ERGODICITY, UNIQUE ERGODICITY AND GROUP EXTENSIONS

1. Generic points, ergodicity and unique ergodicity

Suppose that (X, μ, T) is a measure-preserving system. Throughout this section we will assume that $T : X \to X$ is continuous. We say that a point $x \in X$ is *generic* if

$$\lim_{N\to\infty} S_N f(x) = \int f\,d\mu$$

for all continuous functions $f \in C(X)$. That is, time averages along orbits starting from x always converge to space averages.

Lemma 1.1. *The transformation T is ergodic if and only if almost all points of X are generic.*

Proof. Suppose that T is not ergodic. Then there is a set $A \subseteq X$ with $0 < \mu(A) < 1$ and $T^{-1}A = A$. If $x \in A$ then $S_N 1_A = 1$ for all N, and so $S_N 1_A(x) \not\to \int 1_A\,d\mu$. Since $C(X)$ is dense in $L^1(X)$ we may find a continuous function f, approximating 1_A, such that $S_N f(x) \not\to \int f\,d\mu$. Thus no point in A can be generic.

Conversely, suppose that T is ergodic. For any fixed function $f \in C(X)$ the Birkhoff pointwise ergodic theorem tells us that $S_N f(x) \to f(x)$ for a.e. $x \in X$. Since a countable intersection of sets with full measure has full measure, the same is true simultaneously for any countable set $(f_i)_{i=1}^{\infty}$ of continuous functions on X, that is to say *all* of the time averages $S_N f_i(x)$ tend to the corresponding space averages $\int f_i\,d\mu$ for x in some set E with measure 1. Since $C(X)$ is separable, we may choose this family to be dense in $C(X)$. Since each map S_N is a contraction in L^∞, it follows that $S_N f(x) \to \int f\,d\mu$ for all $x \in E$. In other words, E consists of generic points. $\square$

Definition 1.2 (Unique ergodicity)**.** We say that T is *uniquely ergodic* if *every* point $x \in X$ is generic.

The remainder of this section is devoted to explaining the use of the word "unique" here. In our discussion of ergodic theory so far, the T-invariant measure μ has been regarded as a fixed component of the measure-preserving system (X, μ, T). Sometimes it is convenient to consider the whole space $\mathcal{M}(X)$ of regular Borel probability measures on X. Recall that this is identified, via the Riesz representation theorem, with a closed convex subset of $C(X)^*$. Inside $\mathcal{M}(X)$ we identify the space $\mathcal{M}(X)^T$ of T-invariant measures.

Lemma 1.3. *Suppose that X is a compact metric space and that $T : X \to X$ is continuous. Then $\mathcal{M}(X)^T$ is nonempty.*

Proof. Let x be any point, and define a sequence of Borel measures μ_N, $N = 1, 2, \dots$ by

$$\mu_n := \mathbb{E}_{0\leqslant n<N} \delta_{T^n x},$$

where δ_y denotes the "Dirac delta" measure for which $\delta_y(E) = 1$ if $y \in E$ and is zero otherwise. The measures μ_N are almost T-invariant, and in fact it is clear that

$$| \int (f \circ T - f) \, d\mu_N | = | \frac{1}{N} (f(T^N x) - f(x)) | \leqslant \frac{2\|f\|_\infty}{N},$$

and so $\mu_N \circ T \to \mu_N$ in the weak topology. Since $\mathcal{M}(X)$ is compact in this topology, there must be some weak limit μ of the μ_Ns, and by the preceding discussion this must be T-invariant. □

It can be rather useful to consider this collection $\mathcal{M}(X)^T$ of T-invariant measures geometrically, as a closed and convex subset of $C(X)^*$. Suppose that S is a closed and convex set in some locally convex topological vector space[1] V. We say that $s \in S$ is *extremal* if, whenever $s = \lambda s_1 + (1 - \lambda) s_2$ for some $\lambda \in (0, 1)$, we have $s_1 = s_2 = s$.

