An argument of Cameron and Erdős.

This brief note is to accompany my paper [2] on the Cameron-Erdős conjecture, in which it is shown that the number of sum-free subsets of $[1, n]$ is asymptotically $c2^{n/2}$ if n is even, and $c'2^{n/2}$ if n is odd. In that paper we needed the following result of Cameron and Erdős [1].

Theorem 1 (Cameron–Erdős) *The number of sum-free subsets of $[\lceil (n+1)/3 \rceil, n]$ is asymptotically $c2^{n/2}$ if n is even and $c'2^{n/2}$ if n is odd.*

The purpose of this note is simply to put the argument of Cameron and Erdős in a convenient place (on the web) so that anyone who wishes to read a complete solution of the Cameron–Erdős conjecture may do so with a minimum of trouble.

We shall sketch a proof of Theorem 1 in the case $n = 2r - 1$. The case n even is very similar. Write $\lceil (n+1)/3 \rceil = r - k$, and observe that $k \leq r/3$. The only solutions to $x + y = z$ with $x, y, z \in [r - k, n]$ have $x, y \in [r - k, r + k - 1]$. Thus, if $A \subseteq [r - k, n]$ and if

$$S \;=\; (A \cap [r - k, r + k - 1]) - (r - k) \;\subseteq\; [0, 2k - 1]$$

then A is sumfree precisely if it is disjoint from

$$T \;=\; ((S + S) + 2(r - k)) \cap [1, n].$$

Observe that $2(r - k) \geq r + k$, so that T is disjoint from $[r - k, r + k - 1]$. Hence for a particular S the number of choices for A is precisely

$$2^{n-r-k+1-|((S+S)+2(r-k))\cap[1,n]|} \;=\; 2^{n-r-k+1-|(S+S)\cap[0,2k-1]|}.$$

Thus our task is reduced to proving the following proposition.

Proposition 2 *Let $g(k) = 2^{-k}\sum_S 2^{-|(S+S)\cap[0,2k-1]|}$, where the sum is over all subsets S of $[0, 2k - 1]$. Then $g(k)$ tends to a finite limit as $k \to \infty$.*

Proof. First, write $g(k) = \hat{g}_0(k) + g_0(k)$ where $\hat{g}_0$ and g_0 involve sums over subsets S with $0 \notin S$ and $0 \in S$ respectively. We claim that $\hat{g}_0(k) = g(k - 1)$. To each subset of $[0, 2k - 3]$ corresponds two subsets $S_1 = S + 1$ and $S_2 = S \cup \{2k - 1\}$ of $[1, 2k - 1]$; and

$$(S_1 + S_1) \cap [0, 2k - 1] = (S_2 + S_2) \cap [0, 2k - 1] = ((S + S) \cap [0, 2k - 3]) + 2.$$

Since S_1, S_2 run over all subsets of $[1, 2k - 1]$, we have indeed $\hat{g}_0(k) = g(k - 1)$. Putting $h(k) = g_0(k)$, this shows that $g(k) = g(1) + \sum_{j=2}^{k} h(j)$, and so it suffices to prove that $\sum h(k)$ converges. Write

$$h(k) = h_{00}(k) + h_{01}(k) + h_{10}(k) + h_{11}(k),$$

where $h_{ij}(k)$ are the sums over those sets $S \subseteq [0, 2k-1]$ with $0 \in S$ and

$$(2k \in S+S) \Leftrightarrow i = 1, \qquad (2k+1 \in S+S) \Leftrightarrow j = 1.$$

Each set S gives rise to four sets in the sum for $h(k+1)$, namely S, $S \cup \{2k\}$, $S \cup \{2k+1\}$, $S \cup \{2k, 2k+1\}$. Since $0 \in S$, we have $S \subseteq S+S$. A small calculation allows us to confirm that

$$h(k+1) \ \leq \ \tfrac{9}{8}h_{00}(k) + \tfrac{3}{4}(h_{01}(k) + h_{10}(k)) + \tfrac{1}{2}h_{11}(k),$$

which is at most

$$\tfrac{3}{4}h(k) + \tfrac{3}{8}h_{00}(k). \tag{1}$$

A set S contributes to $h_{00}(k)$ if and only if neither $2k$ nor $2k+1$ is in $S+S$. Thus it contains at most one of each of the pairs $\{1, 2k-1\}$, $\{2k-1, 2\}$, $\{2, 2k-2\}$, $\ldots$, $\{k-1, k+1\}$, and it contains 0 but not k. So S consists of 0 together with an independent set in the path $P = \{1, 2k{-}1, 2, \ldots, k{-}1, k{+}1\}$. Thus certainly $h_{00}(k)$ is at most $\tau(2k{-}2) = \sum_U 2^{-|U|}$, where the sum is over all sets U which do not contain two consecutive elements of $\{1, \ldots, 2k-2\}$.

One can easily set up a recurrence for $\tau(n)$ and solve it, getting

$$\tau(n) \ = \ \frac{3+2\sqrt{3}}{6}\left(\frac{1+\sqrt{3}}{2}\right)^n + \frac{3-2\sqrt{3}}{6}\left(\frac{1-\sqrt{3}}{2}\right)^n.$$

Combining this with (1) yields

$$h(k+1) \ \leq \frac{3}{4}h(k) + C\left(\frac{1+\sqrt{3}}{2}\right)^{2k},$$

from which it is clear that $\sum h(k)$ converges. □

References

[1] Cameron, P.J; Erdős, P. *On the number of sets of integers with various properties*, Number theory (Banff, AB, 1988), 61–79, de Gruyter, Berlin, 1990.

[2] Green, B.J. *The Cameron–Erdős conjecture*, preprint.