Part III Functional Analysis

András Zsák

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Contents

1 The Hahn–Banach extension theorems

Dual space. Let X be a normed space. The *dual space* X^* *of* X is the space of all bounded linear functionals on X. The dual space is a Banach space in the operator norm which is defined for $f \in X^*$ as

$$\|f\| = \sup\big\{|f(x)| : x \in B_X\big\} .$$

Recall that $B_X = \{x \in X : \|x\| \leqslant 1\}$ is the closed unit ball of X and that $S_X = \{x \in X : \|x\| = 1\}$ is the unit sphere of X.

Examples. $\ell_p^* \cong \ell_q$ for $1 \leqslant p < \infty$, $1 < q \leqslant \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. We also have $c_0^* \cong \ell_1$. If H is a Hilbert space then $H^* \cong H$ (conjugate-linear in the complex case).

Notation. 1. Given normed spaces X and Y, we write $X \sim Y$ if X and Y are isomorphic, *i.e.,* when there is a linear bijection $T : X \to Y$ such that both T and T^{-1} are continuous. (Recall that by the Open Mapping Theorem, if X and Y are both complete and T is a continuous linear bijection, then T^{-1} is automatically continuous.)

2. Given normed spaces X and Y, we write $X \cong Y$ if X and Y are isometrically isomorphic, *i.e.,* when there is a surjective linear map $T\colon X \to Y$ such that $\|Tx\| = \|x\|$ for all $x \in X$. It follows that T is a continuous linear bijection and that T^{-1} is also isometric, and hence continuous.

3. For $x \in X$ and $f \in X^*$, we shall sometimes denote $f(x)$, the action of f on x, by $\langle x, f\rangle$. By definition of the operator norm we have

$$|\langle x, f\rangle| = |f(x)| \leqslant \|f\| \cdot \|x\| \ .$$

When X is a Hilbert space and we identify X^* with X in the usual way, then $\langle x, f\rangle$ is the inner product of x and f.

Definition. Let X be a real vector space. A functional $p\colon X \to \mathbb{R}$ is called

- *positive homogeneous* if $p(tx) = tp(x)$ for all $x \in X$ and $t \geqslant 0$, and
- *subadditive* if $p(x + y) \leqslant p(x) + p(y)$ for all $x, y \in X$.

Theorem 1. (Hahn–Banach) Let X be a real vector space and p be a positive homogeneous, subadditive functional on X. Let Y be a subspace of X and $g\colon Y \to \mathbb{R}$ be a linear functional such that $g(y) \leqslant p(y)$ for all $y \in Y$. Then there is a linear functional $f\colon X \to \mathbb{R}$ such that $f\restriction_Y = g$ and $f(x) \leqslant p(x)$ for all $x \in X$.

Zorn's Lemma. This is used in the proof, so we recall it here. Let $(P, \leqslant)$ be a poset (partially ordered set). An element $x \in P$ is an *upper bound* for a subset A of P if $a \leqslant x$ for all $a \in A$. A subset C of P is called a *chain* if it is linearly ordered by $\leqslant$. An element $x \in P$ is a *maximal element* of P if whenever $x \leqslant y$ for some $y \in P$, we have $y = x$. Zorn's Lemma states that if $P \neq \emptyset$ and every non-empty chain in P has an upper bound, then P has a maximal element.

Definition. A *seminorm* on a real or complex vector space X is a functional $p\colon X \to \mathbb{R}$ such that

- $p(x) \geqslant 0$ for all $x \in X$;
- $p(\lambda x) = |\lambda| p(x)$ for all $x \in X$ and for all scalars λ;
- $p(x + y) \leqslant p(x) + p(y)$ for all $x, y \in X$.

Note. "norm" $\implies$ "seminorm" $\implies$ "positive homogeneous+subadditive".

Theorem 2. (Hahn–Banach) Let X be a real or complex vector space and p be a seminorm on X. Let Y be a subspace of X and g be a linear functional on Y such that $|g(y)| \leqslant p(y)$ for all $y \in Y$. Then there is a linear functional f on X such that $f\restriction_Y = g$ and $|f(x)| \leqslant p(x)$ for all $x \in X$.

Remark. It follows from the proof given in lectures that for a complex normed space X, the map $f \mapsto \mathrm{Re}(f)\colon \big(X^*\big)_{\mathbb{R}} \to \big(X_{\mathbb{R}}\big)^*$ is an isometric isomorphism. Here $Y_{\mathbb{R}}$, for a complex vector space Y, denotes the real vector space obtained from Y by restricting scalar multiplication to the reals.

Corollary 3. Let X be a real or complex vector space and p be a seminorm on X. For every $x_0 \in X$ there is a linear functional f on X with $f(x_0) = p(x_0)$ and $|f(x)| \leqslant p(x)$ for all $x \in X$.

Theorem 4. (Hahn–Banach) Let X be a real or complex normed space. Then

(i) Given a subspace Y of X and $g \in Y^*$, there exists $f \in X^*$ with $f\restriction_Y = g$ and $\|f\| = \|g\|$.

(ii) Given a non-zero $x_0 \in X$, there exists $f \in S_{X^*}$ with $f(x_0) = \|x_0\|$.

Remarks. 1. Part (i) can be viewed as a linear version of Tietze's extension theorem. The latter states that if L is a closed subset of a compact Hausdorff space K, and $g\colon L \to \mathbb{R}$ (or $\mathbb{C}$) is continuous, then there is a continuous function $f\colon K \to \mathbb{R}$ (respectively, $\mathbb{C}$) such that $f\restriction_L = g$ and $\|f\|_\infty = \|g\|_\infty$.
2. Part (ii) implies that X^* separates the points of X: given $x \neq y$ in X, there exists $f \in X^*$ such that $f(x) \neq f(y)$. Thus there are "plenty" bounded linear functionals on X.
3. The element $f \in X^*$ in part (ii) is called a *norming functional for* x_0. It shows that

$$\|x_0\| = \max\big\{|\langle x_0, g\rangle| : g \in B_{X^*}\big\} .$$

Another name for f is *support functional at* x_0. Assume X is a real normed space and that $\|x_0\| = 1$. Then $B_X \subset \{x \in X : f(x) \leqslant 1\}$, and hence the hyperplane $\{x \in X : f(x) = 1\}$ can be thought of as a tangent to B_X at x_0.

Bidual. Let X be a normed space. Then $X^{**} = \big(X^*\big)^*$ is called the *bidual* or *second dual of* X. It is the Banach space of all bounded linear functionals on X^* with the operator norm. For $x \in X$ we define $\hat{x}\colon X^* \to \mathbb{R}$ (or $\mathbb{C}$) by $\hat{x}(f) = f(x)$ (*evaluation at* x). Then $\hat{x} \in X^{**}$ and $\|\hat{x}\| \leqslant \|x\|$. The map $x \mapsto \hat{x}\colon X \to X^{**}$ is called the *canonical embedding*.

Theorem 5. The canonical embedding defined above is an isometric isomorphism of X *into* X^{**}.

Remarks. 1. Using the bracket notation, we have

$$\langle f, \hat{x}\rangle = \langle x, f\rangle = f(x) \qquad x \in X,\ f \in X^* .$$

2. The image $\widehat{X} = \{\hat{x} : x \in X\}$ of the canonical embedding is closed in X^{**} if and only if X is complete.
3. In general, the closure of $\widehat{X}$ in X^{**} is a Banach space which contains an isometric copy of X as a dense subspace. We have thus proved that every normed space has a completion.

Definition. X is *reflexive* if the canonical embedding of X into X^{**} is surjective.

Examples. 1. The spaces ℓ_p, $1 < p < \infty$, Hilbert spaces, finite-dimensional spaces are reflexive. Later we prove that the spaces $L_p(\mu)$, $1 < p < \infty$, are also reflexive.
2. The spaces c_0, ℓ_1, $L_1[0,1]$ are not reflexive.

Remark. There are Banach spaces X with $X \cong X^{**}$ which are not reflexive. So for $1 < p < \infty$, it is not sufficient to say that $\ell_p^{**} \cong \ell_q^* \cong \ell_p$ (where q is the conjugate index of p) to deduce that ℓ_p is reflexive. One also has to verify that this isometric isomorphism is indeed the canonical embedding.

Dual operators. Recall that for normed spaces X, Y we denote by $\mathcal{B}(X, Y)$ the space of all bounded linear maps $T \colon X \to Y$. This is a normed space in the operator norm:

$$\|T\| = \sup \big\{ \|Tx\| : x \in B_X \big\} .$$

Moreover, $\mathcal{B}(X, Y)$ is complete if Y is.

For $T \in \mathcal{B}(X, Y)$, the *dual operator* of T is the map $T^* \colon Y^* \to X^*$ defined by $T^*(g) = g \circ T$ for $g \in Y^*$. In bracket notation:

$$\langle x, T^* g \rangle = \langle Tx, g \rangle \qquad x \in X,\ g \in Y^* .$$

T^* is linear and bounded with $\|T^*\| = \|T\|$ (uses Theorem 4(ii)).

Remark. When X and Y are Hilbert spaces, the dual operator T^* corresponds to the adjoint of T after identifying X^* and Y^* with X and Y, respectively.

Example. Let $1 < p < \infty$ and consider the right shift $R \colon \ell_p \to \ell_p$. Then $R^* \colon \ell_q \to \ell_q$ (q the conjugate index of p) is the left shift.

