ORBITS ON THE HEISENBERG NILMANIFOLD

In this set of notes I shall prove a kind of special case of a special case of Ratner's theorem and apply it to study the distribution of bracket polynomials modulo one. In the last lecture of the course I shall state Ratner's theorem in a fairly general setting and sketch some number-theoretic consequences.

1. Basic facts about the Heisenberg group

These all being simple exercises in manipulation of matrices, I leave proofs to the reader. Define

$$G := \{\left(\begin{smallmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{smallmatrix}\right) : x, y, z \in \mathbb{R}\}.$$

This is a connected and simply-connected 2-step nilpotent Lie group. Indeed it is topologically $\mathbb{R}^3$, and its first commutator $[G, G]$ is given by

$$[G, G] = \{\left(\begin{smallmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right) : z \in \mathbb{R}\}.$$

The third commutator $[G, [G, G]]$ contains only the identity, and that is why G is 2-step nilpotent.

Consider also the subgroup

$$\Gamma := \{\left(\begin{smallmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{smallmatrix}\right) : x, y, z \in \Gamma\}.$$

This is discrete and cocompact, which for our purposes means that the quotient G/Γ is a pleasant space. Indeed it is topologically a 3-dimensional cube with sides identified, as we remarked in the first lecture. Given $g \in G$, it acts on the space $\tilde{X} = G/\Gamma$ by multiplication on the left, giving rise to a continuous transformation $\tilde{T} : \tilde{X} \to \tilde{X}$. There is a natural measure on G, namely the Haar measure. The group G is not compact and so it is not a probability measure, but it can be given very explicitly in terms of Lebesgue measure, the integral being defined by

$$\int F\left(\left(\begin{smallmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{smallmatrix}\right)\right) dxdydz.$$

it is easy to see that this measure is invariant under multiplication on the left and on the right by elements of G. This Haar measure descends naturally to a probability measure $\tilde{\mu}$ on $\tilde{X} = G/\Gamma$, and the left-invariance of Haar measure means that $(\tilde{X}, \tilde{\mu}, \tilde{T})$ is a measure-preserving system which we call a *Heisenberg nilsystem*.

Why the tildes? Well, there is a natural projection map from $\tilde{X}$ to the 2-dimensional torus $X = \mathbb{R}^2/\mathbb{Z}^2$. Indeed the factor map $\pi : G \to G/[G, G]$, which sends $\left(\begin{smallmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{smallmatrix}\right)$ to (x, y), factors through quotienting by Γ.

We note that $\tilde{X}$ is in fact a group extension of X by the circle $\mathbb{R}/\mathbb{Z}$. It is quite instructive (and indeed I put this on the first example sheet) to work out a cocycle ρ

which describes the extension explicitly. We will not need to work with such explicit cocycles here.

2. Questions of ergodicity

Theorem 2.1 (Ergodicity of Heisenberg nilflows)**.** *The transformation $\tilde{T}$ is ergodic if and only if T is.*

Proof. It is clear that if T fails to be ergodic then $\tilde{T}$ does too. Conversely, suppose that T is ergodic. Suppose that $f \in L^2(\tilde{X})$ is $\tilde{T}$-invariant, that is to say $f(gx) = f(x)$ for a.e. $x \in \tilde{X}$. We proceed in a fashion somewhat analogous to that used in the proof of Chapter 5, Proposition 2.1, where we took the Fourier transform of f in "vertical fibres".

In our setting these vertical fibres are cosets of the centre $[G, G]$ of G. Suppressing the rôle of the basepoint in X, we define for all $x \in \tilde{X}$ the functions

$$f_n(x) := \int_0^1 f\left(\left(\begin{smallmatrix} 1 & 0 & \theta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right) x\right) e^{-2\pi i n\theta} d\theta.$$

Note that each function $f_n(x)$ transforms according to the law

$$f_n\left(\left(\begin{smallmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right) x\right) = e^{2\pi i n z} f_n(x). \tag{2.1}$$

It is clear that each f_n is $\tilde{T}$-invariant. Up to a twist by $e^{-2\pi i n z}$, the function f_n is the nth vertical Fourier coefficient of f.

When $n = 0$, the function f_0 is actually $[G, G]$-invariant, and so it descends to a function on X. Since T is ergodic this pushed-down function must be constant μ-a.e., and hence f_0 must be constant $\tilde{\mu}$-a.e.

