8 Borel functional calculus and spectral theory

Throughout H is a (non-zero) complex Hilbert space, $\mathcal{B}(H)$ is the C*-algebra of all bounded, linear operators on H, K is a compact Hausdorff space and $\mathcal{B}$ is the Borel σ-field on K.

Operator-valued measures. A *resolution of the identity of H over K* is a map $P\colon \mathcal{B} \to \mathcal{B}(H)$ such that

(i) $P(\emptyset) = 0$, $P(K) = I$;

(ii) $P(E)$ is an orthogonal projection for every $E \in \mathcal{B}$;

(iii) $P(E \cap F) = P(E)P(F)$ for all $E, F \in \mathcal{B}$;

(iv) if $E \cap F = \emptyset$, then $P(E \cup F) = P(E) + P(F)$;

(v) for every $x, y \in H$, the function $P_{x,y}\colon \mathcal{B} \to \mathbb{C}$ defined by

$$P_{x,y}(E) = \big\langle P(E)x, y\big\rangle$$

is a regular complex Borel measure on K.

Example. $H = L_2[0,1]$, $K = [0,1]$ and $P(E)f = \mathbf{1}_E \cdot f$.

Simple properties.

(i) Any two projections $P(E)$ and $P(F)$ commute.

(ii) if $E \cap F = \emptyset$, then $P(E)H \perp P(F)H$.

(iii) if $x \in H$, then $P_{x,x}$ is a positive measure of total mass $P_{x,x}(K) = \|x\|^2$.

(iv) P is finitely additive but, in general, not countably additive. However, for each $x \in H$, the map $E \mapsto P(E)x$ is a countably additive H-valued function on $\mathcal{B}$.

(v) Although P need not be countably additive, we do have $P\big(\bigcup_n E_n\big) = 0$ whenever $P(E_n) = 0$ for all $n \in \mathbb{N}$.

Motivation. Having defined a notion of measure, our next step is to define a notion of integral. To motivate this, consider a compact hermitian operator T. We know (see the notes *Resumé on Hilbert spaces and Spectral Theory*) that $\sigma(T)$ is countable and every non-zero $\lambda \in \sigma(T)$ is an eigenvalue of T. For $\lambda \in \sigma(T)$ let P_λ be the orthogonal projection onto the eigenspace $E_\lambda = \ker(\lambda I - T)$ (which may be zero if $\lambda = 0$ and $\sigma(T)$ is infinite). Then the series $\sum_{\lambda\in\sigma(T)} \lambda P_\lambda$ converges in norm to T.

For $E \subset \sigma(T)$ define $P(E)$ to be the orthogonal projection onto the orthogonal direct sum $\bigoplus_{\lambda\in E} E_\lambda$ (so that $P(E)(x) = \sum_{\lambda\in E} P_\lambda(x)$ for all $x \in H$). It is straightforward to check that P is a resolution of the identity of H over $\sigma(T)$. Now the only sensible notion of integral on the countable set $\sigma(T)$ is given by summation. So in particular, $\int_{\sigma(T)} \lambda \, \mathrm{d}P = \sum_{\lambda \in \sigma(T)} \lambda P_\lambda = T$. The spectral theorem for normal operators (Theorem 4 below) is a far-reaching generalization of this.

***P*-essentially bounded functions.** Let P be a resolution of the identity of H over K. A Borel function $f \colon K \to \mathbb{C}$ is *P-essentially bounded* if there exists $E \in \mathcal{B}$ with $P(E) = 0$ such that f is bounded on $K \setminus E$. We then set

$$\|f\|_\infty = \inf \left\{ \|f\|_{K \setminus E} : \, E \in \mathcal{B}, \; P(E) = 0 \right\} ,$$

where $\|f\|_{K \setminus E} = \sup_{z \in K \setminus E} |f(z)|$. As in the case of scalar measures, the infimum is attained: there is a Borel set E with $P(E) = 0$ such that $\|f\|_\infty = \|f\|_{K \setminus E}$. The set $L_\infty(P)$ of all P-essentially bounded Borel functions on K is a commutative, unital C*-algebra with pointwise operations and norm $\|\cdot\|_\infty$. (Technically, $\|\cdot\|_\infty$ is not a norm as $\|f\|_\infty = 0$ need not imply that $f = 0$. As usual in measure theory, we identify functions f and g if there is a Borel set E with $P(E) = 0$ such that f and g agree on $K \setminus E$, *i.e.*, when $f = g$ P-almost everywhere.)

