1. How many different partial orders (up to isomorphism) are there on a set of 4 elements? How many of these are complete?

2. Which of the following posets (ordered by inclusion) are complete?

(i) the set of all subsets of $\mathbb{N}$ that are finite or have finite complement

(ii) the set of all linearly independent subsets of a vector space V

(iii) the set of all subspaces of a vector space V

3. Let X be a complete poset, and let $f\colon X \to X$ be order-reversing (meaning that $x \leqslant y$ implies $f(x) \geqslant f(y)$). Give an example to show that f need not have a fixed point. Show, however, that there must exist either a fixed point of f or two distinct points x and y with $f(x) = y$ and $f(y) = x$.

4. Use Zorn's Lemma to show that every partial order on a set extends to a total order.

5. Give a direct proof of Zorn's Lemma (not using ordinals and not using the Axiom of Choice) for countable posets.

6. Show that the statement 'for any sets X and Y, either X injects into Y or Y injects into X' is equivalent to the Axiom of Choice (in the presence of the other rules for building sets). [Hint for one direction: Hartogs' Lemma.]

7. Formulate sets of axioms in suitable languages (to be specified) for the following theories.

(i) the theory of fields of characteristic 2

(ii) the theory of posets having no maximal element

(iii) the theory of bipartite graphs

(iv) the theory of algebraically closed fields

(v) the theory of groups of order 60

(vi) the theory of simple groups of order 60

(vii) the theory of real vector spaces

8. Show that $\{x = y, y = z\} \vdash x = z$ by writing down a proof. Show that the relation defined in the proof of the Model existence lemma for first-order theories is an equivalence relation.

9. Prove the sentence $(\forall x)(\neg x = 0 \Rightarrow (\exists y)(x = sy))$ in Peano Arithmetic.

10. Write down axioms (in the language of partial orders) for the theory of total orders that are dense (between any two elements is a third) and have no greatest or least element. Show that every countable model of this theory is isomorphic to $\mathbb{Q}$. Why does it follow that this theory is complete?

11. Show that the theory of fields of positive characteristic is not axiomatisable (in the language of rings with 1), and that the theory of fields of characteristic zero is axiomatisable but not finitely axiomatisable.

12. Is every countable model of Peano Arithmetic isomorphic to $\mathbb{N}$?

13. Let L be the language consisting of a single function symbol f of arity 1. Write down a theory T that asserts that f is a bijection with no finite orbits, and describe the countable models of T. Prove that T is a complete theory.