THE BIG O AND OTHER IMPORTANT NOTATION

PART III PRIME NUMBERS, MICHAELMAS 2004

Analytic number theory is a subject which constantly requires one to estimate and to prove inequalities. Often, obtaining an exact inequality like

$$\pi(x+y) - \pi(x) \leqslant 2y/\log y,$$

valid for any $x, y \geqslant 1$ (here $\pi(x)$ is the number of primes less than or equal to x) is a very difficult business. It can, as in this example, force one to compute extensively as well as come up with new ideas.

In many arguments somewhat crude estimates are all that is required. They might only be valid for sufficiently large values of the parameters, or they might involve unspecified absolute constants. For example it is relatively easy to show that

$$\pi(N) \leqslant CN/\log N$$

for all $N \geqslant 2$, for some absolute constant C.

There are two standard ways of abbreviating this statement somewhat. One could write

$$\pi(N) = O(N/\log N),$$

or one could write

$$\pi(N) \ll N/\log N.$$

Both of these mean that there is an absolute constant C such that $\pi(N) \leqslant CN/\log N$ for sufficiently large N.

Write $d(n)$ for the number of divisors of n. The function d grows more slowly than any power of n, a statement that we write as

$$\text{“}d(n) \ll_\epsilon n^\epsilon$$

for any $\epsilon > 0$”. What this means is that for any fixed $\epsilon > 0$ there is a constant C_ϵ (which will depend on ϵ) such that

$$d(n) \leqslant C_\epsilon n^\epsilon$$

for n sufficiently large. The notion of “n sufficiently large” could also depend on ϵ, but here it does not need to as one can cater for small values of n by enlarging C_ϵ if necessary.

Sometimes, when we are feeling genuinely casual, we will write simply $d(n) \ll_\epsilon n^\epsilon$, without saying that ϵ can be an arbitrary positive quantity. This statement can also be written $d(n) = O_\epsilon(n^\epsilon)$.

We might add a subscript ot two to the $\ll$ or O-notation whenever the *implied constant* C depends on some other parameters. For example if N is regarded as variable whilst

k, δ are though of as constant then

$$(\delta^3 + e^{k/\delta}) \log(N + k + \delta^{-2}) \ll_{k,\delta} \log N.$$

As you can imagine, it would be very tedious to carry the extra baggage of explicit constants around with us in a case like this.

The little o-notation is just as useful as the big O-notation. Often one might have a main term plus a small extra term which one wishes to regard as the error. For example $A(N)$, the number of three-term arithmetic progressions $(x, x+d, x+2d)$, $d \neq 0$, contained in $\{1, \dots, N\}$, is given by the formula

$$A(N) = \begin{cases} \frac{1}{4}N^2 - \frac{1}{2}N & (N \text{ even}) \\ \frac{1}{4}N^2 - \frac{1}{2}N + \frac{1}{4} & (N \text{ odd}). \end{cases}$$

This is somwhat unwieldly, and for most purposes it suffices to know that

$$A(N) = \tfrac{1}{4}N^2 + o(N^2).$$

What does this mean? The notation $o(N^2)$ describes a quantity $E(N)$ which is eventually dominated by any constant multiple of N^2, that is to say

$$\lim_{N \to \infty} |E(N)|/N^2 = 0.$$

We could also write

$$A(N) = (\tfrac{1}{4} + o(1))N^2,$$

or we could plump for the more precise statement

$$A(N) = \tfrac{1}{4}N^2 + O(N).$$

As with big-O, one can attach subscripts to little-o. Thus, for example, if N is a large variable whilst δ and k are fixed then

$$\frac{e^k}{\log(N - \delta^{-7})} = o_{k,\delta}(1).$$

Finally, we discuss asymptotics. The prime number theorem, which will be a focus of this course, says that

$$\pi(N) \sim \frac{N}{\log N},$$

which is read as "$\pi(N)$ is asymptotic to $N/\log N$". This means the same as

$$\pi(N) = (1 + o(1))\frac{N}{\log N},$$

which is equivalent to

$$\pi(N) - \frac{N}{\log N} = o(\frac{N}{\log N}).$$