Lemma 1. (Definition of $\int_K f \, \mathrm{d}P$.) Let P be a resolution of the identity of H over K. Then there is an isometric, unital $*$-isomorphism Φ of $L_\infty(P)$ onto a commutative, unital C*-subalgebra of $\mathcal{B}(H)$ such that

(i) $\langle \Phi(f)x, y \rangle = \int_K f \, \mathrm{d}P_{x,y}$ for every $f \in L_\infty(P), \; x, y \in H$;

(ii) $\|\Phi(f)x\|^2 = \int_K |f|^2 \, \mathrm{d}P_{x,x}$ for every $f \in L_\infty(P), \; x \in H$;

(iii) $S \in \mathcal{B}(H)$ commutes with every $\Phi(f)$ if and only if it commutes with every $P(E)$.

Proof. Let s be a simple function, *i.e.*, $s = \sum_{i=1}^m \alpha_i \mathbf{1}_{E_i}$ for some measurable partition $E_1, \dots, E_m$ of K and some $\alpha_1, \dots, \alpha_m \in \mathbb{C}$. Define

$$\Phi(s) = \sum_{i=1}^m \alpha_i P(E_i) \; . \tag{1}$$

Let $t = \sum_{j=1}^n \beta_j \mathbf{1}_{F_j}$ be another simple function. To check that Φ is well defined, assume that $s = t$ a.e. Then for all i, j, since $s = \alpha_i$ and $t = \beta_j$ on $E_i \cap F_j$, either $P(E_i \cap F_j) = 0$ or $\alpha_i = \beta_j$. It follows that

$$\sum_i \alpha_i P(E_i) = \sum_{i,j} \alpha_i P(E_i \cap F_j) = \sum_{i,j} \beta_j P(E_i \cap F_j) = \sum_j \beta_j P(F_j) \; ,$$

where we used finite additivity of P. This shows that Φ is well defined.

Since orthogonal projections are hermitian, we have $\Phi(\bar{s}) = \Phi(s)^*$. Next, we have

$$\Phi(s)\Phi(t) = \sum_{i,j} \alpha_i \beta_j P(E_i) P(F_j) = \sum_{i,j} \alpha_i \beta_j P(E_i \cap F_j) = \Phi(st) \; .$$

Similarly, $s + t = \sum_{i,j} (\alpha_i + \beta_j) \mathbf{1}_{E_i \cap F_j}$, and hence

$$\begin{aligned} \Phi(s+t) &= \sum_{i,j} (\alpha_i + \beta_j) P(E_i \cap F_j) \\ &= \sum_{i,j} \alpha_i P(E_i \cap F_j) + \sum_{i,j} \beta_j P(E_i \cap F_j) \\ &= \sum_i \alpha_i P(E_i) + \sum_j \beta_j P(F_j) = \Phi(s) + \Phi(t) \ , \end{aligned}$$

where we again used finite additivity of P. Note that $\Phi(\mathbf{1}_K) = P(K) = I$. Thus Φ is a unital $*$-homomorphism on the unital $*$-subalgebra of simple functions.

If $x, y \in H$, then from (1) we get

$$\langle \Phi(s)x, y \rangle = \sum_{i=1}^m \alpha_i \langle P(E_i)x, y \rangle = \sum_{i=1}^m \alpha_i P_{x,y}(E_i) = \int_K s \, \mathrm{d}P_{x,y} \ .$$

Hence, we obtain

$$\|\Phi(s)x\|^2 = \langle \Phi(s)x, \Phi(s)x \rangle = \langle \Phi(s)^* \Phi(s)x, x \rangle = \langle \Phi(|s|^2)x, x \rangle = \int_K |s|^2 \, \mathrm{d}P_{x,x} \ .$$

