

1. Hilbert's Fifth Problem.....

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At the start of the twentieth century David Hilbert, the foremost mathematician of the time, formulated a list of 23 problems to occupy the minds of mathematicians in the coming century. The fifth one asked “Are continuous groups automatically differential groups?” The question as posed is a little vague, but for at least one natural formulation it was solved by Gleason and Yamabe in the 1950s. This theorem states that every locally compact group is a “locally a Lie group” in a certain sense. A Lie group is a rather special type of mathematical object, whereas *a priori* a locally compact topological group is quite general. The theorem is therefore somewhat remarkable.

The techniques they used, as well as the result itself, have very recently become important in additive combinatorics in connection with the study of so-called approximate groups.

This essay is about Hilbert's fifth problem, and the core of it should be an account, not necessarily with absolutely all the details, of the Gleason-Yamabe theorem. You could also include some history of the problem, and sketch links to additive combinatorics or to Gromov's theorem on groups of polynomial growth.

Relevant Courses

None, though Lie Algebras might be a little bit useful. A good grounding in undergraduate analysis will be very helpful. Thankfully, Terence Tao is currently (Michaelmas 2011) giving a course on precisely this topic at UCLA; his notes [1] will be an important resource for this essay but you will need to distill them down quite a bit, and should consult other works too. Interested students should take a look at [1] and then come and talk to me.

References

[1] Terence Tao, *Math 254A: Hilbert's Fifth Problem*,
<http://www.math.ucla.edu/~tao/254a.1.11f/>