Linear Analysis Example Sheet 4 BJG, Michaelmas 2010

1. Let $p : \mathbb{C} \to \mathbb{C}$ is a polynomial. Suppose that $T : X \to X$ is a bounded linear operator on a Hilbert space. Show that the spectrum of $p(T)$ is $\{p(\lambda) : \lambda \in \sigma(T)\}$. Extend this result to rational functions (quotients of two polynomials). How are the spectra of T and T^* related?

2. Suppose that $U : X \to X$ is a unitary map on a Hilbert space X: that is to say, $\langle Ux, Uy\rangle = \langle x, y\rangle$ for all $x, y \in X$. Show that $\sigma(U)$ is contained in the unit circle of the complex plane. By considering $i(U + I)(U - I)^{-1}$, or otherwise, show that $\sigma(U) \neq \emptyset$.

3. Let $T : X \to X$ be a bounded linear operator on a Hilbert space. Show that if one of the maps T, T^*, TT^*, T^*T is compact then they all are.

4. Let $S, T : X \to X$ be bounded linear operators on a Hilbert space. Show that if one of them is compact then ST is compact.

5. Let $T : X \to X$ be a compact self-adjoint operator on a Hilbert space. Show that there is a unique compact self-adjoint operator $T^+ : X \to X$ such that $\|T^+x\| = \|Tx\|$ for all x and T^+ is *positive* in the sense that $\langle T^+x, x\rangle \geqslant 0$ for all $x \in X$.

6. Let $(e_n)_{n\in\mathbb{Z}}$ be a complete orthonormal system in a Hilbert space X, and consider the bounded linear map $T : X \to X$ which maps e_n to e_{n+1}. Describe $\sigma(T)$.

7. A bounded linear operator $T : X \to X$ on a Hilbert space is said to be *normal* if T commutes with T^*. Show that if T is normal then every element of $\sigma(T)$ is an approximate eigenvalue.

8. Is every compact operator on Hilbert space Hilbert-Schmidt?

9. Suppose that T_n are bounded operators on a Hilbert space X such that $T_n x \to 0$ for all $x \in X$. Is it true that $T_n^* x \to 0$?

10. Is there a bounded self-adjoint operator on a Hilbert space with no eigenvalues?

11. Show that any compact set $K \subseteq \mathbb{C}$ is the spectrum of some bounded operator on ℓ_2.

12. Construct a linear operator T on Hilbert space with empty spectrum.