Properties of dual operators. 1. $\big(\mathrm{Id}_X\big)^* = \mathrm{Id}_{X^*}$
2. $(\lambda S + \mu T)^* = \lambda S^* + \mu T^*$ for scalars λ, μ and for $S, T \in \mathcal{B}(X, Y)$. Note that unlike for adjoints there is no complex conjugation here. That is because the identification of a Hilbert space with its dual is conjugate linear in the complex case.
3. $(ST)^* = T^* S^*$ for $T \in \mathcal{B}(X, Y)$ and $S \in \mathcal{B}(Y, Z)$.
4. The map $T \mapsto T^* \colon \mathcal{B}(X, Y) \to \mathcal{B}(Y^*, X^*)$ is an *into* isometric isomorphism.
5. The following diagram

$$\begin{array}{ccc} X & \xrightarrow{\ T\ } & Y \\ \big\downarrow & & \big\downarrow \\ X^{**} & \xrightarrow{\ T^{**}\ } & Y^{**} \end{array}$$

commutes, where the vertical arrows represent canonical embeddings.

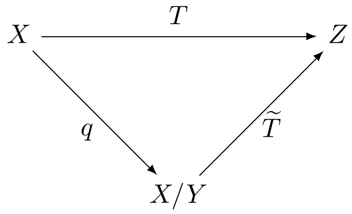
Remark. It follows from the above properties that if $X \sim Y$ then $X^* \sim Y^*$.

Quotient space. Let X be a normed space and Y a closed subspace. Then the quotient space X/Y becomes a normed space in the *quotient norm* defined as follows.

$$\|x + Y\| = \inf \big\{ \|x + y\| : y \in Y \big\} \qquad x \in X .$$

The quotient map $q \colon X \to X/Y$ is a bounded, surjective linear map with $\|q\| \leqslant 1$. It maps the open unit ball $D_X = \{x \in X : \|x\| < 1\}$ onto $D_{X/Y}$. It follows that $\|q\| = 1$ (if $X \neq Y$) and that q is an open map (it maps open sets onto open sets).

If $T\colon X \to Z$ is a bounded linear map with $Y \subset \ker(T)$, then there is a unique map $\widetilde{T}\colon X/Y \to Z$ such that the diagram



commutes, and moreover $\widetilde{T}$ is bounded, linear and $\|\widetilde{T}\| = \|T\|$.

Theorem 6. Let X be a normed space. If X^* is separable, then so is X.

Remark. The converse is false. *E.g.,* $X = \ell_1$ is separable but $X^* \cong \ell_\infty$ is not.

Theorem 7. Every separable normed space is isometrically isomorphic to a subspace of ℓ_∞.

Remarks. 1. The result says that ℓ_∞ is *isometrically universal* for the class $\mathcal{SB}$ of all separable Banach spaces. We will later see that there is a separable space with the same property.
2. A dual version of the above result states that every separable Banach spaces is a quotient of ℓ_1.

Theorem 8. (Vector-valued Liouville) Let X be a complex Banach space and $f\colon \mathbb{C} \to X$ be a bounded, holomorphic function. Then f is constant.

Locally Convex Spaces. A *locally convex space (LCS)* is a pair $(X, \mathcal{P})$, where X is a real or complex vector space and $\mathcal{P}$ is a family of seminorms on X that separates the points of X in the sense that for every $x \in X$ with $x \neq 0$, there is a seminorm $p \in \mathcal{P}$ with $p(x) \neq 0$. The family $\mathcal{P}$ defines a topology on X: a set $U \subset X$ is *open* if and only if for all $x \in U$ there exist $n \in \mathbb{N}$, $p_1, \dots, p_n \in \mathcal{P}$ and $\varepsilon > 0$ such that

$$\big\{ y \in X : p_k(y - x) < \varepsilon \text{ for } 1 \leqslant k \leqslant n \big\} \subset U \ .$$

Remarks. 1. Addition and scalar multiplication are continuous.
2. The topology of X is Hausdorff as $\mathcal{P}$ separates the points of X.
3. A sequence $x_n \to x$ in X if and only if $p(x_n - x) \to 0$ for all $p \in \mathcal{P}$. (The same holds for nets.)
4. For a subspace Y of X define $\mathcal{P}_Y = \{p\restriction_Y \colon p \in \mathcal{P}\}$. Then $(Y, \mathcal{P}_Y)$ is a locally convex space, and the corresponding locally convex topology on Y is nothing else but the subspace topology on Y induced by X.
5. Let $\mathcal{P}$ and $\mathcal{Q}$ be two families of seminorms on X both of which separate the points of X. We say $\mathcal{P}$ and $\mathcal{Q}$ are *equivalent* and write $\mathcal{P} \sim \mathcal{Q}$ if they define the same topology on X.

The topology of a locally convex space $(X, \mathcal{P})$ is metrizable if and only if there is a countable $\mathcal{Q}$ with $\mathcal{Q} \sim \mathcal{P}$.

Definition. A *Fréchet space* is a complete metrizable locally convex space.

Examples. 1. Every normed space $(X, \|\cdot\|)$ is a LCS with $\mathcal{P} = \{\|\cdot\|\}$.
2. Let U be a non-empty, open subset of $\mathbb{C}$, and let $\mathcal{O}(U)$ denote the space of all holomorphic functions $f \colon U \to \mathbb{C}$. For a compact set $K \subset U$ and for $f \in \mathcal{O}(U)$ set $p_K(f) = \sup_{z \in K} |f(z)|$. Set $\mathcal{P} = \{p_K : K \subset U, K \text{ compact}\}$. Then $(\mathcal{O}(U), \mathcal{P})$ is a locally convex space whose topology is the topology of local uniform convergence.

There exist compact sets $K_n \subset U$, $n \in \mathbb{N}$, such that $K_n \subset \operatorname{int}(K_{n+1})$ and $U = \bigcup_n K_n$. Then $\{p_{K_n} : n \in \mathbb{N}\}$ is countable and equivalent to $\mathcal{P}$. Hence $(\mathcal{O}(U), \mathcal{P})$ is metrizable and in fact it is a Fréchet space.

The topology of local uniform convergence is not *normable*: the topology is not induced by a norm. This follows, for example, from Montel's theorem: given a sequence (f_n) in $\mathcal{O}(U)$ such that $(f_n \restriction_K)$ is bounded in $(C(K), \|\cdot\|_\infty)$ for every compact $K \subset U$, there is a subsequence of (f_n) that converges locally uniformly.
3. Fix $d \in \mathbb{N}$ and a non-empty open set $\Omega \subset \mathbb{R}^d$. Let $C^\infty(\Omega)$ denote the space of all infinitely differentiable functions $f \colon \Omega \to \mathbb{R}$. Every *multi-index* $\alpha = (\alpha_1, \dots, \alpha_d) \in \left(\mathbb{Z}_{\geqslant 0}\right)^d$ gives rise to a partial differential operator

$$D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} \dots \left(\frac{\partial}{\partial x_d}\right)^{\alpha_d} .$$

For a multi-index α, a compact set $K \subset \Omega$, and $f \in C^\infty(\Omega)$ define

$$p_{K,\alpha}(f) = \sup \left\{ |(D^\alpha f)(x)| : x \in K \right\} .$$

Set $\mathcal{P} = \left\{p_{K,\alpha} : K \subset \Omega \text{ compact}, \alpha \in (\mathbb{Z}_{\geqslant 0})^d \right\}$. Then $(C^\infty(\Omega), \mathcal{P})$ is a locally convex space. It is a Fréchet space and is not normable.

Lemma 9. Let $(X, \mathcal{P})$ and $(Y, \mathcal{Q})$ be locally convex spaces. Let $T \colon X \to Y$ be a linear map. Then TFAE:

(i) T is continuous.

(ii) T is continuous at 0.

(iii) For all $q \in Q$ there exist $n \in \mathbb{N}$, $p_1, \dots, p_n \in \mathcal{P}$ and $C \geqslant 0$ such that

$$q(Tx) \leqslant C \max_{1 \leqslant k \leqslant n} p_k(x) \qquad \text{for all } x \in X .$$

Dual space. Let $(X, \mathcal{P})$ be a locally convex space. The *dual space* X^* of X is the space of all linear functionals on X that are continuous with respect to the locally convex topology of X.

Lemma 10. Let f be a linear functional on a locally convex space X. Then $f \in X^*$ if and only if $\ker f$ is closed.

Theorem 11. (Hahn–Banach) Let $(X, \mathcal{P})$ be a locally convex space.

(i) Given a subspace Y of X and $g \in Y^*$, there exists $f \in X^*$ with $f \restriction_Y = g$.

(ii) Given a closed subspace Y of X and $x_0 \in X \setminus Y$, there exists $f \in X^*$ such that $f \restriction_Y = 0$ and $f(x_0) \neq 0$.

Remark. It follows that X^* separates the points of X.

2 The dual space of $L_p(\mu)$ and of $C(K)$

L_p spaces. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. For $1 \leqslant p < \infty$ we define $L_p(\Omega, \mathcal{F}, \mu)$ or simply $L_p(\mu)$ to be the real (or complex) vector space of all measurable functions $f\colon \Omega \to \mathbb{R}$ (respectively, $\mathbb{C}$) such that $\int_\Omega |f|^p \,\mathrm{d}\mu < \infty$. This is a normed space in the L_p-norm $\|\cdot\|_p$ defined for $f \in L_p(\mu)$ by

$$\|f\|_p = \left(\int_\Omega |f|^p \,\mathrm{d}\mu\right)^{\frac{1}{p}}$$

provided we identify functions f and g if $f = g$ a.e. (almost everywhere).

