EXAMPLES 2

PART III ADDITIVE NUMBER THEORY, LENT 2007

1. Suppose that y is large. Use the Selberg sieve to show that the number of primes in the interval $[x, x+y]$ is no more than 100 times the number of primes in the interval $[1, y]$, for any $x \geqslant 0$.

2. Fill in the details of the sketch of van der Corput's bound

$$\sum_{x \in I \cap \mathbb{Z}} e(f(x)) = \int_I e(f(x))\,dx + O(1)$$

given in lectures.

3. Show that if $X \geqslant X_0(k)$ is sufficiently large then at least $\sqrt{X}$ of the numbers up to X are representible as a sum of k non-negative kth powers.

4. Suppose that $A, B \subseteq \{1, \dots, N\}$ are sets of size $N/1000$. Show that

$$\Big| \sum_{a \in A} \sum_{b \in B} e(ab\sqrt{2}) \Big| \ll N^{2-c}$$

for some absolute constant $c > 0$.

5. Let $\theta \in [0, 1]$. Prove that there is some $n \leqslant N$ such that $\|n^2\theta\|_{\mathbb{T}} \ll N^{-c}$, for some absolute constant $c > 0$. *Hints: try to use Fourier analysis to do this. This ought to work unless a certain exponential sum involving θ is large. In that case, θ is major arc and one can proceed alternatively – think about the case when $\theta = a/q$ for example.*

6. Is it true that there is k with the following propety: every sufficiently large number can be written as

$$x_2^2 + x_3^3 + \cdots + x_k^k,$$

with $x_2, \dots, x_k \in \mathbb{Z}_{\geqslant 0}$?