1. Let X be a normed space. Show that for $f \in S_{X^*}$ we have $|f(x)| = \operatorname{dist}(x, \ker f)$ for all $x \in X$. Show further that for a closed subspace Y of X and $x_0 \notin Y$ there is $f \in S_{X^*}$ with $Y \subset \ker f$ and $f(x_0) = d(x_0, Y)$.

2. Prove Riesz's lemma: if Y is a proper, closed subspace of a normed space X, then for all $\varepsilon > 0$ there exists $x \in S_X$ with $\operatorname{dist}(x, Y) = \inf\{\|x - y\| : y \in Y\} > 1 - \varepsilon$.

3. Let Y be a closed subspace of a normed space X. Show that the topology on X/Y induced by the quotient norm is the quotient topology induced by the quotient map $q \colon X \to X/Y$. Show further that Y and X/Y are complete if and only if X is complete.

4. Show that every separable Banach space X is the quotient of ℓ_1, *i.e.,* that there is a closed subspace Y of ℓ_1 with $X \cong \ell_1/Y$.

5. Let $T \colon X \to Y$ be a bounded linear map between Banach spaces. Show that
(i) T is an into isomorphism if and only if T^* is onto;
(ii) T^* is an into isomorphism if T is onto; ("only if" also true: see later)
(iii) T^* is injective if and only if $T(X)$ is dense in Y;
(iv) $T(X)$ is closed if and only if $T^*(Y^*)$ is closed.

6. For a subset A of a normed space X, we define the *annihilator of* A as the subset $A^\perp = \{f \in X^* : f(x) = 0 \text{ for all } x \in A\}$ of X^*. Similarly, for $B \subset X^*$, we define the *preannihilator of* B as the subset $B_\perp = \{x \in X : f(x) = 0 \text{ for all } f \in B\}$ of X. Show that $\overline{\operatorname{span}} A = \big(A^\perp\big)_\perp$. Is it true that $\overline{\operatorname{span}} B = \big(B_\perp\big)^\perp$?

7. Let Y be a closed subspace of a normed space X. Show that $Y^* \cong X^*/Y^\perp$ and that $\big(X/Y\big)^* \cong Y^\perp$.

8. Let X be a Banach space. Show that X is reflexive if and only if X^* is reflexive. Show also that if Y is a closed subspace of X, then X is reflexive if and only if Y and X/Y are reflexive.

9. Show that none of the spaces c_0, ℓ_1, ℓ_∞, $L_1[0,1]$ and $L_\infty[0,1]$ is reflexive.

10. Let Ω be a set and $\mathcal{F}$ be a σ-field on Ω. Prove carefully that the set $L_\infty(\Omega, \mathcal{F})$ of all bounded, measurable, scalar-valued functions on Ω is a Banach space in the supremum norm: $\|f\|_\infty = \sup_{\omega \in \Omega} |f(\omega)|$. The aim of this question is to identify $L_\infty(\Omega, \mathcal{F})^*$.

A *finitely additive measure on* $\mathcal{F}$ is a (real or complex) function ν on $\mathcal{F}$ such that $\nu(\emptyset) = 0$ and $\nu(A \cup B) = \nu(A) + \nu(B)$ whenever $A, B \in \mathcal{F}$ and $A \cap B = \emptyset$. The *total variation measure* $|\nu|$ *of* ν is defined as follows.

$$|\nu|(A) = \sup\Big\{ \sum_{k=1}^{n} |\nu(A_k)| : A = \bigcup_{k=1}^{n} A_k \text{ is a measurable partition of } A \Big\} .$$

The *total variation of* ν is $\|\nu\|_1 = |\nu|(\Omega)$. We say ν is *bounded* if $\|\nu\|_1 < \infty$. Show that the space $\operatorname{ba}(\Omega, \mathcal{F})$ of all bounded, finitely additive measures on $\mathcal{F}$ is a Banach space in the total variation norm and that it is isometrically isomorphic to $L_\infty(\Omega, \mathcal{F})^*$.

11. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Show that $L_\infty(\mu)$ is a quotient of $L_\infty(\Omega, \mathcal{F})$. Deduce that $L_\infty(\mu)^*$ is a subspace of $\operatorname{ba}(\Omega, \mathcal{F})$ and identify that subspace.

Some more questions

12. Let Y and Z be closed subspaces of a normed space X of the same finite codimension. Show that there is an isomorphism $T \colon X \to X$ such that $T(Y) = Z$.

13. Let $1 \leqslant p \leqslant \infty$. The *$\ell_p$-direct sum* of a sequence (X_n) of Banach spaces is the space

$$\Big(\bigoplus_{n=1}^{\infty} X_n\Big)_{\ell_p} = \Big\{(x_n) : x_n \in X_n \text{ for all } n \in \mathbb{N},\ \sum_n \|x_n\|^p < \infty\Big\}$$

with norm $\big\|(x_n)\big\| = \big(\sum_n \|x_n\|^p\big)^{\frac{1}{p}}$ when $p < \infty$, and the space

$$\Big(\bigoplus_{n=1}^{\infty} X_n\Big)_{\ell_\infty} = \Big\{(x_n) : x_n \in X_n \text{ for all } n \in \mathbb{N},\ \sup_n \|x_n\| < \infty\Big\}$$