In the case $p = \infty$ we call a measurable function $f\colon \Omega \to \mathbb{R}$ (or $\mathbb{C}$) *essentially bounded* if there is a μ-null set $N \in \mathcal{F}$ such that f is bounded on $\Omega \setminus N$. We then define $L_\infty(\Omega, \mathcal{F}, \mu)$ or simply $L_\infty(\mu)$ to be the real (or complex) vector space of all measurable, essentially bounded functions $f\colon \Omega \to \mathbb{R}$ (respectively, $\mathbb{C}$). This is a normed space in the *essential sup norm* $\|\cdot\|_\infty$ defined for $f \in L_\infty(\mu)$ by

$$\|f\|_\infty = \operatorname{ess\,sup}|f| = \inf\big\{ \sup_{\Omega\setminus N} |f| : N \in \mathcal{F},\ \mu(N) = 0 \big\}$$

having identified functions that are equal a.e. Note that the essential supremum is attained: there is a μ-null set $N \in \mathcal{F}$ such that $\operatorname{ess\,sup}|f| = \sup_{\Omega\setminus N}|f|$.

Remark. Strictly speaking $\|\cdot\|_p$ is a seminorm on $L_p(\mu)$ for $1 \leqslant p \leqslant \infty$. In general, if $\|\cdot\|$ is a seminorm on a real or complex vector space X, then $N = \{z \in X : \|z\| = 0\}$ is a subspace of X, and $\|x + N\| = \|x\|$ defines a norm on the the quotient space X/N. However, we will not do this for $L_p(\mu)$. We prefer to think of elements of $L_p(\mu)$ as functions rather than equivalence classes of functions. One must remember that equality in $L_p(\mu)$ means a.e. equality.

Theorem 1. The space $L_p(\mu)$ is complete for $1 \leqslant p \leqslant \infty$.

Complex measures. Let Ω be a set and $\mathcal{F}$ be a σ-field on Ω. A *complex measure on* $\mathcal{F}$ is a countably additive set function $\nu\colon \mathcal{F} \to \mathbb{C}$. The *total variation measure* $|\nu|$ of ν is defined at $A \in \mathcal{F}$ as follows.

$$|\nu|(A) = \sup\left\{ \sum_{k=1}^{n} |\nu(A_k)| : A = \bigcup_{k=1}^{n} A_k \text{ is a measurable partition of } A \right\}.$$

(*Measurable partition* means that $A_k \in \mathcal{F}$ for all k, and $A_j \cap A_k = \emptyset$ for all $j \neq k$.) It is easy to check that $|\nu|\colon \mathcal{F} \to [0, \infty]$ is a positive measure. (The expression *positive measure* simply means measure but is used for emphasis to distinguish it from complex and signed measures.) It is also straightforward to verify that $|\nu|$ is the smallest positive measure that dominates ν. We will soon see that $|\nu|$ is in fact a finite measure, *i.e.,* that $|\nu|(\Omega) < \infty$ (see the remarks following Theorem 2.) We define the *total variation* $\|\nu\|_1$ *of* ν to be $|\nu|(\Omega)$.

Continuity of measure. A complex measure ν on $\mathcal{F}$ is continuous in the following sense. Given a sequence (A_n) in $\mathcal{F}$,

(i) if $A_n \subset A_{n+1}$ for all $n \in \mathbb{N}$, then $\nu\big(\bigcup A_n\big) = \lim \nu(A_n)$, and

(ii) if $A_n \supset A_{n+1}$ for all $n \in \mathbb{N}$, then $\nu\big(\bigcap A_n\big) = \lim \nu(A_n)$.

Signed measures. Let $(\Omega, \mathcal{F})$ be as before. A *signed measure on* $\mathcal{F}$ is a countably additive set function $\nu\colon \mathcal{F} \to \mathbb{R}$. Every signed measure is in particular a complex measure.

Theorem 2. Let $\nu\colon \mathcal{F} \to \mathbb{R}$ be a signed measure on the σ-field $\mathcal{F}$. Then there is a measurable partition $\Omega = P \cup N$ of Ω such that $\nu(A) \geqslant 0$ for all $A \subset P$, $A \in \mathcal{F}$ and $\nu(A) \leqslant 0$ for all $A \subset N$, $A \in \mathcal{F}$.

Remarks. 1. The partition $\Omega = P \cup N$ is called the *Hahn decomposition* of Ω (or of ν).
2. Define $\nu^+(A) = \nu(A \cap P)$ and $\nu^-(A) = -\nu(A \cap N)$ for $A \in \mathcal{F}$. Then ν^+ and ν^- are finite positive measures satisfying $\nu = \nu^+ - \nu^-$ and $|\nu| = \nu^+ + \nu^-$. These properties uniquely define ν^+ and ν^-. This unique way of writing the signed measure ν as the difference of two finite positive measures is called the *Jordan decomposition of* ν.
3. Let $\nu\colon \mathcal{F} \to \mathbb{C}$ be a complex measure. Then $\mathrm{Re}(\nu)$ and $\mathrm{Im}(\nu)$ are signed measures, so they have Jordan decompositions $\nu_1 - \nu_2$ and $\nu_3 - \nu_4$, respectively. We then obtain the expression $\nu = \nu_1 - \nu_2 + \mathrm{i}\nu_3 - \mathrm{i}\nu_4$ called the Jordan decomposition of ν. It follows that $\nu_k \leqslant |\nu|$ for all k, and $|\nu| \leqslant \nu_1 + \nu_2 + \nu_3 + \nu_4$. From this we deduce that $|\nu|$ is a finite measure.
4. Let ν be a signed measure with Jordan decomposition $\nu^+ - \nu^-$ (as in **2.** above). Then for each $A \in \mathcal{F}$ we have $\nu^+(A) = \sup\{\nu(B) : B \subset A,\ B \in \mathcal{F}\}$. This motivates the proof of Theorem 2.

Absolute continuity. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $\nu\colon \mathcal{F} \to \mathbb{C}$ be a complex measure. We say ν *is absolutely continuous with respect to* μ, and write $\nu \ll \mu$, if $\nu(A) = 0$ whenever $A \in \mathcal{F}$ and $\mu(A) = 0$.

Remarks. 1. If $\nu \ll \mu$, then $|\nu| \ll \mu$. It follows that if $\nu = \nu_1 - \nu_2 + \mathrm{i}\nu_3 - \mathrm{i}\nu_4$ is the Jordan decomposition of ν, then $\nu_k \ll \mu$ for all k.
2. If $\nu \ll \mu$, then for all $\varepsilon > 0$ there exists $\delta > 0$ such that for every $A \in \mathcal{F}$ we have $\mu(A) < \delta \implies |\nu(A)| < \varepsilon$.

Example. Let $f \in L_1(\mu)$ and define $\nu(A) = \int_A f \,\mathrm{d}\mu$ for $A \in \mathcal{F}$. Then by dominated convergence, ν is a complex or signed measure. Clearly $\nu \ll \mu$.

Definition. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. A set $A \in \mathcal{F}$ is called *σ-finite* if there is a sequence (A_n) in $\mathcal{F}$ such that $A = \bigcup_n A_n$ and $\mu(A_n) < \infty$ for all $n \in \mathbb{N}$. We say that the measure μ (or the measure space $(\Omega, \mathcal{F}, \mu)$) is σ-finite if Ω is a σ-finite set (equivalently every $A \in \mathcal{F}$ is σ-finite).

Theorem 3. (Radon–Nikodym) Let $(\Omega, \mathcal{F}, \mu)$ be a σ-finite measure space and $\nu\colon \mathcal{F} \to \mathbb{C}$ be a complex measure on $\mathcal{F}$ which is absolutely continuous with respect to μ. Then there exists a unique $f \in L_1(\mu)$ such that $\nu(A) = \int_A f \,\mathrm{d}\mu$ for all $A \in \mathcal{F}$. Moreover, f takes values in $\mathbb{C}/\mathbb{R}/\mathbb{R}^+$ according to whether ν is a complex/signed/positive measure, respectively.

Remarks. 1. The proof shows that for an arbitrary complex measure ν on $\mathcal{F}$ there is a decomposition $\nu = \nu_1 + \nu_2$, where ν_1 is absolutely continuous with respect to μ (and hence of the form $\int_A f \,\mathrm{d}\mu$ for some $f \in L_1(\mu)$), and ν_2

is *orthogonal* to μ, written $\nu_2 \perp \mu$, which means that there is a measurable partition $\Omega = P \cup N$ of Ω with $\mu(P) = 0$ and $\nu_2(A) = 0$ for all $A \subset N$. This decomposition is called the *Lebesgue decomposition of ν with respect to μ.*
2. The unique $f \in L_1(\mu)$ in the theorem is called the *Radon–Nikodym derivative of ν with respect to μ* and is denoted by $\frac{\mathrm{d}\nu}{\mathrm{d}\mu}$. It can be shown that a measurable function g is ν-integrable if and only if $g \cdot \frac{\mathrm{d}\nu}{\mathrm{d}\mu}$ is μ-integrable in which case we have

$$\int_\Omega g \,\mathrm{d}\nu = \int_\Omega g \cdot \frac{\mathrm{d}\nu}{\mathrm{d}\mu} \,\mathrm{d}\mu \ .$$

The dual space of L_p. We fix a measure space $(\Omega, \mathcal{F}, \mu)$. Let $1 \leqslant p < \infty$ and let q be the conjugate index of p (*i.e.,* $1 < q \leqslant \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$). For $g \in L_q = L_q(\mu)$ define

$$\varphi_g \colon L_p \to \text{scalars} \ , \qquad \varphi_g(f) = \int_\Omega fg \,\mathrm{d}\mu \ .$$

By Hölder's inequality we have φ_g is well-defined and $|\varphi_g(f)| \leqslant \|f\|_p \cdot \|g\|_q$. By linearity of integration, φ_g is linear, and hence $\varphi_g \in L_p^*$ with $\|\varphi_g\| \leqslant \|g\|_q$. We have thus obtained a function

$$\varphi \colon L_q \to L_p^* \ , \quad g \mapsto \varphi_g \ .$$

This is linear by linearity of integration and bounded with $\|\varphi\| \leqslant 1$.

