1. Let K be a compact Hausdorff space, and (f_n) a bounded sequence in $C(K)$. Show that (f_n) is weakly null if and only if it is pointwise null.

2. Show that a Banach space X with its weak topology is a completely regular topological space: given a weakly closed subset C of X and $p \notin C$, there is a continuous function $f\colon (X,w) \to [0,1]$ such that $f(p) = 1$ and $f = 0$ on C.

3. Show the following quantitative version of Lemma 3.3. Let X be a normed space, $f, g_1, g_2, \dots, g_n \in X^*$ and $\varepsilon > 0$. Assume that the restriction of f to $\bigcap_{i=1}^n \ker g_i$ has norm at most ε. Deduce that $d\big(f, \operatorname{span}\{g_1, \dots, g_n\}\big) \leqslant \varepsilon$.

4. Show that the weak and norm topologies on a normed space X coincide if and only if $\dim X < \infty$. Show that the weak topology of an infinite-dimensional normed space and the w^*-topology of the dual space of an infinite-dimensional Banach space are not metrizable.

5. When is the canonical embedding $X^* \to X^{***}$ w^*-to-w^* continuous?

6. Let $T\colon X \to Y$ be a bounded linear map between Banach spaces. Show that if T^* is an into isomorphism, then T is onto (*cf.* Q5 on Sheet 1).

7. Let $T\colon X \to Y$ be a linear map between normed spaces. Show that the following are equivalent.
(i) T is norm-to-norm continuous.
(ii) T is weak-to-weak continuous.
(iii) T is norm-to-weak continuous.

Show further that T is weak-to-norm continuous if and only if T is bounded and of finite-rank.

8. Let $T\colon Y^* \to X^*$ be a bounded linear map between dual spaces. Show that the following are equivalent.
(i) T is w^*-to-w^* continuous.
(ii) There is a bounded linear map $S\colon X \to Y$ with $S^* = T$.
(iii) $T^*(X) \subset Y$.

9. Prove that the closed unit ball B_X of a normed space X is weakly closed. Use this to show that if every norm-Cauchy sequence in X is weakly convergent then X is a Banach space.

10. Show that in ℓ_∞ the set $\{e_n : n \in \mathbb{N}\} \cup \{0\}$ is weakly compact but not norm compact.

11. Let Y be a closed subspace of a Banach space X and $q\colon X \to X/Y$ be the quotient map. Show that q is an open map with respect to the weak topologies of X and X/Y.

12. Show that in ℓ_1 every weakly null sequence is norm null. Rosenthal's beautiful ℓ_1-theorem states that in a Banach space that does not contain an isomorphic copy of ℓ_1 every bounded sequence has a weakly Cauchy subsequence. (A sequence (x_n) in a normed space X is *weakly Cauchy* if $\big(f(x_n)\big)$ is convergent for all $f \in X^*$.) Deduce that in every infinite-dimensional Banach space not containing ℓ_1 there is a normalized, weakly null sequence, *i.e.,* a sequence (x_n) with $\|x_n\| = 1$ for all $n \in \mathbb{N}$ and $x_n \xrightarrow{w} 0$.

13. Show that every weakly null sequence in ℓ_2 has a subsequence (x_n) whose Cesàro sum is zero: $\frac{1}{n}\sum_{i=1}^n x_i$ converges to zero in norm. Show that this fails in *Schreier space* which is defined to be the completion of the space c_{00} of eventually zero scalar sequences with respect to the norm

$$\big\|(a_i)\big\| = \sup\Big\{\sum_{i=1}^k |a_{n_i}| : k \leqslant n_1 < n_2 < \dots < n_k\Big\}\ .$$

Some more questions

14. Let X be a topological space with topology τ. Let $\mathcal{F}$ be a family of functions such that each $f \in \mathcal{F}$ is a function (not necessarily continuous) from X to some topological space Y_f. Assume that for every topological space Z a function $g\colon Z \to X$ is continuous if and only if $f \circ g$ is continuous for all $f \in \mathcal{F}$. Show that $\tau = \sigma(X, \mathcal{F})$.

15. Let K be a locally compact Hausdorff space, and let $C_0(K)$ be the space of continuous functions $f\colon K \to \mathbb{C}$ *vanishing at infinity* : the set $\{x \in K : |f(x)| \geqslant \varepsilon\}$ is compact for all $\varepsilon > 0$. Identify the dual space of $C_0(K)$.

16. Let $0 < p < 1$ and define $L_p(0,1)$ to be the space of Lebesgue measurable functions $f\colon [0,1] \to \mathbb{R}$ for which $\int_0^1 |f(x)|^p \, \mathrm{d}x < \infty$. Show that

$$d(f,g) = \int_0^1 |f(x) - g(x)|^p \, \mathrm{d}x$$

defines a metric on $L_p(0,1)$ (as usual, we identify functions that are equal a.e.). Identify the dual space of $L_p(0,1)$, *i.e.,* the space of linear functionals on $L_p(0,1)$ that are continuous with respect to d.

17. Use the Riesz Representation Theorem to show that there is a unique regular Borel measure λ on $\mathbb{R}$ such that $\lambda\big([a,b]\big) = b - a$ for all $a < b$ in $\mathbb{R}$.

18. (i) Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. A set $N \subset \Omega$ is a *null set* if there exists $A \in \mathcal{F}$ such that $N \subset A$ and $\mu(A) = 0$. Let $\mathcal{N}$ be the collection of null sets. If $\mathcal{N} \subset \mathcal{F}$, then we say μ is a *complete measure* or that $(\Omega, \mathcal{F}, \mu)$ is a *complete measure space.* Now set

$$\overline{\mathcal{F}} = \{A \triangle N : A \in \mathcal{F},\ N \in \mathcal{N}\} \quad \text{and} \quad \bar{\mu}(A \triangle N) = \mu(A)\ , \quad A \in \mathcal{F},\ N \in \mathcal{N}\ .$$

Show that $(\Omega, \overline{\mathcal{F}}, \bar{\mu})$ is a complete measure space (called the *completion of* $(\Omega, \mathcal{F}, \mu)$).

(ii) Let K be a compact Hausdorff space, and let μ^* be the outer measure defined in the proof of the Riesz Representation Theorem corresponding to some positive linear functional on $C(K)$. Let $\mathcal{M}$ be the set of μ^*-measurable sets and let $\mu_1 = \mu^*\restriction_{\mathcal{M}}$. Show that $(K, \mathcal{M}, \mu_1)$ is the completion of $(K, \mathcal{B}, \mu)$.

19. Let X be a real or complex vector space with a topology such that addition, scalar multiplication are continuous, and the sets $\{x\}$, $x \in X$, are closed. Assume further that for every neighbourhood V of 0 there is an open, convex, balanced neighbourhood U of 0 with $U \subset V$ (note: convex is enough). Show that X is a LCS, *i.e.,* the topology of X is induced by a family of seminorms on X that separates the points of X.