1. Show that the Axiom of Separation is deducible from the Axiom of Replacement. Show also that the Pair-Set Axiom is deducible from the Axioms of Empty-Set, Power-Set and Replacement.

2. Is it true that if x is a transitive set then the relation $\in$ on x is a transitive relation? Does the converse hold?

3. Show that $(\forall x)(\forall y)(x^+ = y^+ \Rightarrow x = y)$ holds in ZF.

4. Let F be a function-class that is an automorphism of $(V, \in)$. Show that F must be the identity.

5. What is the rank of $\{2, 3, 6\}$? What is the rank of $\{\{2, 3\}, \{6\}\}$. Work out the ranks of $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ using your favourite constructions of these objects from ω.

6. A set x is called *hereditarily finite* if each member of $\mathrm{TC}(\{x\})$ is finite. Prove that the class HF of hereditarily finite sets coincides with V_ω. Which of the axioms of ZF are satisfied in the structure HF, *i.e.*, the set HF, with the relation $\in_{\mathrm{HF}} = \in_V \cap (\mathrm{HF} \times \mathrm{HF})$?

7. Which of the axioms of ZF are satisfied in the structure $V_{\omega+\omega}$?

8. What is the cardinality of the set of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$?

9. Is there an ordinal α such that $\omega_\alpha = \alpha$?

10. Explain why, for each $n \in \omega$, there is no surjection from $\aleph_n$ to $\aleph_{n+1}$. Use this fact to show that there is no surjection from $\aleph_\omega$ to $\aleph_\omega^{\aleph_0}$, and deduce that $2^{\aleph_0} \neq \aleph_\omega$.

11. If ZF is consistent then, by Downward Löwenheim–Skolem, it has a countable model. Doesn't this contradict the fact that, for example, the power-set of ω is uncountable?

12. Prove (in ZF) that a countable union of countable sets cannot have cardinality $\aleph_2$.

The remaining questions are on the final, non-examinable chapter of the course.

13. A function between Polish spaces is *Borel* if the inverse image of every Borel set is Borel. Show that images and inverse images of analytic sets under a Borel function are analytic. Show further that if $f\colon X \to Y$ is a continuous function between Polish spaces and f is injective on a Borel set $B \subset X$, then $f(B)$ is Borel.

14. Prove that a set A (in some Polish space) is analytic if and only if there exist closed sets $A_{n_1,\dots,n_k}$ (indexed by finite sequences of positive integers) such that

$$A = \bigcup_{\mathbf{n} \in \mathcal{N}} \bigcap_{k=1}^{\infty} A_{n_1,\dots,n_k} \ .$$

15. Show that the set of functions in the Polish space $C[0, 1]$ that are differentiable everywhere is analytic.