Theorem 4. Let $(\Omega, \mathcal{F}, \mu)$, p, q, φ be as above.

(i) If $1 < p < \infty$, then φ is an isometric isomorphism. Thus $L_p^* \cong L_q$.

(ii) If $p = 1$ and in addition μ is σ-finite, then φ is an isometric isomorphism. Thus, in this case, we have $L_1^* \cong L_\infty$.

Corollary 5. For any measure space $(\Omega, \mathcal{F}, \mu)$ and for $1 < p < \infty$, the Banach space $L_p(\mu)$ is reflexive.

***C*(*K*) spaces.** For the rest of this chapter K is a compact Hausdorff space. We shall be interested in the following spaces and sets.

$C(K) = \{f \colon K \to \mathbb{C} : f \text{ is continuous}\}$

$C^{\mathbb{R}}(K) = \{f \colon K \to \mathbb{R} : f \text{ is continuous}\}$

$C^+(K) = \{f \in C(K) : f \geqslant 0 \text{ on } K\}$

$M(K) = C(K)^* = \{\varphi \colon C(K) \to \mathbb{C} : \varphi \text{ is linear and continuous}\}$

$M^{\mathbb{R}}(K) = \{\varphi \in M(K) : \varphi(f) \in \mathbb{R} \text{ for all } f \in C^{\mathbb{R}}(K)\}$

$M^+(K) = \{\varphi \colon C(K) \to \mathbb{C} : \varphi \text{ is linear and } \varphi(f) \geqslant 0 \text{ for all } f \in C^+(K)\}$

Some comments on the above list:

(i) $C(K)$ is a complex Banach space in the uniform norm: $\|f\|_\infty = \sup_K |f|$.

(ii) $C^{\mathbb{R}}(K)$ is a real Banach space in the uniform norm.

(iii) $C^+(K)$ is a subset of $C(K)$. By "$f \geqslant 0$ on K" we of course mean that $f(x) \geqslant 0$ for all $x \in K$.

(iv) $M(K)$ is the dual space of $C(K)$, which is a Banach space in the operator norm.

(v) $M^{\mathbb{R}}(K)$ is a closed, real-linear subspace of $M(K)$. We shall soon see that it can be identified with the dual space of $C^{\mathbb{R}}(K)$.

(vi) Elements of $M^+(K)$ are called *positive linear functionals*. We shall see that these are automatically continuous.

Note. Our aim is to describe the dual spaces $M(K)$ and $M^{\mathbb{R}}(K)$. The main conclusion of the next lemma is that it is sufficient to focus on $M^+(K)$.

Lemma 6.

(i) $\forall\, \varphi \in M(K)\ \exists$ unique $\varphi_1, \varphi_2 \in M^{\mathbb{R}}(K)$ such that $\varphi = \varphi_1 + \mathrm{i}\varphi_2$.

(ii) $\varphi \mapsto \varphi\restriction_{C^{\mathbb{R}}(K)} \colon M^{\mathbb{R}}(K) \to \big(C^{\mathbb{R}}(K)\big)^*$ is an isometric isomorphism.

(iii) $M^+(K) \subset M(K)$ and $M^+(K) = \big\{\varphi \in M(K) : \|\varphi\| = \varphi(\mathbf{1}_K)\big\}$.

(iv) For every $\varphi \in M^{\mathbb{R}}(K)$ there exist unique $\varphi^+, \varphi^- \in M^+(K)$ such that

$$\varphi = \varphi^+ - \varphi^- \qquad \text{and} \qquad \|\varphi\| = \|\varphi^+\| + \|\varphi^-\| \ .$$

Topological preliminaries. 1. A compact Hausdorff space is *normal*: given disjoint closed subsets E and F of K, there exist disjoint open subsets U and V of K such that $E \subset U$ and $F \subset V$. Equivalently, given $E \subset U \subset K$ with E closed and U open, there is an open set $V \subset K$ such that $E \subset V \subset \overline{V} \subset U$.
2. Urysohn's lemma: given disjoint closed subsets E and F of K, there is a continuous function $f\colon K \to [0,1]$ such that $f = 0$ on E and $f = 1$ on F.
3. The notation $f \prec U$ will mean that U is an open subset of K, f is a continuous function $K \to [0,1]$, and the *support of* f,

$$\operatorname{supp} f = \overline{\{x \in K : f(x) \neq 0\}}$$

is contained in U. The notation $E \prec f$ will mean that E is a closed subset of K, f is a continuous function $K \to [0,1]$, and $f = 1$ on E.

Note that Urysohn's lemma is equivalent to the following statement. Given $E \subset U \subset K$ with E closed and U open, there is an f with $E \prec f \prec U$.

Lemma 7. Let E and $U_1, \dots, U_n$ be subsets of K with E closed, U_j open, and $E \subset \bigcup_{j=1}^n U_j$. Then

(i) there exist open sets $V_1, \dots, V_n$ with $E \subset \bigcup_{j=1}^n V_j$ and $\overline{V}_j \subset U_j$ for all j;

(ii) there exist functions $f_j \prec U_j$ such that $0 \leqslant \sum_{j=1}^n f_j \leqslant 1$ on K, and $\sum_{j=1}^n f_j = 1$ on E.

Borel sets and measures. Let X be a Hausdorff topological space and $\mathcal{G}$ be the collection of open subsets of X. The *Borel σ-field on X* is defined to be $\mathcal{B} = \sigma(\mathcal{G})$, the σ-field generated by $\mathcal{G}$, *i.e.,* the smallest σ-field on X containing $\mathcal{G}$. Equivalently, $\mathcal{B}$ is the intersection of all σ-fields on X that contain $\mathcal{G}$. Members of $\mathcal{B}$ are called *Borel sets*.

A *Borel measure on X* is a measure on $\mathcal{B}$. Given a (positive) Borel measure μ on X, we say μ is *regular* if the following hold:

(i) $\mu(E) < \infty$ for all compact $E \subset X$;

(ii) $\mu(A) = \inf\{\mu(U) : A \subset U \in \mathcal{G}\}$ for all $A \in \mathcal{B}$;

(iii) $\mu(U) = \sup\{\mu(E) : E \subset U, E \text{ compact}\}$ for all $U \in \mathcal{G}$.

A complex Borel measure ν on X is defined to be *regular* if $|\nu|$ is a regular measure.

Note that if X is compact Hausdorff, then a Borel measure μ on X is regular if and only if

$$\mu(X) < \infty \quad \text{and} \quad \mu(A) = \inf\{\mu(U) : A \subset U \in \mathcal{G}\} \quad \forall\, A \in \mathcal{B}$$

which in turn is equivalent to

$$\mu(X) < \infty \quad \text{and} \quad \mu(A) = \sup\{\mu(E) : E \subset A,\ E \text{ closed}\} \quad \forall\, A \in \mathcal{B}\ .$$

Integration with respect to complex measures. Let Ω be a set, $\mathcal{F}$ be a σ-field on Ω, and ν be a complex measure on $\mathcal{F}$. A measurable function $f\colon \Omega \to \mathbb{C}$ is *ν-integrable* if $\int_\Omega |f|\,\mathrm{d}|\nu| < \infty$. In that case we define

$$\int_\Omega f\,\mathrm{d}\nu = \int_\Omega f\,\mathrm{d}\nu_1 - \int_\Omega f\,\mathrm{d}\nu_2 + \mathrm{i}\int_\Omega f\,\mathrm{d}\nu_3 - \mathrm{i}\int_\Omega f\,\mathrm{d}\nu_4\ ,$$

where $\nu = \nu_1 - \nu_2 + \mathrm{i}\nu_3 - \mathrm{i}\nu_4$ is the Jordan decomposition of ν. Note that f is ν-integrable if and only if f is ν_k-integrable for each k.

The following properties are easy to check:

1. $\int_\Omega \mathbf{1}_A\,\mathrm{d}\nu = \nu(A)$ for all $A \in \mathcal{F}$.

2. Linearity: given ν-integrable functions f, g and complex numbers a, b, the function $af + bg$ is ν-integrable, and $\int_\Omega (af + bg)\,\mathrm{d}\nu = a\int_\Omega f\,\mathrm{d}\nu + b\int_\Omega g\,\mathrm{d}\nu$.

3. Dominated convergence (D.C.): let f_n, $n \in \mathbb{N}$, be ν-integrable functions that converge a.e. to a measurable function f. Assume that there exists $g \in L_1(|\nu|)$ with $|f_n| \leqslant g$ for all n. Then f is ν-integrable and $\int_\Omega f_n\,\mathrm{d}\nu \to \int_\Omega f\,\mathrm{d}\nu$.

4. $\left|\int_\Omega f\,\mathrm{d}\nu\right| \leqslant \int_\Omega |f|\,\mathrm{d}|\nu|$ for all ν-integrable f.