It follows that

$$\|\Phi(s)x\|^2 \leqslant \|s\|_\infty^2 \cdot P_{x,x}(K) = \|s\|_\infty^2 \cdot \|x\|^2 \ ,$$

and thus $\|\Phi(s)\| \leqslant \|s\|_\infty$. Conversely, if $s \neq 0$, then there exists j such that $|\alpha_j| = \|s\|_\infty$ and $P(E_j) \neq 0$. Then we can pick a unit vector $x \in P(E_j)(H)$ and, since the projections $P(E_i)$ are pairwise orthogonal, we have $\Phi(s)(x) = \alpha_j x$. This shows that $\|\Phi(s)\| \geqslant |\alpha_j| = \|s\|_\infty$, and hence

$$\|\Phi(s)\| = \|s\|_\infty \ . \tag{2}$$

Now let $f \in L_\infty(P)$. Then there is a sequence (s_n) of simple functions with $\|f - s_n\|_\infty \to 0$. By (2), the sequence $(\Phi(s_n))$ is Cauchy in $\mathcal{B}(H)$, and hence converges to some operator $\Phi(f) \in \mathcal{B}(H)$. This does not depend on the choice of (s_n): if (t_n) is another sequence of simple functions converging to f, then again by (2), we have $\|\Phi(s_n) - \Phi(t_n)\| = \|s_n - t_n\|_\infty \to 0$. The fact that Φ so defined is an isometric, unital $*$-homomorphism of $L_\infty(P)$ to $\mathcal{B}(H)$ satisfying (i) and (ii) follows easily by what has already been proved for simple functions.

Finally, if $S \in \mathcal{B}(H)$ commutes with every $P(E)$, $E \in \mathcal{B}$, then it commutes with $\Phi(s)$ for every simple function s, and hence by continuity it commutes with every $\Phi(f)$, $f \in L_\infty(P)$. The converse assertion is trivial since $P(E) = \Phi(\mathbf{1}_E)$ for every $E \in \mathcal{B}$. □

Note. The identity $\langle \Phi(f)x, y \rangle = \int_K f \, \mathrm{d}P_{x,y}$ uniquely defines $\Phi(f)$. We shall denote $\Phi(f)$ by $\int_K f \, \mathrm{d}P$. So (i) in the lemma becomes

$$\left\langle \left(\int_K f \, \mathrm{d}P \right) x, y \right\rangle = \int_K f \, \mathrm{d}P_{x,y} \ .$$

Bounded Borel functions. We let $L_\infty(K)$ denote the set of all bounded Borel functions $f\colon K \to \mathbb{C}$. This is a commutative, unital C*-algebra equipped with the 'sup norm' $\|f\|_K$. Note that if P is a resolution of the identity of H over K, then $L_\infty(K) \subset L_\infty(P)$ and the inclusion is a norm-decreasing unital $*$-homomorphism.

Theorem 2. (Spectral theorem for commutative C*-algebras.) Let A be a commutative, unital C*-subalgebra of $\mathcal{B}(H)$ and let $K = \Phi_A$. Then there is a unique resolution of the identity P of H over K such that

$$T = \int_K \widehat{T}\,\mathrm{d}P \qquad \text{for every } T \in A\ ,$$

where $\widehat{T}$ is the Gelfand transform of T. Moreover,

(i) $P(U) \neq 0$ for every non-empty open subset U of K; and for $S \in \mathcal{B}(H)$,

(ii) S commutes with every $T \in A$ if and only if it commutes with every $P(E)$ $(E \in \mathcal{B})$.

Remark. The Gelfand map $T \mapsto \widehat{T}$ is an isometric, unital $*$-isomorphism from A onto $C(K)$ (Gelfand-Naimark theorem). The inverse Gelfand map $\widehat{T} \mapsto T$ is an isometric, unital $*$-isomorphism $C(K) \to A \subset \mathcal{B}(H)$. The theorem states that the inverse Gelfand map can be represented as an intergral with respect to an operator-valued measure. So we can think of this theorem as an operator version of the RRT (Riesz Representation Theorem). We will indeed deduce it from the usual scalar RRT.