Lemma 1.4. *The extremal points of $\mathcal{M}(X)^T$ are precisely the ergodic T-invariant measures on X.*

***Proof.* Suppose that $\mu \in \mathcal{M}(X)^T$ fails to be ergodic. Then there is a set $A \subseteq X$ with $0 < \mu(A) < 1$ and $T^{-1}A = A$. We have $\mu = \mu(A)\mu_1 + (1 - \mu(A))\mu_2$, where $\mu_1(E) := \mu(E \cap A)/\mu(A)$ and $\mu_2(E) := \mu(E \cap A^c)/\mu(A^c)$. Both μ_1 and μ_2 are T-invariant, and it is clear that neither of them is equal to μ.

To establish the converse we use the Radon-Nikodym theorem. Suppose that $\mu \in \mathcal{M}(X)^T$ is ergodic, yet that $\mu = \lambda\mu_1 + (1 - \lambda)\mu_2$ for some $\lambda \in (0, 1)$ and some T-invariant measures $\mu_1, \mu_2 \in \mathcal{M}(X)^T$. Then μ is absolutely continuous with respect to μ_1, that is to say if $\mu(E) = 0$ then $\mu_1(E) = 0$.

It follows from the Radon-Nikodym theorem that there is a $g \in L^1(X)$ (the Radon-Nikodym derivative of μ with respect to μ_1) such that $\mu_1(E) = \int_E g \, d\mu$. Now for any E we have

$$\mu_1(E) = \mu_1(T^{-1}E) = \int_{T^{-1}E} g \, d\mu = \int_E g \circ T \, d\mu,$$

and so

$$\int_E g \, d\mu = \int_E g \circ T \, d\mu$$

for all E. It follows easily $g - g \circ T$ is orthogonal to everything in L^∞, and so $g = g \circ T$ a.e.. But T is ergodic, and so this implies that g is constant a.e.; it follows that $\mu_1(E) = \mu(E)$, and so μ is indeed extremal.**

We have established that the invariant Borel probability measures $\mathcal{M}(X)^T$ form a compact, convex, non-empty subset of $C(X)^*$, and that the extremal points of this subset are precisely the ergodic T-invariant measures on X. Recalling the Krein-Milman theorem, which states that a closed convex set is the convex closure of its extremal points, we see that there must be at least one ergodic T-invariant measure on T.

In particular if there is just one measure in $\mathcal{M}(X)^T$ then it must be ergodic.

Lemma 1.5. *The system (X, μ, T) is uniquely ergodic if and only if $\mathcal{M}(X)^T = \{\mu\}$.*

Remark. Usually the condition that $\mathcal{M}(X)^T$ consists of just one point is given as the definition of unique ergodicity.

---

[1] I would not worry overly much about exactly what this means; the space $C(X)^*$ certainly is one.

Proof. One direction is not hard. Suppose that $\mu_1, \mu_2 \in \mathcal{M}(X)^T$, and that every point of X is generic for both μ_1 and μ_2. Then, for every $f \in C(X)$ and for an arbitrary $x \in X$ we have

$$S_N f(x) \to \int f\, d\mu_1, \int f\, d\mu_2.$$

In particular $\int f\, d\mu_1 = \int f\, d\mu_2$ for all $f \in C(X)$, which is easily seen to imply that $\mu_1 = \mu_2$.

The converse direction is not especially hard either. If $\mathcal{M}(X)^T = \{\mu\}$, we are to show that *every* time average $S_N f(x)$ converges to $\overline{f} = \int f\, d\mu$. This is a type of ergodic theorem – indeed it is a very strong one, albeit in quite a special situation. To prove it, we employ a decomposition of the sort that was successful in proving both the von Neumann and pointwise ergodic theorems. In this setting we shall take $f \in C(X)$ and show that for any $\varepsilon > 0$ we may decompose it as

$$f = \overline{f} + \partial g + h, \tag{1.1}$$

where $g \in C(X)$ and $\|h\|_\infty \leqslant \varepsilon$. Once this is shown we have, for any $x \in X$,

$$|S_N f(x) - \overline{f}| \leqslant |\frac{1}{N}(g(x) - g(T^N x))| + \|h\|_\infty \leqslant \frac{2}{N}\|g\|_\infty + \|h\|_\infty.$$

If N is large enough, this is less than 2ε. Since ε was arbitrary, we must have $S_N f(x) \to \overline{f}$ and so x is generic.