Note. Let ν be a complex Borel measure on K. For $f \in C(K)$ we have

$$\int_K |f|\,\mathrm{d}|\nu| \leqslant \|f\|_\infty \cdot |\nu|(K)\ ,$$

and hence f is ν-integrable. The function $\varphi\colon C(K) \to \mathbb{C}$ given by $\varphi(f) = \int_K f\,\mathrm{d}\nu$ is linear and bounded with $\|\varphi\| \leqslant \|\nu\|_1$. Thus $\varphi \in M(K)$. Note that if ν is a signed measure, then $\varphi \in M^{\mathbb{R}}(K)$, and if ν is a positive measure, then $\varphi \in M^+(K)$.

Theorem 8. (Riesz Representation Theorem) For every $\varphi \in M^+(K)$ there is a unique regular Borel measure μ on K that represents φ:

$$\varphi(f) = \int_K f \, \mathrm{d}\mu \qquad \text{for all } f \in C(K) \ .$$

Moreover, we have $\mu(K) = \|\varphi\|$.

Corollary 9. For every $\varphi \in M(K)$ there is a unique regular complex Borel measure ν on K that represents φ:

$$\varphi(f) = \int_K f \, \mathrm{d}\nu \qquad \text{for all } f \in C(K) \ .$$

Moreover, we have $\|\varphi\| = \|\nu\|_1$, and if $\varphi \in M^{\mathbb{R}}(K)$ then ν is a signed measure.

Corollary 10. The space of regular complex Borel measures on K is a complex Banach space in the total variation norm, and it is isometrically isomorphic to $M(K) = C(K)^*$. The space of regular signed Borel measures on K is a real Banach space in the total variation norm, and it is isometrically isomorphic to $M^{\mathbb{R}}(K) \cong \big(C^{\mathbb{R}}(K)\big)^*$.

3 Weak topologies

Let X be a set and $\mathcal{F}$ be a family of functions such that each $f \in \mathcal{F}$ is a function $f\colon X \to Y_f$ where Y_f is a topological space. The *weak topology* $\sigma(X, \mathcal{F})$ *of* X is the smallest topology on X with respect to which every $f \in \mathcal{F}$ is continuous.

Remarks. 1. The family $\mathcal{S} = \{f^{-1}(U) : f \in \mathcal{F},\ U \subset Y_f \text{ an open set}\}$ is a sub-base for $\sigma(X, \mathcal{F})$. This means that $\sigma(X, \mathcal{F})$ is the topology generated by $\mathcal{S}$, *i.e.,* the smallest topology containing $\mathcal{S}$. Equivalently, the family of finite intersections of members of $\mathcal{S}$ is a base for $\sigma(X, \mathcal{F})$, *i.e.,* $\sigma(X, \mathcal{F})$ consists of arbitrary unions of finite intersections of members of $\mathcal{S}$. To spell this out, a set $V \subset X$ is open in the weak topology $\sigma(X, \mathcal{F})$ if and only if for all $x \in V$ there exist $n \in \mathbb{N}$, $f_1, \dots, f_n \in \mathcal{F}$ and open sets $U_j \subset Y_{f_j}$, $1 \leqslant j \leqslant n$, such that

$$x \in \bigcap_{j=1}^{n} f_j^{-1}(U_j) \subset V \ .$$

To put it in another way, $V \subset X$ is open in the weak topology if and only if for all $x \in V$ there exist $n \in \mathbb{N}$, $f_1, \dots, f_n \in \mathcal{F}$ and open neighbourhoods U_j of $f_j(x)$ in Y_{f_j}, $1 \leqslant j \leqslant n$, such that $\bigcap_{j=1}^{n} f_j^{-1}(U_j) \subset V$.
2. Suppose that $\mathcal{S}_f$ is a sub-base for the topology of Y_f for each $f \in \mathcal{F}$. Then $\{f^{-1}(U) : f \in \mathcal{F},\ U \in \mathcal{S}_f\}$ is a sub-base for $\sigma(X, \mathcal{F})$.
3. If Y_f is a Hausdorff space for each $f \in \mathcal{F}$ and $\mathcal{F}$ separates the points of X (for all $x \neq y$ in X there exists $f \in \mathcal{F}$ such that $f(x) \neq f(y)$), then $\sigma(X, \mathcal{F})$ is Hausdorff.
4. Let $Y \subset X$ and set $\mathcal{F}_Y = \{f\restriction_Y\colon f \in \mathcal{F}\}$. Then $\sigma(X, \mathcal{F})\restriction_Y = \sigma(Y, \mathcal{F}_Y)$, *i.e.,* the subspace topology on Y induced by the weak topology $\sigma(X, \mathcal{F})$ of X is the same as the weak topology on Y defined by $\mathcal{F}_Y$.

5. (Universal property) Let Z be a topological space and $g\colon Z \to X$ a function. Then g is continuous if and only if $f \circ g\colon Z \to Y_f$ is continuous for every $f \in \mathcal{F}$. This universal property characterizes the weak topology (*cf.* Examples Sheet 2).

Examples. 1. Let X be a topological space, $Y \subset X$ and $\iota\colon Y \to X$ be the inclusion map. Then the weak topology $\sigma(Y, \{\iota\})$ is the subspace topology of Y induced by X.
2. Let Γ be an arbitrary set, and let X_γ be a topological space for each $\gamma \in \Gamma$. Let X be the Cartesian product $\prod_{\gamma\in\Gamma} X_\gamma$. Thus

$$X = \{x : x \text{ is a function with domain } \Gamma\text{, and } x(\gamma) \in X_\gamma \text{ for all } \gamma \in \Gamma\}\ .$$

For $x \in X$ we often write x_γ instead of $x(\gamma)$, and think of x as the "Γ-tuple" $(x_\gamma)_{\gamma\in\Gamma}$. For each γ we consider the function $\pi_\gamma\colon X \to X_\gamma$ given by $x \mapsto x(\gamma)$ (or $(x_\delta)_{\delta\in\Gamma} \mapsto x_\gamma$) called *evaluation at* γ or *projection onto* X_γ. The *product topology on* X is the weak topology $\sigma(X, \{\pi_\gamma : \gamma \in \Gamma\})$. Note that $V \subset X$ is open if and only if for all $x = (x_\gamma)_{\gamma\in\Gamma} \in V$ there exist $n \in \mathbb{N}$, $\gamma_1, \dots, \gamma_n \in \Gamma$ and open neighbourhoods U_j of x_{γ_j} in X_{γ_j} such that

$$\big\{y = (y_\gamma)_{\gamma\in\Gamma} : y_{\gamma_j} \in U_j \text{ for } 1 \leqslant j \leqslant n\big\} \subset V\ .$$

Proposition 1. Assume that X is a set, and for each $n \in \mathbb{N}$ we are given a metric space (Y_n, d_n) and a function $f_n\colon X \to Y_n$. Further assume that $\mathcal{F} = \{f_n : n \in \mathbb{N}\}$ separates the points of X. Then the weak topology $\sigma(X, \mathcal{F})$ of X is metrizable.

Remark. Without the assumption that $\mathcal{F}$ separates the points of X, we can conclude that $\sigma(X, \mathcal{F})$ is pseudo-metrizable.

Theorem 2. (Tychonov) The product of compact topological spaces is compact in the product topology.

Weak topologies on vector spaces. Let E be a real or complex vector space. Let F be a subspace of the space of all linear functionals on E that separates the points of E (for all $x \in E$, $x \neq 0$, there exists $f \in F$ such that $f(x) \neq 0$). Consider the weak topology $\sigma(E, F)$ on E. Note that $U \subset E$ is open if and only if for all $x \in U$ there exist $n \in \mathbb{N}$, $f_1, \dots, f_n \in F$ and $\varepsilon > 0$ such that

$$\{y \in E : |f_i(y - x)| < \varepsilon \text{ for } 1 \leqslant i \leqslant n\} \subset U\ .$$

For $f \in F$ define $p_f\colon E \to \mathbb{R}$ by $p_f(x) = |f(x)|$. Set $\mathcal{P} = \{p_f : f \in F\}$. Then $\mathcal{P}$ is a family of seminorms on E that separates the points of E, and the topology of the locally convex space $(E, \mathcal{P})$ is precisely $\sigma(E, F)$. In particular, $\sigma(E, F)$ is a Hausdorff topology with respect to which addition and scalar multiplication are continuous.

Lemma 3. Let E be as above. Let $f, g_1, g_2, \dots, g_n$ be linear functionals on E. If $\bigcap_{j=1}^n \ker g_j \subset \ker f$, then $f \in \operatorname{span}\{g_1, \dots, g_n\}$.

Proposition 4. Let E and F be as above. Then a linear functional f on E is $\sigma(E, F)$-continuous if and only if $f \in F$. In other words $(E, \sigma(E, F))^* = F$.

Note. Recall that $(E, \sigma(E,F))^*$ denotes the dual space of the locally convex space $(E, \sigma(E,F))$, *i.e.,* the space of linear functionals that are continuous with respect to $\sigma(E,F)$.

Examples. The following two examples of weak topologies on vector spaces are the central objects of interest in this chapter.