Suppose now that $n \neq 0$ and, as a hypothesis for contradiction, that f_n does not vanish a.e.

Let $h \in G$ be arbitrary, and define $\theta : G \to \mathbb{R}/\mathbb{Z}$ by

$$[h, g] = \left(\begin{smallmatrix} 1 & 0 & \theta(h) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right).$$

Then we have

$$f_n(hgx) = f_n([h, g]ghx) = e^{2\pi i n\theta(h)} f_n(ghx) = e^{2\pi i n\theta(h)} f_n(hx).$$

Writing $F_h(x) := f_n(hx)\overline{f_n(x)}$, we therefore have

$$F_h(gx) = e^{2\pi i n\theta(h)} F_h(x).$$

However it follows from (2.1) that F_h is $[G, G]$-invariant, and so it too descends to a function on X. This pushed-down function is an eigenfunction of T with eigenvalue $\lambda = e^{-2\pi i n\theta(h)}$. Such an eigenfunction clearly has integral zero unless $\lambda = 1$, and so if $n\theta(h) \neq 0$ in $\mathbb{R}/\mathbb{Z}$ then

$$\int_{\tilde{X}} f_n(hx)\overline{f_n(x)} d\tilde{\mu} = 0.$$

Now by Lusin's theorem (I recommend you work through the details, but I will not examine them) the map $\psi : G \to \mathbb{C}$ defined by

$$\psi(h) = \int_{\tilde{X}} f_n(hx)\overline{f_n(x)}\, d\tilde{\mu}$$

is continuous. Since $\psi(\mathrm{id}) = \|f_n\|_2^2$, which we are supposing to be nonzero, there must be some neighbourhood V of id such that $\psi(h) \neq 0$ for all $h \in V$. For these h we must have $n\theta(h) = 0$ (in $\mathbb{R}/\mathbb{Z}$). Suppose that $n \neq 0$. Since $\theta(\mathrm{id}) = 0$, it follows by continuity (and the fact that G is connected) that $\theta(h) = 0$ for all $h \in V$. This is manifestly a contradiction; one may compute quite explictly, for example, that if

$$g = \begin{pmatrix} 1 & \alpha & \gamma \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix}$$

then

$$\theta\left(\begin{pmatrix} 1 & h_1 & h_3 \\ 0 & 1 & h_2 \\ 0 & 0 & 1 \end{pmatrix}\right) = \alpha h_2 - \beta h_1,$$

which does not vanish in any neighbourhood of the identity.

We have shown that all of the vertical Fourier coefficients of f vanish a.e. except apart from the zeroth mode, which is constant a.e. It follows that f is constant a.e. (for a rigorous justification, which I certainly do not intend to examine, see the starred remarks at the end of Chapter 5, proof of Proposition 2.1). □

As a consequence of this we obtain the following somewhat amusing result.

Proposition 2.2. *Suppose that α, β and 1 are independent over $\mathbb{Q}$. Then for any γ the bracket quadratic $n\gamma + \frac{1}{2}n(n-1)\alpha\beta - n\alpha\lfloor n\beta \rfloor$ is equidistributed modulo 1.*

Proof. Simply observe that if $g = \begin{pmatrix} 1 & \alpha & \gamma \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix}$ then the top right corner of g^n, when reduced $(\mathrm{mod}\,\Gamma)$ to lie in the fundemental domain, is $n\gamma + \frac{1}{2}n(n-1)\alpha\beta - n\alpha\lfloor n\beta \rfloor$. The result follows immediately from the pointwise ergodic theorem. □

Slightly more natural bracket polynomials such as $n\sqrt{2}\lfloor n\sqrt{3} \rfloor$ are also equidistributed modulo 1, but to see this one needs to study orbits on a slightly more elaborate nilmanifold (or alternatively study more complicated types of orbits called *polynomial orbits* on the Heisenberg nilmanifold). Somewhat curiously the bracket quadratic $n\sqrt{2}\lfloor n\sqrt{2} \rfloor$ is *not* equidistributed modulo one, as you might care to check. This may also be understood in terms of nilmanifolds; see Terry Tao's blog post of September 25th 2007 and the ensuing comments for more details!