It remains to establish the decomposition (1.1).

There is one (slightly) technical ingredient, which is the observation that if T is uniquely ergodic with $\mathcal{M}(X)^T = \{\mu\}$ then the only T-invariant *signed* measures are of the form $C\mu$ for some $C \in \mathbb{R}$. **To see this, suppose that ν is a T-invariant signed measure, and consider a Hahn-Jordan decomposition $X = X_+ \cup X_-$ of X (see the updated introductory notes on measure theory). Then it is not hard to see, using the T-invariance of ν, that the symmetric difference $(T^{-1}X_+)\triangle X_+$ is ν-null in the sense that any subset E of this set has $\nu(E) = 0$, and hence $\nu_+(E) = \nu_-(E) = 0$. The same is true for $(T^{-1}X_-)\triangle X_-$. It follows that for any Borel set E we have

$$\nu_+(T^{-1}E) = \frac{\nu(T^{-1}E \cap X_+)}{\nu(X_+)} = \frac{\nu(T^{-1}(E \cap X_+)}{\nu(X_+)} = \frac{\nu(E \cap X_+)}{\nu(X_+)} = \nu_+(E),$$

and so ν_+ is a T-invariant probability measure, and so by unique ergodicity we must have $\nu_+ = \mu$. Similarly $\nu_- = \mu$, and so $\nu = \nu(X_+)\nu_+ - \nu(X_-)\nu_-$ is equal to $C\mu$ where $C = \nu(X_+) - \nu(X_-)^{**}$.

Now write I for the closed subspace of $C(X)$ spanned by coboundaries ($\partial g = g - g \circ T$. Suppose that $\Lambda \in C(X)^*$ is a functional which vanishes on I. By the (signed version of) the Riesz representation theorem we have $\Lambda f = \int f\, d\nu$ for some signed Borel measure ν on X. The fact that Λ vanishes on coboundaries implies that $\int (g - g \circ T)\, d\nu = 0$ for all $g \in C(X)$. Taking limits, the same is true for all $g \in L^1(X)$ and so ν is a T-invariant signed measure. By the preceding discussion we must therefore have $\nu = C\mu$, and in particular the space of Λ which annihilate I is 1-dimensional. It follows that I has codimension one in $C(X)$. However the constant function 1 does not lie in I, as $\int \partial g\, d\mu = 0$ for all g. It follows that $C(X) = I \oplus \mathbb{R}1$, which is precisely the assertion that a decomposition (1.1) always exists. $\square$

As we have already remarked it is follows immediately that if $\mathcal{M}(X)^T$ consists of a single measure then this measure must be ergodic. It is perhaps worth remarking explicitly that if $\mathcal{M}(X)^T$ does *not* consist of a single measure then it must contain at least two *ergodic* measures, since a closed convex subset of $C(X)^*$ of size greater than one must have at least two extremal points by another application of the Krein-Milman theorem.

2. GROUP EXTENSIONS, ERGODICITY AND UNIQUE ERGODICITY

In this section we discuss group extensions in the context of measure-preserving systems. Some of what we say is valid for extensions by any compact topological group K. For simplicity, and with applications in mind, we restrict to the case $K = \mathbb{T} = \mathbb{R}/\mathbb{Z}$ in these notes.

Suppose that (X, μ, T) is a measure-preserving system and let $\rho : X \to \mathbb{T}$ be a measurable map (or cocycle). Then we may define the group extension system $(\tilde{X}, \tilde{\mu}, \tilde{T})$ by taking $\tilde{X} = X \times \mathbb{T}$, $\tilde{\mu} = \mu \times m$ and $\tilde{T}(x, \theta) = (Tx, \theta + \rho(x))$. Here, of course, m is the Lebesgue measure on the circle $\mathbb{T}$. It is trivial to check that $\tilde{\mu}$ is $\tilde{T}$-invariant.