1. Let X be a normed space. The *weak topology on* X is the weak topology $\sigma(X, X^*)$ on X. Note that X^* separates the points of X by Hahn–Banach. We shall sometimes denote X with the weak topology by (X, w). Open sets in the weak topology are called *weakly open* or *w-open.* Note that $U \subset X$ is w-open if and only if for all $x \in U$ there exist $n \in \mathbb{N}$, $f_1, \dots, f_n \in X^*$ and $\varepsilon > 0$ such that

$$\{y \in X : |f_i(y - x)| < \varepsilon \text{ for } 1 \leqslant i \leqslant n\} \subset U \ .$$

2. Let X be a normed space. The *weak-star topology (or w^*-topology) on* X^* is the weak topology $\sigma(X^*, X)$ on X^*. Here we identify X with its image in X^{**} under the canonical embedding. We shall sometimes denote X^* with the w^*-topology by (X^*, w^*). Open sets in the weak-star topology are called *w^*-open.* Note that $U \subset X^*$ is w^*-open if and only if for all $f \in U$ there exist $n \in \mathbb{N}$, $x_1, \dots, x_n \in X$ and $\varepsilon > 0$ such that

$$\{g \in X^* : |g(x_i) - f(x_i)| < \varepsilon \text{ for } 1 \leqslant i \leqslant n\} \subset U \ .$$

Properties. 1. (X, w) and (X^*, w^*) are locally convex spaces, so in particular they are Hausdorff, and addition and scalar multiplication are continuous.
2. $\sigma(X, X^*) \subset \|\cdot\|$-topology with equality if and only if $\dim X < \infty$.
3. $\sigma(X^*, X) \subset \sigma(X^*, X^{**}) \subset \|\cdot\|$-topology with equality in the second inclusion if and only if $\dim X < \infty$, and with equality in the first inclusion if and only if X is reflexive (*cf.* Proposition 5 below).
4. If Y is a subspace of X, then $\sigma(X, X^*) \restriction_Y = \sigma(Y, Y^*)$. This follows from Remark 4 on page 12 and from the Hahn–Banach theorem.

Similarly, we have $\sigma(X^{**}, X^*)\restriction_X = \sigma(X, X^*)$, *i.e.,* the subspace topology on X induced by the w^*-topology of X^{**} is the weak topology of X. Thus, the canonical embedding $X \to X^{**}$ is a w-to-w^*-homeomorphism from X onto its image (as well as being an isometric isomorphism).

Proposition 5. Let X be a normed space. Then

(i) a linear functional f on X is weakly continuous (*i.e.,* continuous with respect to the weak topology on X) if and only if $f \in X^*$. Briefly, $(X, w)^* = X^*$;

(ii) a linear functional φ on X^* is w^*-continuous (*i.e.,* continuous with respect to the weak-star topology on X^*) if and only if $\varphi \in X$, *i.e.,* there exists $x \in X$ with $\varphi = \hat{x}$. Briefly, $(X^*, w^*)^* = X$.

It follows that on X^* the weak and weak-star topologies coincide, *i.e.,* we have $\sigma(X^*, X^{**}) = \sigma(X^*, X)$ if and only if X is reflexive.

Definition. Let X be a normed space.
A set $A \subset X$ is *weakly bounded* if $\{f(x) : x \in A\}$ is bounded for all $f \in X^*$. Equivalently, for each w-open ngbd U of 0, there exists $\lambda > 0$ such that $A \subset \lambda U$.

A set $B \subset X^*$ is *weak-star bounded* if $\{f(x) : f \in B\}$ is bounded for all $x \in X$. Equivalently, for each w^*-open ngbd U of 0, there exists $\lambda > 0$ such that $B \subset \lambda U$.

Principal of Uniform Boundedness (PUB). If X is a Banach space, Y is a normed space, $\mathcal{T} \subset \mathcal{B}(X,Y)$ is pointwise bounded ($\sup\{\|Tx\| : T \in \mathcal{T}\} < \infty$ for all $x \in X$), then $\mathcal{T}$ is uniformly bounded ($\sup\{\|T\| : T \in \mathcal{T}\} < \infty$).

Proposition 6. Let X be a normed space.

(i) Any weakly bounded set in X is bounded in norm.

(ii) If X is complete then any weak-star bounded set in X^* is bounded in norm.

Notation. 1. We write $x_n \xrightarrow{w} x$ and say x_n *converges weakly to* x if $x_n \to x$ in the weak topology (in some normed space X). This happens if and only if $\langle x_n, f\rangle \to \langle x, f\rangle$ for all $f \in X^*$.
2. We write $f_n \xrightarrow{w^*} f$ and say f_n *converges weak-star to* f if $f_n \to f$ in the weak-star topology (in some dual space X^*). This happens if and only if $\langle x, f_n\rangle \to \langle x, f\rangle$ for all $x \in X$.

A consequence of PUB. Let X be a Banach space, Y a normed space, and (T_n) a sequence in $\mathcal{B}(X,Y)$ that converges pointwise to a function T. Then $T \in \mathcal{B}(X,Y)$, $\sup\|T_n\| < \infty$ and $\|T\| \leqslant \liminf\|T_n\|$.

Proposition 7. Let X be a normed space.

(i) If $x_n \xrightarrow{w} x$ in X, then $\sup\|x_n\| < \infty$ and $\|x\| \leqslant \liminf\|x_n\|$.

(ii) If $f_n \xrightarrow{w^*} f$ in X^* and X is complete, then $\sup\|f_n\| < \infty$ and $\|f\| \leqslant \liminf\|f_n\|$.

The Hahn–Banach Separation Theorems. Let $(X, \mathcal{P})$ be a locally convex space. Let C be a convex subset of X with $0 \in \operatorname{int} C$. Define

$$\mu_C \colon X \to \mathbb{R} , \qquad \mu_C(x) = \inf\{t > 0 : x \in tC\} .$$

This function is well-defined and is called the *Minkowski functional* or *gauge functional* of C.

Example. If X is a normed space and $C = B_X$, then $\mu_C = \|\cdot\|$.

Lemma 8. The function μ_C is a positive homogeneous, subadditive functional. Moreover, we have

$$\{x \in X : \mu_C(x) < 1\} \subset C \subset \{x \in X : \mu_C(x) \leqslant 1\}$$

with equality in the first inclusion when C is open and equality in the second inclusion when C is closed.

Remark. If C is symmetric in the case of real scalars, or balanced in the case of complex scalars, then μ_C is a seminorm. If in addition C is bounded, then μ_C is a norm.

Theorem 9. (Hahn–Banach separation theorem) Let $(X, \mathcal{P})$ be a locally convex space and C be an open convex subset of X with $0 \in C$. Let $x_0 \in X \setminus C$. Then there exists $f \in X^*$ such that $f(x_0) > f(x)$ for all $x \in C$ if the scalar field is $\mathbb{R}$, and $\mathrm{Re} f(x_0) > \mathrm{Re} f(x)$ for all $x \in C$ if the scalar field is $\mathbb{C}$.

Remark. From now on, for the rest of this chapter, we assume that the scalar field is $\mathbb{R}$. It is straightforward to modify the statements of theorems, their proofs, etc, in the case of complex scalars.

Theorem 10. (Hahn–Banach separation theorem) Let $(X, \mathcal{P})$ be a locally convex space and A, B be non-empty disjoint convex subsets of X.

(i) If A is open, then there exists $f \in X^*$ and $\alpha \in \mathbb{R}$ such that $f(x) < \alpha \leqslant f(y)$ for all $x \in A$, $y \in B$.

(ii) If A is compact and B is closed, then there exists $f \in X^*$ such that $\sup_A f < \inf_B f$.

Theorem 11. (Mazur) Let C be a convex subset of a normed space X. Then $\overline{C}^{\|\cdot\|} = \overline{C}^w$, *i.e.,* the norm-closure and weak-closure of C are the same. It follows that C is norm-closed if and only if C is weakly closed.

Corollary 12. Assume that $x_n \xrightarrow{w} 0$ in a normed space X. Then for all $\varepsilon > 0$ there exists $x \in \mathrm{conv}\{x_n : n \in \mathbb{N}\}$ with $\|x\| < \varepsilon$.

Remark. It follows that there exist $p_1 < q_1 < p_2 < q_2 < \dots$ and convex combinations $\sum_{i=p_n}^{q_n} t_i x_i$ that converge to zero in norm. In some but not all cases a stronger conclusion can be obtained: see Examples Sheet 2, Question 13.

Theorem 13. (Banach–Alaoglu) The dual ball B_{X^*} is w^*-compact for any normed space X.

Proposition 14. Let X be a normed space and K be a compact Hausdorff space.

(i) X is separable if and only if (B_{X^*}, w^*) is metrizable.

(ii) $C(K)$ is separable if and only if K is metrizable.

Remarks. 1. If X is separable, then (B_{X^*}, w^*) is a compact metric space. In particular, B_{X^*} is w^*-sequentially compact.
2. If X is separable, then X^* is w^*-separable. It is an easy consequence of Mazur's theorem that X is separable if and only if X is weakly separable. Thus, the previous statement reads: if X is w-separable, then X^* is w^*-separable. The converse is false, *e.g.,* for $X = \ell_\infty$ (see the remark following Goldstine's theorem).
3. The proof shows that any compact Hausdorff space K is a subspace of $(B_{C(K)^*}, w^*)$.
4. The proof shows that every normed space X embeds isometrically into $C(K)$ for some compact Hausdorff space. In particular, this holds with $K = (B_{X^*}, w^*)$ (*cf.* Theorem 19 below).

Proposition 15. X^* is separable if and only if (B_X, w) is metrizable.

Theorem 16. (Goldstine) $\overline{B_X}^{w^*} = B_{X^{**}}$, *i.e.,* the w^*-closure of the unit ball B_X of a normed space X in the second dual X^{**} is $B_{X^{**}}$.

Remark. It follows from Goldstine that $\overline{X}^{w^*} = X^{**}$. Thus, if X is separable, then X^{**} is w^*-separable. For example, ℓ_∞^* is w^*-separable.

Theorem 17. Let X be a Banach space. Then TFAE.

