1. Let K be an arbitrary set. Let A be an algebra of complex-valued functions on K with pointwise operations, and assume that $\|\cdot\|$ is a complete algebra norm on A. Prove that $A \subset \ell_\infty(K)$ and that $\sup_K |f| \leqslant \|f\|$ for all $f \in A$.

2. Let A be a unital Banach algebra and $x, y \in A$. Show that $\sigma_A(xy)\setminus\{0\} = \sigma_A(yx)\setminus\{0\}$. Can it happen that $\sigma_A(xy) \neq \sigma_A(yx)$? Show that the *commutator* $xy - yx$ of x and y cannot be a non-zero scalar multiple of the identity.

3. Verify that the only characters on the uniform algebra $\mathcal{R}(K)$, where K is a non-empty compact subset of $\mathbb{C}$, are the point evaluations δ_w with $w \in K$. Similarly, show that $\Phi_{A(\Delta)} = \{\delta_w : w \in \Delta\}$ and that $\Phi_W = \{\delta_w : w \in \mathbb{T}\}$, where $A(\Delta)$ is the disc algebra and W is the Wiener algebra.

4. Let $A = \{f \in C(\Delta) : \exists\, g \in A(\Delta),\ g{\restriction_{\mathbb{T}}} = f{\restriction_{\mathbb{T}}}\}$, where $\Delta = \{z \in \mathbb{C} : |z| \leqslant 1\}$, $\mathbb{T} = \partial\Delta$ and $A(\Delta)$ is the disc algebra. Prove that A is a closed subalgebra of $C(\Delta)$ and determine Φ_A. To which well known topological space is Φ_A homeomorphic?

5. Give an example of 2×2 matrices x, y with $r(xy) > r(x)r(y)$ and $r(x+y) > r(x)+r(y)$.

6. Let K be a compact Hausdorff space, and let A be a subalgebra of $C(K)$ that contains the constant functions and separates the points of K. Assume that A is a Banach algebra in some norm $\|\cdot\|$. Prove that $\delta\colon K \to \Phi_A$, $k \mapsto \delta_k$, is a homeomorphism of K into Φ_A. Deduce that A is semisimple. What can you say about the Gelfand map if A is one of $C(K)$, $A(\Delta)$, W or $\mathcal{R}(K)$?

7. Consider $V = L_1[0,1]$ with the L_1-norm and with multiplication given by the "chopped-off" convolution:

$$f * g(x) = \int_0^x f(t)g(x-t)\,\mathrm{d}t\ .$$

Verify that V is a non-unital commutative Banach algebra. Let $A = V_+$ be the unitization of V. What is the Gelfand map of A?

8. Let A be a unital Banach algebra. For $x \in A$ define $e^x = \sum_{n=0}^\infty \frac{x^n}{n!}$. Show that $e^{x+y} = e^x e^y$ whenever x, y are commuting elements. Show that $\sigma_A(e^x) = \{e^\lambda : \lambda \in \sigma_A(x)\}$. Show further that the connected component G_0 of the topological group $G = G(A)$ that contains $\mathbf{1}$ is the subgroup of G generated by $\{e^x : x \in A\}$.

9. Let A be a commutative, unital Banach algebra, $x \in A$, and U an open subset of $\mathbb{C}$ with $U \supset \sigma_A(x)$. Recall that the holomorphic functional calculus is given by

$$\Theta_x(f) = \frac{1}{2\pi\mathrm{i}}\int_\Gamma f(z)(z1 - x)^{-1}\,\mathrm{d}z\ ,$$

where Γ is a cycle in U that encloses $\sigma_A(x)$ but does not enclose any point of $\mathbb{C}\setminus U$. Use Lemma 6.3 to show that if A is semisimple, then Θ_x is multiplicative. By considering a second cycle Γ' in U that encloses $[\Gamma] \cup \{z \in \mathbb{C} : n(\Gamma, z) \neq 0\}$ but does not enclose any point of $\mathbb{C}\setminus U$, show directly that Θ_x is multiplicative in the general case.

10. Let A be a unital Banach algebra and let $x \in A$. Show that

(i) if $\sigma_A(x)$ is disconnected, then A contains a non-trivial idempotent (*i.e.,* not 0 or 1);

(ii) if $\sigma_A(x) \cap (-\infty, 0] = \emptyset$ then $x = e^y$ for some $y \in A$.

11. Let $\|\cdot\|$ and $\|\cdot\|'$ be C^*-norms on a $*$-algebra A. Prove that $\|\cdot\| = \|\cdot\|'$. Deduce that a $*$-isomorphism between C^*-algebras is isometric.