Motivated by the corresponding results we proved for dynamical systems, we might ask about conditions under which ergodicity and unique ergodicity lift in group extension systems. The results are very similar to the result we proved about lifting minimality in the context of dynamical systems.

Proposition 2.1 (Lifting of ergodicity). *Suppose that T is ergodic. Then $\tilde{T}$ is ergodic unless there is some $m \in \mathbb{Z} \setminus \{0\}$ such that $m\rho$ is a coboundary, that is to say there is a measurable function $F : X \to \mathbb{T}$ such that $m\rho(x) = F(Tx) - F(x)$.*

Remark. Given a specific cocycle ρ, it is somewhat harder to rule out the possibility that $m\rho$ is a cocycle than it was in the topological dynamics setting, since F is allowed to be an arbitrary *measurable* function.

Proof. Suppose that $\tilde{T}$ is not ergodic. Then there is a nonconstant $\tilde{T}$-invariant function $f = f(x, \theta)$ which we may assume to be in $L^\infty(\tilde{X})$. The key idea is to consider a Fourier expansion in the vertical direction, which we may write informally as

$$f(x, \theta) \sim \sum_{m \in \mathbb{Z}} \hat{f}(x, m) e^{2\pi i m\theta}, \tag{2.1}$$

where $\hat{f}(x, m) := \int f(x, \theta) e^{-2\pi i m\theta}\, d\theta$. Continuing to proceed informally, we have

$$f \circ \tilde{T}(x, \theta) \sim \sum_{m \in \mathbb{Z}} \hat{f}(Tx, m) e^{2\pi i m\rho(x)} e^{2\pi i m\theta}.$$

Since $f = f \circ \tilde{T}$, uniqueness of the Fourier expansion ought to imply that

$$e^{2\pi i m\rho(x)} \hat{f}(Tx, m) = \hat{f}(x, m) \tag{2.2}$$

for every m. For each $m \neq 0$, write $S_m := \{x : \hat{f}(x, m) = 0\}$. The preceding equation shows that $T^{-1} S_m = S_m$, and so by ergodicity $\mu(S_m) = 0$ or 1 for each m. If $\mu(S_m) = 1$ for all $m \neq 0$ then $\hat{f}(x, m) = 0$ for all $m \neq 0$ and for a.e. x; from the Fourier expansions this ought to imply that f is constant, contrary to assumption. Suppose then that there is some $m \neq 0$ with $\mu(S_m) = 0$, that is to say $\hat{f}(x, m) \neq 0$ for a.e. x. We then have $m\rho(x) = F(Tx) - F(x)$ where $F(x) = \frac{1}{2\pi i} \log \hat{f}(x, m)$; it does not matter which choice of logarithm we take (so long as it is measurable), for this equation is taken modulo one.

**Let us now justify the preceding argument, or rather show how it may be made rigorous. The main worry, of course, lies in clarifying the sense in which the vertical Fourier expansions converge as x varies. In place of (2.1) we consider

$$\sigma_M f(x, \theta) := \sum_{m \leqslant M} \hat{f}(x, m)(1 - \frac{m}{M})e^{2\pi i\theta m},$$

which corresponds to convolving $f(x, \theta)$ in vertical fibres with the Fejér kernel $K_M(\theta)$. Now from standard Fourier analysis on $\mathbb{T}$ (cf. the first example sheet) we have

$$\sigma_M f(x, \cdot) \to f(x, \cdot)$$

in $L^2(\mathbb{T})$ for a.e. x (in particular, for those x for which $f(x, \cdot) \in L^2(\mathbb{T})$). It follows that

$$\|\sigma_M f - f\|^2_{L^2(\tilde{X})} = \int_X \|\sigma_M f(x, \cdot) - f(x, \cdot)\|^2_{L^2(\mathbb{T})}\, d\mu(x) \to 0,$$

that is to say $\sigma_M f \to f$ in $L^2(\tilde{X})$. To justify this one could use the dominated convergence theorem and the fact that $\|\sigma_M f\|_\infty \leqslant \|f\|_\infty$ (it is here that it is important that we are using the Fejér Kernel rather than the Dirichlet kernel).