(i) X is reflexive.

(ii) (B_X, w) is compact.

(iii) X^* is reflexive.

Remark. It follows that if X is separable and reflexive, then (B_X, w) is a compact metric space, and hence sequentially compact.

Lemma 18. For every non-empty compact metric space K, there is a continuous surjection $\varphi\colon \{0,1\}^{\mathbb{N}} \to K$, where $\{0,1\}^{\mathbb{N}}$ is given the product topology.

Note. $\{0,1\}^{\mathbb{N}}$ is homeomorphic to the middle-third Cantor set Δ via the map $(\varepsilon_i) \mapsto \sum_i (2\varepsilon_i)3^{-i}$.

Theorem 19. Every separable Banach space embeds isometrically into $C[0,1]$. Thus the separable space $C[0,1]$ is isometrically universal for the class of separable Banach spaces.

4 Convexity

Let X be a real (or complex) vector space and K be a convex subset of X. We say a point $x \in K$ is an *extreme point of* K if whenever $x = (1-t)y + tz$ for some $y, z \in K$ and $t \in (0,1)$, then $y = z = x$. We denote by $\operatorname{Ext} K$ the set of all extreme points of K.

Examples. $\operatorname{Ext} B_{\ell_1^2} = \{\pm e_1, \pm e_2\}$, $\operatorname{Ext} B_{\ell_2^2} = S_{\ell_2^2}$, $\operatorname{Ext} B_{c_0} = \emptyset$.

Theorem 1. (Krein–Milman) Let $(X, \mathcal{P})$ be a locally convex space and K a compact convex subset of X. Then $K = \overline{\operatorname{conv}} \operatorname{Ext} K$. In particular, $\operatorname{Ext} K \neq \emptyset$ provided K is not empty.

Corollary 2. Let X be a normed space. Then $B_{X^*} = \overline{\operatorname{conv}}^{w^*} \operatorname{Ext} B_{X^*}$. In particular, $\operatorname{Ext} B_{X^*} \neq \emptyset$.

Remark. It follows that c_0 is not a dual space, *i.e.,* there is no normed space X with $c_0 \cong X^*$.

Definition. Let $(X, \mathcal{P})$ be a locally convex space and $K \subset X$ be a non-empty convex compact set. A *face of* K is a non-empty convex compact subset F of K such that for all $y, z \in K$ and $t \in (0,1)$, if $(1-t)y + tz \in F$, then $y, z \in F$.

Examples. 1. K is a face of K. For $x \in K$, we have $x \in \operatorname{Ext} K$ if and only if $\{x\}$ is a face of K.
2. For $f \in X^*$ and $\alpha = \sup_K f$, the set $F = \{x \in K : f(x) = \alpha\}$ is a face of K. Note that throughout this chapter we will use real scalars in statements of results and in their proofs. Obvious modifications yield the complex case, so here for example one would replace f by $\mathrm{Re} f$ in the definition of α and F.
3. If F is a face of K and E is a face of F, then E is a face of K. In particular, if F is a face of K and $x \in \operatorname{Ext} F$, then $x \in \operatorname{Ext} K$.

Lemma 3. Let $(X, \mathcal{P})$ be a locally convex space, $K \subset X$ be compact, and let $x_0 \in K$. Then for every neighbourhood V of x_0 there exist $n \in \mathbb{N}$, $f_1, \dots, f_n \in X^*$ and $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ such that

$$x_0 \in \{x \in X : f_i(x) < \alpha_i,\ 1 \leqslant i \leqslant n\} \cap K \subset V .$$

Lemma 4. Let $(X, \mathcal{P})$ be a locally convex space, $K \subset X$ be compact and convex, and let $x_0 \in \operatorname{Ext} K$. Then the *slices of* K *at* x_0 form a neighbourhood base of x_0 in K, *i.e.,* for every neighbourhood V of x_0 there exist $f \in X^*$ and $\alpha \in \mathbb{R}$ such that

$$x_0 \in \{x \in X : f(x) < \alpha\} \cap K \subset V .$$

Theorem 5. (Partial converse of Krein–Milman) Let $(X, \mathcal{P})$ be a locally convex space and $K \subset X$ be a non-empty compact convex set. If $S \subset K$ satisfies $K = \overline{\operatorname{conv}} S$, then $\overline{S} \supset \operatorname{Ext} K$.

Remark. Note that $S \supset \operatorname{Ext} K$ need not hold. For example, any dense subset S of $S_{\ell_2^2}$ satisfies $\overline{\operatorname{conv}} S = B_{\ell_2^2}$. Note also that $\operatorname{Ext} K$ need not be closed in general. For example, if K is the convex hull of $\{(x, 0) : x \in S_{\ell_2^2}\} \cup \{(-1, 0, \pm 1)\}$ in $\mathbb{R}^3$, then $\operatorname{Ext} K$ is not closed.

Example. Let K be a compact Hausdorff space. Then

$$\operatorname{Ext} B_{C(K)^*} = \{\lambda \delta_k : k \in K,\ \lambda \text{ a scalar with } |\lambda| = 1\} .$$

Recall that δ_k is the point mass at k, *i.e.,* the evaluation at k.

Theorem 6. (Banach–Stone) Let K and L be compact Hausdorff spaces. Then $C(K) \cong C(L)$ if and only if K and L are homeomorphic.

5 Banach algebras

A real or complex *algebra* is a real or, respectively, complex vector space A with a multiplication $A \times A \to A$, $(a, b) \mapsto ab$ satisfying

(i) $(ab)c = a(bc)$ for all $a, b, c \in A$;

(ii) $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$ for all $a, b, c \in A$;

(iii) $(\lambda a)b = a(\lambda b) = \lambda(ab)$ for all $a, b \in A$ and all scalars λ.

The algebra A is *unital* if there is an element $\mathbf{1} \in A$ such that $\mathbf{1} \neq 0$ and $a\mathbf{1} = \mathbf{1}a = a$ for all $a \in A$. This element is unique and is called the *unit of* A.

An *algebra norm on* A is a norm $\|\cdot\|$ on A such that $\|ab\| \leqslant \|a\|\|b\|$ for all $a, b \in A$. Note that multiplication is continuous with respect to the norm (as are addition and scalar multiplication). A *normed algebra* is an algebra together with an algebra norm on it. A *Banach algebra* is a complete normed algebra.

A *unital normed algebra* is a normed algebra A with an element $\mathbf{1} \in A$ such that $\|\mathbf{1}\| = 1$ and $a\mathbf{1} = \mathbf{1}a = a$ for all $a \in A$. Note that if A is a normed algebra containing an element $\mathbf{1} \neq 0$ such that $a\mathbf{1} = \mathbf{1}a = a$ for all $a \in A$, then

$$|||a||| = \sup\{\|ab\| : \|b\| \leqslant 1\}$$

defines an equivalent norm on A that makes A a unital normed algebra. A *unital Banach algebra* is a complete unital normed algebra.

A linear map $\theta\colon A \to B$ between algebras A and B is a *homomorphism* if $\theta(ab) = \theta(a)\theta(b)$ for all $a, b \in A$. If A and B are unital algebras with units $\mathbf{1}_A$ and $\mathbf{1}_B$, respectively, and in addition $\theta(\mathbf{1}_A) = \mathbf{1}_B$ then θ is called a *unital homomorphism*. In the category of normed algebras an *isomorphism* is a continuous homomorphism with a continuous inverse, however, homomorphisms will not be assumed continuous.

Note. From now on the scalar field is always the field of complex numbers.

Examples. 1. If K is a compact Hausdorff space, then $C(K)$ is a commutative unital Banach algebra with pointwise multiplication and the uniform norm.
2. Let K be a compact Hausdorff space. A *uniform algebra on* K is a closed subalgebra of $C(K)$ that contains the constant functions and separates the points of K. For example, the *disc algebra*

$$A(\Delta) = \big\{f \in C(\Delta) : f \text{ is holomorphic on the interior of } \Delta\big\}$$

is a uniform algebra on the unit disc $\Delta = \{z \in \mathbb{C} : |z| \leqslant 1\}$. More generally, for a non-empty compact subset K of $\mathbb{C}$, the following are uniform algebras on K:

$$\mathcal{P}(K) \subset \mathcal{R}(K) \subset \mathcal{O}(K) \subset A(K) \subset C(K)$$

where $\mathcal{P}(K)$, $\mathcal{R}(K)$ and $\mathcal{O}(K)$ are the closures in $C(K)$ of, respectively, polynomials, rational functions without poles in K, and functions holomorphic on some open neighbourhood of K, whereas $A(K)$ is the algebra of continuous functions on K that are holomorphic on the interior of K. We will later see that $\mathcal{R}(K) = \mathcal{O}(K)$ always holds, whereas $\mathcal{R}(K) = \mathcal{P}(K)$ if and only if $\mathbb{C} \setminus K$ is connected (Runge's theorem). In general, $\mathcal{R}(K) \neq A(K)$, and $A(K) = C(K)$ if and only if K has empty interior.
3. $L_1(\mathbb{R})$ with the L_1-norm and convolution as multiplication defined by

$$f * g(x) = \int_{-\infty}^{\infty} f(t)g(x-t)\,\mathrm{d}t$$

is a commutative Banach algebra without a unit (Riemann–Lebesgue lemma).
4. If X is a Banach space, then the algebra $\mathcal{B}(X)$ of bounded linear maps on X is a unital Banach algebra with composition as multiplication and the operator norm. It is non-commutative if $\dim X > 1$. An important special case is when X is a Hilbert space, in which case $\mathcal{B}(X)$ is a C^*-algebra (see Chapter 7).