with norm $\big\|(x_n)\big\| = \sup_n \|x_n\|$ when $p = \infty$. We also define the c_0-direct sum to be the subspace $\big(\bigoplus_{n=1}^{\infty} X_n\big)_{c_0}$ of $\big(\bigoplus_{n=1}^{\infty} X_n\big)_{\ell_\infty}$ consisting of sequences (x_n) with $\|x_n\| \to 0$. Show that $\big(\bigoplus_{n=1}^{\infty} X_n\big)_{\ell_p}^{*} \cong \big(\bigoplus_{n=1}^{\infty} X_n^{*}\big)_{\ell_q}$ where $1 \leqslant p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, and that $\big(\bigoplus_{n=1}^{\infty} X_n\big)_{c_0}^{*} \cong \big(\bigoplus_{n=1}^{\infty} X_n^{*}\big)_{\ell_1}$.

14. Let $(X, \|\cdot\|)$ be a normed space. Let Y be a subspace of X, and $|||\cdot|||$ be a norm on Y that is equivalent to $\|\cdot\|$ on Y. Show that $|||\cdot|||$ extends to an equivalent norm on X.

15. Let X be a Banach space. Show that TFAE:
(i) If Y and Z are Banach spaces and $X \sim Y \subset Z$ then Y is complemented in Z.
(ii) Given Banach spaces $Y \subset Z$ and given $T \in \mathcal{B}(Y, X)$, there exists $\widetilde{T} \in \mathcal{B}(Z, X)$ such that $\widetilde{T}\restriction_Y = T$.

Such a space X is called *injective*. Show that $\ell_\infty(\Gamma)$, the space of bounded scalar functions on the set Γ with the supremum norm, is injective.

(X is called *λ-injective* if in (i) the subspace Y is λ-complemented in Z whenever $X \cong Y \subset Z$, or equivalently, in (ii) we have $\|\widetilde{T}\| \leqslant \lambda \|T\|$. Note that $\ell_\infty(\Gamma)$ is 1-injective.)

16. In lectures we proved that ℓ_∞ is isometrically universal for the class of separable Banach spaces. Show that c_0 is almost isometrically universal for the class of finite-dimensional normed spaces: for every finite-dimensional space E and for every $\varepsilon > 0$, there is a linear map $T \colon E \to T(E) \subset c_0$ with $\|T\| \cdot \|T^{-1}\| \leqslant 1 + \varepsilon$. Is there a separable, reflexive space with the same property?

17. Let $(X, \mathcal{P})$ be a locally convex space. Prove that X is metrizable if and only if there is a countable family $\mathcal{Q}$ of seminorms on X equivalent to $\mathcal{P}$.

18. A subset A of a locally convex space is *bounded* if for every neighbourhood V of 0 there a scalar λ with $A \subset \lambda V$. Consider the locally convex space $\mathcal{O}(U)$ of analytic functions on a non-empty open subset U of $\mathbb{C}$ with the topology of local uniform convergence. Show that $\mathcal{O}(U)$ is a Fréchet space. Show that a subset A of $\mathcal{O}(U)$ is bounded if and only if for every compact $K \subset U$ the set $\{f\restriction_K \colon f \in A\}$ is bounded in $(C(K), \|\cdot\|_\infty)$. Prove Montel's theorem: every bounded sequence in $\mathcal{O}(U)$ has a convergent subsequence. Deduce that $\mathcal{O}(U)$ is not normable.

19. Let $d \in \mathbb{N}$ and let Ω be a non-empty open subset of $\mathbb{R}^d$. Prove that the locally convex space $C^\infty(\Omega)$ is a Fréchet space and that it is not normable.

20. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $f \in L_1(\mu)$. Consider the complex measure $\nu(A) = \int_A f \,\mathrm{d}\mu$, $A \in \mathcal{F}$, that satisfies $\nu \ll \mu$. Show that $|\nu|(A) = \int_A |f| \,\mathrm{d}\mu$ for $A \in \mathcal{F}$.

Show also that a measurable function g on Ω is ν-integrable if and only if gf is μ-integrable, in which case $\int g \,\mathrm{d}\nu = \int gf \,\mathrm{d}\mu$.

21. Let Ω be a set, $\mathcal{F}$ a σ-field on Ω, and ν a complex measure on $\mathcal{F}$. Show that $\nu \ll |\nu|$ and that $\left|\frac{\mathrm{d}\nu}{\mathrm{d}|\nu|}\right| = 1$ $|\nu|$-almost everywhere. Show also that there is a measurable function $g \colon \Omega \to \mathbb{C}$ such that $|g| = 1$ $|\nu|$-a.e. and $|\nu|(A) = \int_A g \,\mathrm{d}\nu$ for all $A \in \mathcal{F}$.

22. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, and let $\nu \colon \mathcal{F} \to \mathbb{C}$ be a complex measure. Show that there exist unique complex measures ν_1 and ν_2 on $\mathcal{F}$ such that $\nu = \nu_1 + \nu_2$, $\nu_1 \ll \mu$ (ν_1 absolutely continuous with respect to μ) and $\nu_2 \perp \mu$ (ν_2 orthogonal to μ). [Recall that μ and ν_2 are said to be *orthogonal* if there is a measurable partition $\Omega = A \cup B$ such that $\mu(B) = |\nu_2|(A) = 0$.]