Similarly we have $\sigma_M(f \circ T) \to f \circ T = f$, and so

$$\sum_{m \leqslant M} (1 - \frac{|m|}{M})(\hat{f}(x, m) - e^{2\pi i m\rho(x)}\hat{f}(Tx, m))e^{2\pi i\theta m} \to 0$$

in $L^2(\tilde{X})$. By Bessel's inequality this can only be the case if (2.2) is satisfied for all m.** □

We turn now to conditions for lifting of unique ergodicity. Of course the lifted system $(\tilde{X}, \tilde{T}, \tilde{\mu})$ has no chance of being uniquely ergodic unless it is ergodic, and it is also clear that the base system (X, T, μ) must be uniquely ergodic (since any T-invariant Borel measure μ' on X may be lifted to a $\tilde{T}$-invariant Borel measure $\mu' \times m$ on $\tilde{X}$). It turns out that these conditions are also sufficient.

Proposition 2.2 (Lifting of unique ergodicity)**.** *Suppose that (X, T, μ) is uniquely ergodic and that $(\tilde{X}, \tilde{T}, \tilde{\mu})$ is ergodic. Then $(\tilde{X}, \tilde{T}, \tilde{\mu})$ is also uniquely ergodic.*

Proof. From Lemma 1.1 we know that $\tilde{\mu}$-a.e point $(x, \theta) \in \tilde{X}$ is generic. The set $\tilde{A}$ of generic points is easily seen to be invariant under the vertical rotations R_α, and hence must be of the form $A \times \mathbb{T}$ for some $A \subseteq X$. Since $\tilde{\mu}(\tilde{A}) = 1$ we must of course have $\mu(A) = 1$. Suppose that $\tilde{T}$ is not uniquely ergodic, and that $\tilde{\mu}'$ is some other ergodic $\tilde{T}$-invariant measure on $\tilde{X}$. Then the set $\tilde{A}'$ of generic points for $\tilde{\mu}'$ must be of the form $A' \times \mathbb{T}$, and it must be disjoint from $\tilde{A}$ (cf. the proof of Lemma 1.5). By Lemma 1.1 we must have $\tilde{\mu}'(\tilde{A}') = 1$. Now it is clear that the projection of $\tilde{\mu}'$ to X is T-invariant, and so by unique ergodicity of the base this projection must be μ. That is, $\tilde{\mu}'(E \times \mathbb{T}) = \mu(E)$ for all measurable $E \subseteq X$. Taking $E = A'$, we obtain $\mu(A') = 1$. Thus A and A' cannot be disjoint, a contradiction. □

3. UNIQUE ERGODICITY OF SKEW TORI

Recall the definition of the skew torus system: given a dimension d and some $\alpha \in \mathbb{R}$, we consider the map $T : \mathbb{T}^d \to \mathbb{T}^d$ given by

$$T(\theta_1, \dots, \theta_d) = (\theta_1 + \alpha, \theta_2 + \theta_1, \dots, \theta_d + \theta_{d-1}).$$

It is not hard to see that this is a measure-preserving system when $\mathbb{T}^d$ is endowed with Lebesgue measure; just as in the setting of topological dynamical systems, it may be realised as a tower of group extensions, the system at level $j+1$ being an extension of that at level j by the cocycle $\rho(\theta_1, \dots, \theta_j) = \theta_j$.