Proof of Theorem 2. Given $x, y \in H$, consider the map $\widehat{T} \mapsto \langle Tx, y\rangle$ defined on $C(K)$. This is a bounded linear functional on $C(K)$ of norm at most $\|x\|\|y\|$. By the RRT, there is a unique regular complex Borel measure $\mu_{x,y}$ on K such that

$$\langle Tx, y\rangle = \int_K \widehat{T}\,\mathrm{d}\mu_{x,y} \qquad \text{for every } T \in A\ , \tag{3}$$

and moreover $\|\mu_{x,y}\|_1$ (the total variaton norm of $\mu_{x,y}$) is at most $\|x\|\|y\|$. Note that if $\widehat{T}$ is a real function, then T is hermitian, and so

$$\int_K \widehat{T}\,\mathrm{d}\mu_{y,x} = \langle Ty, x\rangle = \overline{\langle Tx, y\rangle} = \int_K \widehat{T}\,\mathrm{d}\overline{\mu_{x,y}}\ .$$

Thus, by the uniqueness in the RRT, we have $\mu_{y,x} = \overline{\mu_{x,y}}$. We also have

$$\int_K \widehat{T}\,\mathrm{d}\mu_{\lambda x+y,z} = \langle T(\lambda x+y), z\rangle = \lambda\langle Tx, z\rangle + \langle Ty, z\rangle = \lambda\int_K \widehat{T}\,\mathrm{d}\mu_{x,z} + \int_K \widehat{T}\,\mathrm{d}\mu_{y,z}$$

for all $x, y, z \in H$ and $\lambda \in \mathbb{C}$. So again, by the uniqueness in the RRT, we obtain $\mu_{\lambda x+y,z} = \lambda\mu_{x,z} + \mu_{y,z}$ for all $x, y, z \in H$ and $\lambda \in \mathbb{C}$. It follows that for fixed $f \in L_\infty(K)$, the map $(x, y) \mapsto \int_K f\,\mathrm{d}\mu_{x,y}$ is a bounded sesquilinear form on H of norm at most $\|f\|_K$, and moreover this is a hermitian form when f is real. Hence there is a unique operator $\Psi(f) \in \mathcal{B}(H)$ of norm at most $\|f\|_K$ such that

$$\big\langle \Psi(f)x, y\big\rangle = \int_K f\,\mathrm{d}\mu_{x,y} \qquad \text{for every } x, y \in H\ .$$

By (3), we have

$$\langle \Psi(\widehat{T})x, y\rangle = \int_K \widehat{T}\,\mathrm{d}\mu_{x,y} = \langle Tx, y\rangle$$

for all $x, y \in H$, and hence $\Psi(\widehat{T}) = T$. This shows that Ψ is an extension to $L_\infty(K)$ of the inverse Gelfand transform. Ψ is easily seen to be linear and, when f is real, $\Psi(f)$ is hermitian, and so $\Psi(\bar{f}) = \Psi(f)^*$ for every $f \in L_\infty(K)$.

We next show that $\Psi(fg) = \Psi(f)\Psi(g)$ for $f, g \in L_\infty(K)$ completing the proof that Ψ is a norm-decreasing, unital $*$-homomorphism from $L_\infty(K)$ to $\mathcal{B}(H)$. Firstly, for $S, T \in A$, we have $\widehat{ST} = \widehat{S}\widehat{T}$, and so

$$\int_K \widehat{S}\widehat{T}\,\mathrm{d}\mu_{x,y} = \langle STx, y\rangle = \int_K \widehat{S}\,\mathrm{d}\mu_{Tx,y} \qquad (x, y \in H)\ .$$

Since this holds for all $S \in A$, by the uniqueness in the RRT, $\widehat{T}\,\mathrm{d}\mu_{x,y} = \mathrm{d}\mu_{Tx,y}$ as measures. Hence for all $f \in L_\infty(K)$, we have

$$\int_K f\widehat{T}\,\mathrm{d}\mu_{x,y} = \int_K f\,\mathrm{d}\mu_{Tx,y} = \big\langle \Psi(f)Tx, y\big\rangle = \big\langle Tx, \Psi(f)^*y\big\rangle = \int_K \widehat{T}\,\mathrm{d}\mu_{x,\Psi(f)^*y}\ .$$

As this holds for every $T \in A$, we get $f\,\mathrm{d}\mu_{x,y} = \mathrm{d}\mu_{x,\Psi(f)^*y}$ as measures (again by the uniqueness in the RRT). Hence for every $g \in L_\infty(K)$,

$$\begin{aligned}\big\langle \Psi(f)\Psi(g)x, y\big\rangle = \big\langle \Psi(g)x, \Psi(f)^*y\big\rangle &= \int_K g\,\mathrm{d}\mu_{x,\Psi(f)^*y} = \int_K gf\,\mathrm{d}\mu_{x,y}\\ &= \big\langle \Psi(fg)x, y\big\rangle\ .\end{aligned}$$

Since this holds for every $x, y \in H$, we obtain $\Psi(fg) = \Psi(f)\Psi(g)$, as required.

