EXAMPLES 2

PART III ERGODIC THEORY, LENT 2008

1. Let X be a finite set endowed and let $T : X \to X$ be a permutation. When is T ergodic with respect to the uniform probability measure? When is T uniquely ergodic?

2. Is there a uniform version of Khintchine's theorem? That is, does there exist a function $F : [0,1] \times (0,1] \to \mathbb{R}$ such that, whenever (X, μ, T) is a measure-preserving system and $A \subseteq X$ is a measurable set then any interval of length at most $F(\mu(A), \epsilon)$ contains an n such that $\mu(A \cap T^{-n}A) > \mu(A)^2 - \epsilon$?

3. Is it possible for the set of generic points of a measure-preserving transformation to have measure $1/2$?

4. Let $M \in \mathrm{SL}_d(\mathbb{Z})$, and consider the toral automorphism $T_M : \mathbb{T}^d \to \mathbb{T}^d$ defined by $x \mapsto Mx (\bmod 1)$. Give necessary and sufficient conditions on M in order for this transformation to be ergodic.

5. Is the $\times 2$ map on the unit circle uniquely ergodic?

6. Consider the space $X = \{0,1\}^{\mathbb{N}}$ with the product toplogy. Show that there is a Borel probability measure on X such that the measure of the cylinder set $C_{a_1,\dots,a_k} = \{x : x_1 = a_1, \dots, x_k = a_k\}$ is 2^{-k}. Show that the right shift map $T : X \to X$ defined by $(Tx)_n = x_{n+1}$ is measure-preserving and ergodic. (*Hint: binary expansions.*)

7. Let S^* be the Hardy-Littlewood maximal operator. Using the weak type $(1,1)$ bound we proved in lectures, viz that $\mu(x : S^*f(x) \geqslant \lambda) \leqslant \|f\|_{\ell^1}/\lambda$, together with the trivial bound $\|S^*f\|_\infty \leqslant \|f\|_{\ell^\infty}$, show that for every $p \in (1,\infty)$ there is a constant C_p such that $\|S^*f\|_p \leqslant C_p\|f\|_{\ell^p}$. (*Hint: split $f(x)$ into two parts according to whether $f(x)$ is large or not. Treat one part using the weak $(1,1)$ estimate and the other using the ℓ^∞ estimate. Look up "Marcinkiewicz" in Rudin's red book if you have trouble.*) By following the lines of the proof of the pointwise ergodic theorem, deduce that if $f \in L^p$ and if $T : X \to X$ is ergodic then $S_N f \to \overline{f}$ in L^p.

8. Is it true that if T is ergodic then so is T^2?

9. Given a real number x, write $p_n(x)/q_n(x)$ for the nth convergent of x, defined using the continued fraction expansion for x. Show that for almost all $x \in [0,1]$ we have $n^{-1}\log q_n(x) \to \pi^2/12\log 2$.

Reading exercise: look up the more usual, rather slick, proof of the maximal ergodic theorem.

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