Proposition 3.1. *Suppose that $\alpha \notin \mathbb{Q}$. Then the skew torus system defined above is uniquely ergodic.*

By Proposition 2.2 and induction on the dimension d it suffices to show that the system is ergodic. We prove this directly, rather than appeal to Proposition 2.1 and try to show that no multiple of ρ is a coboundary. The homotopy argument we used earlier in the course is no longer valid, as the function F defining the coboundary $F \circ T - F$ can be *measurable* and not merely continuous. Though measurable functions are "almost" continuous by Lusin's theorem (Littlewood's second principle) I have not managed to push through anything like a clean argument.

Let (X, T, μ) be the skew-torus system at level d and suppose that $f \in L^2(X)$ satisfies $f = f \circ T$ a.e.. We use Fourier analysis, the rigorous justification of which proceeds along very similar lines to those seen in several previous arguments. Expanding f as a Fourier series

$$f(\theta) \sim \sum_{r \in \mathbb{Z}^d} a_r e^{2\pi i r \cdot \theta}, \tag{3.1}$$

we have

$$f \circ T(\theta) \sim \sum_{r \in \mathbb{Z}^d} a_r e^{2\pi i r_1 \alpha} e^{2\pi i M r \cdot \theta},$$

where

$$M(r_1, \dots, r_d)^T = (r_1 + r_2, r_2 + r_3, \dots, r_d)^T.$$

Since the matrix M has an inverse M^{-1} with integer entries, we may rewrite (3.1) as

$$f(\theta) \sim \sum_{r \in \mathbb{Z}^d} a_{Mr} e^{2\pi i M r \cdot \theta}.$$

Comparing coefficients, we see that $a_{Mr} = a_r e^{2\pi i r_1 \alpha}$ for all $r \in \mathbb{Z}^d$. As a consequence of this we see that $|a_r| = |a_{Mr}| = |a_{M^2 r}| = \dots$; since $\sum |a_r|^2 = \|f\|_2^2 < \infty$, we must have $a_r = 0$ unless $M^n r = r$ for some $n \neq 0$. However it is not hard to see by induction that

$$M^n(r_1, \dots, r_d)$$
$$= (r_1 + \binom{n}{1} r_2 + \binom{n}{2} r_3 + \dots + \binom{n}{d-1} r_d, r_2 + \binom{n}{1} r_3 + \dots + \binom{n}{d-2} r_d, \dots, r_d).$$

Thus we can only have $M^n r = r$ if $r_2 = \dots = r_d = 0$. Then, however, we may return to the relation $a_{Mr} = e^{2\pi i r_1 \alpha} a_r$, which becomes simply $a_r = e^{2\pi i r_1 \alpha} a_r$. Since α is irrational, we must have $a_r = 0$ for all such r except $r = 0$. It follows that f is constant a.e., and so T is indeed ergodic. □

In the next section we will apply the following result.

Proposition 3.2 (Exponential sums). *Suppose that* $p(n)$ *is a polynomial of degree* d *with an irrational leading coefficient. Then* $\mathbb{E}_{0\leqslant n<N}e^{2\pi i p(n)} \to 0$ *as* $N \to \infty$*.*

Proof. Recall the relationship between the skew torus system and polynomials, viz that if the leading coefficient of p is γ and if $\alpha = d!\gamma$ then

$$T^n(\Delta^{d-1}p(0), \dots, \Delta p(0), p(0)) = (\Delta^{d-1}p(n), \dots, \Delta p(n), p(n)).$$

Let $f : X \to \mathbb{R}$ be the continuous function defined by $f(\theta) = e^{2\pi i\theta_d}$. Since the skew torus system is uniquely ergodic, the point $x_0 := (\Delta^{d-1}p(0), \dots, \Delta p(0), p(0))$ is generic, and so

$$\mathbb{E}_{0\leqslant n<N}e^{2\pi i p(n)} = S_N f(x_0) \to \int_X f\, d\mu = 0,$$

as required. □

4. REFERENCES

Much of the material in these notes is again from H. Furstenberg, *Recurrence in Ergodic Theory and Combinatorial Number Theory.* I benefitted from discussing some of these issues with B. Host, particularly Proposition 2.1.