EXAMPLES 1

PART III ERGODIC THEORY, LENT 2008

1. Suppose that p is a polynomial with at least one irrational coefficient other than the constant coefficient. Show that the set $(p(n)(\bmod 1))_{n\in\mathbb{N}}$ is dense in $\mathbb{T}$.

2. This exercise uses the fact that every continuous map $f : \mathbb{T}^d \to \mathbb{T}$ is homotopic to a unique map $L_{a_1,\dots,a_d}$ with $a_1, \dots, a_d \in \mathbb{Z}$, where $L_{a_1,\dots,a_d}(\theta_1, \dots, \theta_d) := a_1\theta_1 + \cdots + a_d\theta_d$ (prove it.....) Write $a_i = \Pi_i(f)$.

Consider the following generalised skew torus system $T : \mathbb{T}^d \to \mathbb{T}^d$, defined by a map of the form

$$T(\theta_1, \dots, \theta_d) = (\theta_1 + \gamma, \theta_2 + f_1(\theta_1), \theta_3 + f_2(\theta_1, \theta_2), \dots, \theta_d + f_{d-1}(\theta_1, \dots, \theta_{d-1})),$$

where each $f_i : \mathbb{T}^i \to \mathbb{T}$ is a continuous function with the property that $\Pi_i(f) \neq 0$. If γ is irrational, show that this system is minimal.

3. Suppose that $p_1, \dots, p_k$ are polynomials. Show that the set of $n \in \mathbb{N}$ satisfying $\|p_i(n) - p_i(0)\|_{\mathbb{T}} \leqslant \varepsilon$ for $i = 1, \dots, k$ is syndetic.

4. Give an example of a topological dynamical system which cannot be decomposed into disjoint minimal subsystems.

5. The purpose of this exercise is to guide you through some of the basic Fourier theory we used in lectures. Write $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, and define the convolution $f * g$ of two functions on $\mathbb{T}$ by $f * g(\theta) := \int f(\theta - \theta')\overline{g(\theta')}\, d\theta'$. Define $\widehat{f}(r) := \int f(\theta)e^{-2\pi i\theta r}\, d\theta$.

(1) Define the Nth Fejér kernel $K_N : \mathbb{T} \to \mathbb{R}$ by $K_N(\theta) = \frac{1}{2N+1}(e^{-2\pi iN\theta} + \cdots + 1 + e^{2\pi i\theta} + \cdots + e^{2\pi iN\theta})^2$. Let $f \in C(\mathbb{T})$. Show that $f * K_N \to f$ uniformly. [*Hint. Divide the range of* θ' *in the definition of convolution into two parts*]

(2) Suppose that $f \in L^1(\mathbb{T})$. Show that $f * K_N \to f$ in L^1. [*Hint. Approximate* f *by a continuous function and use the previous part*]

(3) Suppose that $f \in L^1(\mathbb{T})$. Write $f * K_N$ in terms of the Fourier coefficients $\widehat{f}(r)$, $|r| \leqslant N$. [*Hint. You can guess what the formula has to be by "knowing" about the expansion of a function as a Fourier series and about the Fourier coefficients of convolutions. Now prove it rigorously*]

(4) (Uniqueness.) Show that if $f, g \in L^1(\mathbb{T})$ and if $\widehat{f}(r) = \widehat{g}(r)$ for all $r \in \mathbb{Z}$ then $f = g$ almost everywhere.

(5) Suppose that $f \in C^\infty(\mathbb{T})$. Show that the Fourier expansion $\sum_{r\in\mathbb{Z}} \widehat{f}(r)e^{2\pi i\theta r}$ converges uniformly to f [*Hint: recall that* $\widehat{f}(r)$ *is rapidly-decaying. This makes it possible to examine the infinite sum here, and in particular to compute its Fourier coefficients.*]

(6) Suppose that $f \in L^2(\mathbb{T})$. Show that the partial sums $\sum_{|r|\leqslant N} \widehat{f}(r)e^{2\pi i\theta r}$ converge to f in L^2. [*Hint. The exponentials* $e^{2\pi i\theta r}$ *are an orthonormal sequence in* $L^2(\mathbb{T})$.

Use a little Hilbert space theory to reduce to showing that if $\langle f, e^{2\pi i\theta 2}\rangle = 0$ *for all* r *then* $f = 0$ *a.e. Conclude using the previous part.*]

6. Deduce the multiple recurrence theorem from Van der Waerden's theorem.

7. Use van der Waerden's theorem to give an alternative proof that, for every $\varepsilon > 0$, there is an $n \in \mathbb{N} \setminus \{0\}$ such that $|\{n^2\sqrt{2}\}| < \varepsilon$.

8. Show that there is some $\delta > 0$ such that, for all $n \in \mathbb{N}$, the number of $n \leqslant N$ for which $n^2\sqrt{2}(\mathrm{mod}\,1) \in [0, \frac{1}{2})$ is $N/2 + O(N^{1-\delta})$.

9. We say that a set $B \subseteq \mathbb{Z}$ is *thick* if it contains arbitrarily long intervals. A *piecewise syndetic* set is the intersection of a thick set with a syndetic set. Show that if $\mathbb{Z}$ is partitioned into finitely many sets then at least one of these is piecewise syndetic. [*Hint. Why is this question here?*]

10. Consider the Heisenberg nilmanifold G/Γ introduced at the start of the course. Show that this may be exhibited as a group extension system $X \times_\sigma \mathbb{T}$, where $X = \mathbb{R}^2/\mathbb{Z}^2$, at least if one allows the cocycle σ to be only *piecewise* continuous. Can σ be chosen to be continuous?

11$^+$. Let (X, T) be a minimal dynamical system, let $x \in X$ and let U be an open set containing x. Is it true that for each L the set of n such that $T^n x, \dots, T^{Ln} x$ all lie in U is a syndetic set?

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