$, we choose $a_{s(n)}$ from the set $\phi(a_n)$. The *axiom of dependent choice* states that this is always possible.

The axiom of dependent choice is used in the proof of Baire's category theorem (where?).

The following exercises show that if Baire's category theorem is true, then the axiom of dependent choice must hold. Suppose that $X$ is a non-empty set, and that $\phi$ is a mapping from $X$ into the set of non-empty subsets of $X$. Let $X_n = X$ for $n \in \mathbf{Z}^+$, and let $P = \prod_{n=0}^{\infty} X_n$. Give each $X_n$ the discrete metric, and give $P$ a uniform product metric $d$.

1. Show that $(P, d)$ is a complete metric space.
2. If $n \in \mathbf{Z}^+$, let $V_n = \{f \in P : \text{there exists } k > n \text{ with } f_k \in \phi(f_n)\}$. Show that $V_n$ is open and dense in $(P, d)$.
3. If Baire's category theorem is true, there exists $f \in \cap_{n=0}^{\infty} V_n$. If $n \in \mathbf{Z}^+$, let $j(n) = \inf\{k : k > n, f_k \in \phi_n\}$. Use recursion to show there exists a an increasing sequence $(c_n)_{n=0}^{\infty}$ such that $c_0 = 0$ and $f(c_{n+1}) \in \phi_{c_n}$ for $n \in \mathbf{N}^+$.
4. Show that the axiom of dependent choice holds.

This was proved by the New Zealand mathematician Robert Goldblatt in 1983.