We are now ready to define an operator-valued measure by setting $P(E) = \Psi(\mathbf{1}_E)$ for $E \in \mathcal{B}$. It is routine to verify that P is a resolution of the identity of H over K. Note that

$$P_{x,y}(E) = \langle \Psi(\mathbf{1}_E)x, y\rangle = \int_K \mathbf{1}_E\,\mathrm{d}\mu_{x,y} = \mu_{x,y}(E)$$

for all $E \in \mathcal{B}$, and thus $P_{x,y} = \mu_{x,y}$. It follows that for $f \in L_\infty(K)$,

$$\int_K f\,\mathrm{d}P_{x,y} = \big\langle \Psi(f)x, y\big\rangle \qquad \text{for every } x, y \in H\ ,$$

and hence $\int_K f\,\mathrm{d}P = \Psi(f)$. In particular,

$$\int_K \widehat{T}\,\mathrm{d}P = T$$

holds for every $T \in A$, as required.

For the uniqueness of P, note that by the RRT, the requirement

$$\langle Tx, y\rangle = \int_K \widehat{T}\,\mathrm{d}P_{x,y} \qquad \text{for all } T \in A$$

uniquely determines the measure $P_{x,y}$ for every $x, y \in H$. Since $\langle P(E)x, y\rangle = P_{x,y}(E)$, the projections $P(E)$ are also uniquely determined.

Now for the 'moreover' parts. If U is a non-empty open subset of K, then by Urysohn's lemma we can find a non-zero continuous function $f\colon K \to [0,1]$ whose support is contained in U. By the Gelfand-Naimark Theorem, $f = \widehat{T}^2$ for some positive operator $T \in A$. Then $T \neq 0$, and so we can then choose $x \in H$ with $Tx \neq 0$. Observe that

$$0 < \|Tx\|^2 = \langle T^2 x, x\rangle = \int_K f \,\mathrm{d}P_{x,x} \leqslant P_{x,x}(U) \ .$$

This establishes (i). To see (ii), let $S \in \mathcal{B}(H)$ and note that

$$\langle STx, y\rangle = \langle Tx, S^*y\rangle = \int_K \widehat{T}\,\mathrm{d}P_{x,S^*y} \quad \text{and} \quad \langle TSx, y\rangle = \int_K \widehat{T}\,\mathrm{d}P_{Sx,y} \ .$$

Thus, by the uniqueness in the RRT, S commutes with every $T \in A$ if and only if the measures P_{x,S^*y} and $P_{Sx,y}$ are the same for all $x, y \in H$. Since $P_{x,S^*y}(E) = \langle P(E)x, S^*y\rangle = \langle SP(E)x, y\rangle$ and $P_{Sx,y}(E) = \langle P(E)Sx, y\rangle$ for every $E \in \mathcal{B}$, these measures are the same if and only if S commutes with every $P(E)$. □

Exponentials in Banach algebras. Let A be a unital Banach algebra. Then for $x \in A$ we define $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. The sum converges absolutely, and hence it converges in A. It is easy to check that if $x, y \in A$ commute then $e^{x+y} = e^x e^y$ (see Examples Sheet 4, question 8).

Lemma 3. (Fuglede-Putnam-Rosenblum) Let A be a unital C*-algebra and $x, y, z \in A$. Assume that x and y are normal and $xz = zy$. Then $x^*z = zy^*$. In particular, if z commutes with x, then it commutes with x^*.