12. A *Banach $*$-algebra* is a Banach algebra with an involution satisfying $\|x^*\| = \|x\|$ for every x. Let $\theta\colon A \to B$ be a $*$-homomorphism from a Banach $*$-algebra A to a C^*-algebra B. Show that $\|\theta(x)\| \leqslant \|x\|$ for all $x \in A$. [*Hint*: First consider the unital case and then use the result of Question 17 about unitization.]

13. Show that $T \in \mathcal{B}(H)$ is positive if and only if $\langle Tx, x\rangle \geqslant 0$ for all $x \in H$.

14. (*Continuous Functional Calculus*) Let $T \in \mathcal{B}(H)$ be a normal operator and $K = \sigma(T)$. Prove that there is a unique unital $*$-homomorphism $f \mapsto f(T)\colon C(K) \to \mathcal{B}(H)$ such that $z(T) = T$, where $z(\lambda) = \lambda$ for all $\lambda \in K$.

Some more questions

15. Show that the direct sum $A \oplus B$ of C^*-algebras A and B is a C^*-algebra with coordinate-wise operations and with $\|(x, y)\| = \max\big\{\|x\|, \|y\|\big\}$.

16. A *double centralizer* for a C^*-algebra A is a pair (L, R) of bounded linear maps on A such that for all $a, b \in A$ we have

$$L(ab) = L(a)b, \quad R(ab) = aR(b) \quad \text{and} \quad R(a)b = aL(b)\ .$$

Let $M(A)$ be the set of all double centralizers for A. Show the following.

(i) For each $c \in A$, the pair (L_c, R_c), where $L_c(a) = ca$ and $R_c(a) = ac$ for all $a \in A$, is a double centralizer for A.

(ii) If (L, R) is a double centralizer for A, then $\|L\| = \|R\|$.

(iii) $M(A)$ is a closed subspace of $\mathcal{B}(A) \oplus \mathcal{B}(A)$.

(iv) $M(A)$ is a C^*-algebra with multiplication and involution defined by

$$(L_1, R_1)(L_2, R_2) = (L_1L_2, R_2R_1) \qquad (L, R)^* = (R^*, L^*)\ ,$$

where for a bounded linear map $T\colon A \to A$, we set $T^*(a) = \big(T(a^*)\big)^*$, $a \in A$.

17. Using the previous two questions, show that if A is a C^*-algebra, then there is a (necessarily unique) C^*-norm on its unitization A_+. [*Hint*: consider separately the cases whether A is unital or not.] Show that this norm extends the norm on A.

18. Let $D = \{z \in \mathbb{C} : |z| < 1\}$, and let $f_n\colon D \to \mathbb{C}$, $n \in \mathbb{N}$, be a sequence of bounded analytic functions converging pointwise to $f\colon D \to \mathbb{C}$. Must f be analytic?

19. Show the following converse to Runge's theorem: if $\mathbb{C} \setminus K$ is not connected, then there is a rational function with poles outside K that cannot be uniformly approximated on K be polynomials.

Deduce that if A is a unital Banach Algebra generated by a single element x, and K is the spectrum of x in A, then $\mathbb{C} \setminus K$ is connected.

20. Let A be a unital Banach algebra. Let $x \in A$ and assume that $\sigma_A(x)$ contains no real number $t \leqslant 0$. Prove that there is a unique element $y \in A$ satisfying $y^3 = x$ and $|\arg \lambda| < \pi/3$ for all $\lambda \in \sigma_A(y)$.

21. Let $A = C(\mathbb{R})$, the algebra of complex-valued continuous functions on $\mathbb{R}$. Prove that $\Phi_A = \{\delta_x : x \in \mathbb{R}\}$. Assume that A has an (incomplete) algebra norm $\|\cdot\|$, and let $\Phi_c = \{\phi \in \Phi_A : \phi \text{ is } \|\cdot\|\text{-continuous}\}$. Show that there exists a compact set $K \subset \mathbb{R}$ such that $\Phi_c = \{\delta_x : x \in K\}$. Deduce that $C(\mathbb{R})$ cannot be given any algebra norm.

22. (i) (*A theorem of Kaplansky.*) Let K be a compact, Hausdorff space, and let $A = C(K)$ with the supremum norm $\|\cdot\|$. Let $\|\cdot\|_1$ be some (possibly incomplete) algebra-norm on A. Show that $\|f\| \leqslant \|f\|_1$ for all $f \in A$.

(ii) Let A and B be unital C^*-algebras and let $\theta\colon A \to B$ be an injective, unital $*$-homomorphism. Prove that θ is an isometry onto a C^*-subalgebra of B.