Resumé on Hilbert spaces and Spectral Theory

1 Hilbert spaces

I assume you know what a Hilbert space is and that you are familiar with basic Hilbert space geometry (Parallelogram Law, orthogonality, Pythagoras's Theorem, orthogonal complements, etc). We recall the following result.

1.1 If Y is a closed subspace of a Hilbert space H, then H is the orthogonal direct sum of Y and $Y^\perp$ (written $H = Y \oplus Y^\perp$).

1.2 Corresponding to the orthogonal decomposition $Y \oplus Y^\perp$ of H is the map

$$P\colon H \to H \quad \text{given by} \quad P(y+z) = y \quad (y \in Y, z \in Y^\perp).$$

This is a bounded linear map with $\operatorname{im} P = Y$ and $\ker P = Y^\perp$. It is called the *orthogonal projection* of H onto Y.

1.3 Riesz Representation Theorem (Yes, another one.) For each $f \in H^*$ there is a unique $y \in H$ such that $f(x) = \langle x, y\rangle$ for all $x \in H$. The map $f \mapsto y$ is an isometric, conjugate-linear isomorphism of H^* onto H.

1.4 A *sesquilinear form* on a (complex) Hilbert space H is a map $\theta\colon H \times H \to \mathbb{C}$ satisfying for all $x, y, z \in H$ and $\lambda, \mu \in \mathbb{C}$ that

(i) $\theta(\lambda x + \mu y, z) = \lambda\theta(x, z) + \mu\theta(y, z)$ (linearity in first variable), and

(ii) $\theta(x, \lambda y + \mu z) = \bar{\lambda}\theta(x, y) + \bar{\mu}\theta(x, z)$ (conjugate-linearity in second variable).

The sesquilinear form θ is called a *hermitian form* if in addition it satisfies

(iii) $\theta(y, x) = \overline{\theta(x, y)}$ for all $x, y \in H$.

Note that (i) and (iii) imply (ii). A sesquilinear form θ is *bounded* if there is a constant $C \geqslant 0$ such that

$$|\theta(x, y)| \leqslant C\|x\|\|y\| \qquad \text{for all } x, y \in H\ .$$

This is equivalent to θ being continuous. The smallest C that works is denoted by $\|\theta\|$. Note that

$$\|\theta\| = \sup\left\{|\theta(x, y)| : x, y \in H,\ \|x\| \leqslant 1,\ \|y\| \leqslant 1\right\}\ .$$

E.g., the inner product $\langle\cdot,\cdot\rangle$ is a bounded hermitian form on H with norm 1.

1.5 Theorem Let θ be a bounded sesquilinear form on H. Then there is a unique map $T\colon H \to H$ such that

$$(1) \qquad \theta(x, y) = \langle Tx, y\rangle \qquad \text{for all } x, y \in H\ .$$

Moreover, $T \in \mathcal{B}(H)$ and $\|T\| = \|\theta\|$.

Proof. Fix $x \in H$. The map $y \mapsto \overline{\theta(x, y)}$ is a bounded linear map of norm at most $\|\theta\|\|x\|$. By the Riesz Representation Theorem, there exists some $Tx \in H$ such that $\overline{\theta(x, y)} = \langle y, Tx\rangle$ for all $y \in H$. This defines a map $T\colon H \to H$ satisfying (1). Given $x, y, z \in H$ and $\lambda, \mu \in \mathbb{C}$, we have

$$\begin{aligned}\langle T(\lambda x + \mu y), z\rangle &= \theta(\lambda x + \mu y, z) = \lambda\theta(x, z) + \mu\theta(y, z)\\ &= \lambda\langle Tx, z\rangle + \mu\langle Ty, z\rangle\\ &= \langle \lambda Tx + \mu Ty, z\rangle\end{aligned}$$

Since z was arbitrary, it follows that $T(\lambda x + \mu y) = \lambda Tx + \mu Ty$, and T is linear. Next,

$$\|Tx\|^2 = \langle Tx, Tx\rangle = \theta(x, Tx) \leqslant \|\theta\|\|x\|\|Tx\| \ .$$

Hence T is bounded with $\|T\| \leqslant \|\theta\|$. Conversely,

$$|\theta(x, y)| = |\langle Tx, y\rangle| \leqslant \|Tx\|\|y\| \leqslant \|T\|\|x\|\|y\|$$

by Cauchy-Schwarz. Thus $\|\theta\| = \|T\|$.

Finally, to show uniqueness, assume that $\langle Tx, y\rangle = \langle Sx, y\rangle$ for all x, y. Putting $y = Tx - Sx$ yields $\|Tx - Sx\| = 0$, and hence $T = S$. □

1.6 Adjoints For $T \in \mathcal{B}(H)$ there is a unique map $T^*\colon H \to H$ such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x, y \in H$. Moreover, $T^* \in \mathcal{B}(H)$ and $\|T^*\| = \|T\|$.

Proof. Apply Theorem 1.5 to $\theta(x, y) = \langle x, Ty\rangle$. □

1.7 Remark $T^*\colon H \to H$ is called the *adjoint* of T. By the Riesz Representation Theorem, T^* can be viewed as a map $H^* \to H^*$, and then it is nothing else but the dual operator of T.