Proof. We have $x^n z = zy^n$ for all $n \geqslant 0$, and hence $e^{\overline{\lambda}x}z = ze^{\overline{\lambda}y}$ (*i.e.*, $z = e^{-\overline{\lambda}x}ze^{\overline{\lambda}y}$) for $\lambda \in \mathbb{C}$. Since x and y are normal, by the comment about the exponential function above, we have

$$f(\lambda) = e^{\lambda x^*} z e^{-\lambda y^*} = e^{\lambda x^* - \overline{\lambda}x} z e^{\overline{\lambda}y - \lambda y^*} \qquad \text{for all } \lambda \in \mathbb{C} \ .$$

Since involution is continuous, the elements $e^{\lambda x^* - \overline{\lambda}x}$ and $e^{\overline{\lambda}y - \lambda y^*}$ are unitary, and so of norm 1. It follows that f is a bounded analytic function on $\mathbb{C}$. By the vector-valued version of Liouville's theorem, we have $f(\lambda) = f(0) = z$ for all λ, *i.e.*, $e^{\lambda x^*}z = ze^{\lambda y^*}$ for all $\lambda \in \mathbb{C}$. Equating coefficients yields $x^*z = zy^*$, as required. □

Theorem 4. (Spectral theorem for normal operators.) Let $T \in \mathcal{B}(H)$ be a normal operator. Then there is a unique resolution P of the identity of H over $\sigma(T)$ such that

$$T = \int_{\sigma(T)} \lambda \,\mathrm{d}P \ .$$

Moreover, $S \in \mathcal{B}(H)$ commutes with every projection $P(E)$ ($E \in \mathcal{B}$) if and only if $ST = TS$.

This integral representation of T is called the *spectral decomposition* of T. The orthogonal projections $P(E)$, $E \in \mathcal{B}$, are called *spectral projections.*

Proof. Let A be the closed subalgebra of $\mathcal{B}(H)$ generated by I, T, T^*. Since T is normal, A is a commutative, unital C*-subalgebra of $\mathcal{B}(H)$, and moreover, $\sigma_A(T) = \sigma(T)$ by Corollary 7.3. By Lemma 7.2, a character $\varphi \in \Phi_A$ is uniquely determined by its value $\varphi(T)$ at T. It follows that the map $\varphi \mapsto \varphi(T)$ is a homeomorphism from Φ_A onto $\sigma(T)$, which allows us to identify Φ_A with $\sigma(T)$. Note that under this identification, the functions $\widehat{T}$ and $\widehat{T^*}$ on Φ_A correspond to the functions $\lambda \mapsto \lambda$ and $\lambda \mapsto \bar{\lambda}$ on $\sigma(T)$, respectively. The existence of P now follows from Theorem 2.

To see uniqueness, if P is a resolution of the identity of H over $\sigma(T)$ such that $T = \int_{\sigma(T)} \lambda \, \mathrm{d}P$, then by Lemma 1 we have

$$p(T, T^*) = \int_{\sigma(T)} p(\lambda, \bar{\lambda}) \, \mathrm{d}P \tag{4}$$

for every complex polynomial p in two variables. By the Stone–Weierstrass theorem, these polynomials are dense in $C(\sigma(T))$, so by the uniqueness in the RRT, the measures $P_{x,y}$ are uniquely determined by (4) for every $x, y \in H$. This in turn uniquely determines P.

Finally, if $ST = TS$, then $ST^* = T^*S$ by Lemma 3. So $ST = TS$ if and only if S commutes with every element of A, which in turn is equivalent to S commuting with every projection $P(E)$ ($E \in \mathcal{B}$) by Theorem 2. □

Theorem 5. (Borel functional calculus for a normal operator) Let $T \in \mathcal{B}(H)$ be a normal operator and let $K = \sigma(T)$. For $f \in L_\infty(K)$ define

$$f(T) = \int_K f \, \mathrm{d}P \ ,$$

where P is the resolution of the identity over $K = \sigma(T)$ given by Theorem 4. The map $f \mapsto f(T)$ has the following properties:

(i) it is a unital $*$-homomorphism from $L_\infty(K)$ to $\mathcal{B}(H)$ with $z(T) = T$ (where $z(\lambda) = \lambda,\ \lambda \in K$);

(ii) $\|f(T)\| \leqslant \|f\|_K$ for all $f \in L_\infty(K)$ with equality for $f \in C(K)$;

(iii) if $S \in \mathcal{B}(H)$ and $ST = TS$, then $Sf(T) = f(T)S$ for all $f \in L_\infty(K)$.