1.8 Properties of adjoints Let $S, T \in \mathcal{B}(H)$ and $\lambda, \mu \in \mathbb{C}$.

(i) $(\lambda S + \mu T)^* = \bar{\lambda} S^* + \bar{\mu} T^*$

(ii) $(ST)^* = T^*S^*$

(iii) $T^{**} = T$

(iv) $\|T^*T\| = \|T\|^2$

1.9 An operator $T \in \mathcal{B}(H)$ is

(i) *hermitian* if $T^* = T$

(ii) *unitary* if $TT^* = T^*T = I$

(iii) *normal* if $TT^* = T^*T$.

Examples of hermitian operators include orthogonal projections. An operator is unitary if and only if it is isometric and surjective. Examples of normal operators include all hermitian and unitary operators.

1.10 Note If θ in Theorem 1.5 is hermitian, then the corresponding operator T is also hermitian.

2 Spectral Theory

Let X be a (non-zero) complex Banach space and $T \in \mathcal{B}(X)$. The *spectrum* of T is the set

$$\sigma(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ not invertible}\} \ .$$

This is a special case of the spectrum of an element of a unital Banach algebra as defined in the course. In particular, the spectrum of T is a non-empty, compact subset of $\{\lambda \in \mathbb{C} : |\lambda| \leqslant \|T\|\}$. Moreover the *spectral radius* $r(T)$ of T satisfies

$$r(T) = \sup_{\lambda \in \sigma(T)} |\lambda| = \lim_{n\to\infty} \|T^n\|^{1/n} \ .$$

2.1 We say λ is an *approximate eigenvalue* of T if there exists a sequence (x_n) in X with $\|x_n\| = 1$ for all $n \in \mathbb{N}$ such that

$$(\lambda I - T)x_n \to 0 \quad \text{as} \quad n \to \infty \ .$$

The sequence (x_n) is an *approximate eigenvector* for λ. The *approximate point spectrum* of T is the set of all approximate eigenvalues of T, and is denoted by $\sigma_{\mathrm{ap}}(T)$. We also define the *point spectrum* of T to be the set of all eigenvalues of T and denote it by $\sigma_{\mathrm{p}}(T)$. One clearly has

$$\sigma_{\mathrm{p}}(T) \subset \sigma_{\mathrm{ap}}(T) \subset \sigma(T) \ .$$

In general, these inclusions can be strict and the point spectrum can be empty (unlike the spectrum). However, we have the following result. Here ∂A denotes the boundary of a set A in a topological space X: $\partial A = \overline{A} \setminus A^{\circ}$.

2.2 Theorem We have $\partial\sigma(T) \subset \sigma_{\mathrm{ap}}(T)$. In particular, $\sigma_{\mathrm{ap}}(T) \neq \emptyset$.

Proof. Let $\lambda \in \partial\sigma(T)$. Then there is a sequence $\lambda_n \notin \sigma(T)$ converging to λ. It follows from a result proved in lectures that

$$\big\|(\lambda_n I - T)^{-1}\big\| \to \infty \quad \text{as} \quad n \to \infty \ .$$

Thus, there is a sequence (x_n) of unit vectors such that

$$\big\|(\lambda_n I - T)^{-1} x_n\big\| \to \infty \quad \text{as} \quad n \to \infty \ .$$

Set

$$y_n = \frac{(\lambda_n I - T)^{-1} x_n}{\big\|(\lambda_n I - T)^{-1} x_n\big\|} \ .$$

It is easy to check that (y_n) is an approximate eigenvector for λ. $\square$

2.3 Theorem Let $T \in \mathcal{B}(X)$ be a compact operator. Let $\lambda \in \sigma_{\mathrm{ap}}(T)$ and $\lambda \neq 0$. Then λ is an eigenvalue of T.

2.4 From now on H is a (non-zero) complex Hilbert space. For $T \in \mathcal{B}(H)$ we have

$$\sigma(T^*) = \{\bar{\lambda} : \lambda \in \sigma(T)\} \ .$$

If T is hermitian then $\sigma(T) \subset \mathbb{R}$. It follows that $\sigma(T) = \sigma_{\mathrm{ap}}(T)$. This latter fact holds also when T is a normal operator.

2.5 Theorem Let $T \in \mathcal{B}(H)$ be a compact hermitian operator. Then $\sigma(T)$ is countable and if $\lambda \in \sigma(T)$, $\lambda \neq 0$, then λ is an eigenvalue whose eigenspace is finite-dimensional: $\dim\ker(\lambda I - T) < \infty$.

2.6 Spectral Theorem Let $T \in \mathcal{B}(H)$ be a compact hermitian operator. Then there is an orthonormal sequence $x_1, x_2, \dots$ (finite or infinite) of eigenvectors of T with non-zero eigenvalues $\lambda_1, \lambda_2, \dots$, respectively, such that

$$T\Big(\sum a_n x_n + z\Big) = \sum \lambda_n a_n x_n$$

for all scalars a_n, and all $z \in \{x_n : n \in \mathbb{N}\}^{\perp}$.

2.7 Remark The above holds for compact normal operators as well.

2.8 Definition $T \in \mathcal{B}(H)$ is *positive* if T is hermitian and $\sigma(T) \subset \mathbb{R}^+$.

2.9 Theorem $T \in \mathcal{B}(H)$ is positive if and only if $\langle Tx, x\rangle \geqslant 0$ for all $x \in H$.