(iv) $\sigma(f(T)) \subset \overline{f(K)}$ for all $f \in L_\infty(K)$.

Proof. Note that the map $f \mapsto f(T) \colon L_\infty(K) \to \mathcal{B}(H)$ is the composite of the inclusion $L_\infty(K) \subset L_\infty(P)$ (which is a norm-decreasing unital $*$-homomorphism) and the map Φ of Lemma 1 (which is an isometric, unital $*$-isomorphism into $\mathcal{B}(H)$). The restriction of this map to $C(K)$ is the inverse Gelfand map for the unital C*-subalgebra A of $\mathcal{B}(H)$ generated by T (having identified Φ_A with K as in Theorem 4). Properties (i) and (ii) now follow directly from Lemma 1 and Theorems 2 and 4. Next, (iii) follows from Theorem 4 and Lemma 1. Finally, (iv) follows from the fact that a unital $*$-homomorphism maps invertibles to invertibles, and that the spectrum of f in $L_\infty(K)$ is $\overline{f(K)}$. □

Polar decomposition of normal operators. Let $T \in \mathcal{B}(H)$ be a normal operator. Then $T = RU$, where R is positive, U is unitary, and R, U, T pairwise commute.

Proof. Define functions r and u on $\sigma(T)$ by

$$r(\lambda) = |\lambda| \quad \text{and} \quad u(\lambda) = \begin{cases} \frac{\lambda}{|\lambda|} & \text{if } \lambda \neq 0 \ , \\ 1 & \text{if } \lambda = 0 \ . \end{cases}$$

We then apply Borel functional calculus to obtain operators $R = r(T)$ and $U = u(T)$. Since r is positive and u is unitary in $L_\infty(\sigma(T))$, it follows that R is positive and U is unitary in $\mathcal{B}(H)$.

Since $\lambda = r(\lambda)u(\lambda)$ for all $\lambda \in \sigma(T)$, it follows that $T = RU$. Finally, R, U, T pairwise commute since $L_\infty(\sigma(T))$ is commutative. □

Representation of unitary operators. Let $U \in \mathcal{B}(H)$ be a unitary operator. Then $U = e^{iQ}$ for some hermitian operator Q.

Proof. Since U is unitary, $\sigma(U) \subset \mathbb{T}$. Next, there is a bounded, Borel function $f \colon \mathbb{T} \to \mathbb{R}$ such that $e^{if(t)} = t$ for all $t \in \mathbb{T}$ (a Borel branch of logarithm). Apply Borel functional calculus to obtain a hermitian operator $Q = f(U)$. Let $f_n(t) = \sum_{k=0}^{n} \frac{(if(t))^k}{k!}$ $(t \in \mathbb{T})$. Then f_n converges uniformly to the function $t \mapsto t$ on $\mathbb{T}$, and so in particular on $\sigma(U)$. It follows that

$$e^{iQ} = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(iQ)^k}{k!} = \lim_{n\to\infty} f_n(U) = U \ .$$

□

Connectedness of $G(\mathcal{B}(H))$. The group of all invertible operators in $\mathcal{B}(H)$ is connected. Moreover, every invertible operator is the product of two exponentials.

Proof. Let $T \in G(\mathcal{B}(H))$. An application of the Gelfand-Naimark theorem gave polar decomposition $T = RU$, where $R = (TT^*)^{1/2}$ is an invertible positive operator, and $U = R^{-1}T$ is a unitary operator. Since R is positive and invertible, $\sigma(R) \subset (0, \infty)$, and so $\lambda \mapsto \log \lambda$ is a continuous function on $\sigma(R)$. Hence, by Gelfand-Naimark, $R = e^S$ for a hermitian operator S. By the previous application $U = e^{iQ}$ for some other hermitian operator Q. Thus $T = e^S e^{iQ}$ is a product of two exponentials. (Note that S and Q need not commute.)

Finally, the map $t \mapsto e^{tS}e^{itQ}$ $(t \in [0,1])$ is a continuous path in $G(\mathcal{B}(H))$ from I to T. So $G(\mathcal{B}(H))$ is connected. □