Elementary constructions. 1. If A is a unital algebra with unit $\mathbf{1}$, then a *unital subalgebra* of A is a subalgebra B of A with $\mathbf{1} \in B$. If A is a normed algebra, then the closure of a subalgebra of A is a subalgebra of A.
2. The *unitization* of an algebra A is the vector space direct sum $A_+ = A \oplus \mathbb{C}$ with multiplication $(a, \lambda)(b, \mu) = (ab+\lambda b+\mu a, \lambda\mu)$. Then A_+ is a unital algebra with unit $\mathbf{1} = (0, 1)$ and A is isomorphic to the ideal $\{(a, 0) : a \in A\}$. Identifying this ideal with A we can write $A_+ = \{a + \lambda\mathbf{1} : a \in A,\ \lambda \in \mathbb{C}\}$.

When A is a normed algebra, its unitization A_+ becomes a unital normed algebra with norm $\|a + \lambda\mathbf{1}\| = \|a\| + |\lambda|$ for $a \in A$, $\lambda \in \mathbb{C}$. Note that A is then a closed ideal of A_+. When A is a Banach algebra, so is A_+.
3. The closure of an ideal of a normed algebra A is an ideal of A. If J is a closed ideal of A, then the quotient algebra A/J is a normed algebra with the quotient norm. If A is a unital normed algebra and J is a closed proper ideal of A, then A/J is a unital normed algebra in the quotient norm.
4. The (Banach space) completion $\widetilde{A}$ of a normed algebra A is a Banach algebra with multiplication defined as follows. Given $a, b \in \widetilde{A}$ choose sequences (a_n) and (b_n) in A converging to a and b, respectively, and set $ab = \lim_{n\to\infty} a_n b_n$. One of course needs to check that this is well-defined and has the required properties.
5. Let A be a unital Banach algebra. For $a \in A$, the map $L_a \colon A \to A$, $b \mapsto ab$, is a bounded linear operator. The map $a \mapsto L_a \colon A \to \mathcal{B}(A)$ is an isometric unital homomorphism. It follows that every Banach algebra is isometrically isomorphic to a closed subalgebra of $\mathcal{B}(X)$ for some Banach space X.

Lemma 1. Let A be a unital Banach algebra, and let $a \in A$. If $\|\mathbf{1} - a\| < 1$, then a is invertible, and moreover $\|a^{-1}\| \leqslant \frac{1}{1-\|\mathbf{1}-a\|}$.

Notation. We write $G(A)$ for the group of invertible elements of a unital algebra A.

Corollary 2. Let A be a unital Banach algebra.

(i) $G(A)$ is an open subset of A.

(ii) $x \mapsto x^{-1}$ is a continuous map on $G(A)$.

(iii) If (x_n) is a sequence in $G(A)$ and $x_n \to x \notin G(A)$, then $\|x_n^{-1}\| \to \infty$.

(iv) If $x \in \partial G(A)$, then there is a sequence (z_n) in A such that $\|z_n\| = 1$ for all $n \in \mathbb{N}$ and $z_n x \to 0$ and $x z_n \to 0$ as $n \to \infty$. It follows that x has no left or right inverse in A, or in any unital Banach algebra that contains A as a (not necessarily unital) subalgebra.

Definition. Let A be an algebra and $x \in A$. The *spectrum* $\sigma_A(x)$ *of* x *in* A is defined as follows: if A is unital, then

$$\sigma_A(x) = \{\lambda \in \mathbb{C} : \lambda\mathbf{1} - x \notin G(A)\} ,$$

and if A is non-unital then $\sigma_A(x) = \sigma_{A_+}(x)$.

Examples. 1. If $A = M_n(\mathbb{C})$ then $\sigma_A(x)$ is the set of eigenvalues of x.
2. If $A = C(K)$, K compact Hausdorff, then $\sigma_A(f) = f(K)$, the set of values taken by the function f.

3. If X is a Banach space and $A = \mathcal{B}(X)$, then for $T \in A$ the spectrum of T in A has the usual meaning: $\lambda \in \sigma_A(T)$ if and only if $\lambda I - T$ is not an isomorphism.

Theorem 3. Let A be a Banach algebra and $x \in A$. Then $\sigma_A(x)$ is a non-empty compact subset of $\{\lambda \in \mathbb{C} : |\lambda| \leqslant \|x\|\}$.

Corollary 4. (Gelfand–Mazur) A complex unital normed division algebra is isometrically isomorphic to $\mathbb{C}$.

Definition. Let A be a Banach algebra and $x \in A$. The *spectral radius* $r_A(x)$ *of* x *in* A is defined as

$$r_A(x) = \sup\{|\lambda| : \lambda \in \sigma_A(x)\} \ .$$

This is well-defined by Theorem 3 and satisfies $r_A(x) \leqslant \|x\|$.

Note. If x, y are commuting elements of a unital algebra A, then xy is invertible if and only if x and y are both invertible.

Lemma 5. (The Spectral Mapping Theorem for polynomials.) Let A be a unital Banach algebra and let $x \in A$. Then for a complex polynomial $p = \sum_{k=0}^n a_k z^k$, we have

$$\sigma_A(p(x)) = \{p(\lambda) : \lambda \in \sigma_A(x)\} \ ,$$

where $p(x) = \sum_{k=0}^n a_k x^k$ and $x^0 = \mathbf{1}$.

Theorem 6. (The Beurling–Gelfand Spectral Radius Formula.) Let A be a Banach algebra and $x \in A$. Then $r_A(x) = \lim\|x^n\|^{1/n} = \inf\|x^n\|^{1/n}$.

Theorem 7. Let A be a unital Banach algebra, let B be a closed unital subalgebra of A, and let $x \in B$. Then

$$\sigma_B(x) \supset \sigma_A(x) \qquad \text{and} \qquad \partial\sigma_B(x) \subset \sigma_A(x) \ .$$

It follows that $\sigma_B(x)$ is the union of $\sigma_A(x)$ together with some of the bounded components of $\mathbb{C} \setminus \sigma_A(x)$.

Proposition 8. Let A be a unital Banach algebra and C be a maximal commutative subalgebra of A. Then C is a closed unital subalgebra of A and $\sigma_C(x) = \sigma_A(x)$ for all $x \in C$.

Definition. A non-zero homomorphism $\varphi\colon A \to \mathbb{C}$ on an algebra A is called a *character* on A. We denote by Φ_A the set of all characters on A. Note that if A is unital, then $\varphi(\mathbf{1}) = 1$ for all $\varphi \in \Phi_A$.

Lemma 9. Let A be a Banach algebra and $\varphi \in \Phi_A$. Then φ is continuous and $\|\varphi\| \leqslant 1$. Moreover, if A is unital then $\|\varphi\| = 1$.

Lemma 10. Let A be a unital Banach algebra and J be a proper ideal of A. Then the ideal $\overline{J}$ is also proper. It follows that every maximal ideal of A is closed.

Notation. For an algebra A we denote by $\mathcal{M}_A$ the set of all its maximal ideals.

Theorem 11. Let A be a commutative unital Banach algebra. Then the map $\varphi \mapsto \ker\varphi$ is a bijection $\Phi_A \to \mathcal{M}_A$.

Corollary 12. Let A be a commutative unital Banach algebra and $x \in A$.

(i) $x \in G(A)$ if and only if $\varphi(x) \neq 0$ for all $\varphi \in \Phi_A$.

(ii) $\sigma_A(x) = \{\varphi(x) : \varphi \in \Phi_A\}$.

(iii) $r_A(x) = \sup\{|\varphi(x)| : \varphi \in \Phi_A\}$.

Corollary 13. Let A be a Banach algebra and let x and y be commuting elements of A. Then $r_A(x+y) \leqslant r_A(x) + r_A(y)$ and $r_A(xy) \leqslant r_A(x) r_A(y)$.

Examples. 1. Let K be a compact Hausdorff space and $A = C(K)$. Then $\Phi_A = \{\delta_k : k \in K\}$. (Recall that $\delta_k(f) = f(k)$ for $k \in K$ and $f \in C(K)$.)
2. Let $K \subset \mathbb{C}$ be non-empty and compact. Then $\Phi_{\mathcal{R}(K)} = \{\delta_w : w \in K\}$.
3. Let $A = A(\Delta)$ be the disc algebra. Then $\Phi_A = \{\delta_w : w \in \Delta\}$.
4. The *Wiener algebra* $W = \{f \in C(\mathbb{T}) : \sum_{n\in\mathbb{Z}} |\hat{f}_n| < \infty\}$ is a commutative unital Banach algebra under pointwise operations and norm $\|f\|_1 = \sum_{n\in\mathbb{Z}} |\hat{f}_n|$. It is isomorphic to the Banach algebra $\ell_1(\mathbb{Z})$ with pointwise vector space operations, the ℓ_1-norm, and convolution as multiplication: $(a * b)_n = \sum_{j+k=n} a_j b_k$. The characters on W are again given by point evaluations: $\Phi_W = \{\delta_z : z \in \mathbb{T}\}$. Hence by Corollary 12(i) we obtain Wiener's theorem: if $f \in C(\mathbb{T})$ has absolutely summing Fourier series and $f(z) \neq 0$ for all $z \in \mathbb{T}$, then $1/f$ also has absolutely summing Fourier series.

Definition. Let A be a commutative unital Banach algebra. Then Φ_A is a w^*-closed subset of B_{A^*}, and hence it is w^*-compact and Hausdorff. The w^*-topology on Φ_A is called the *Gelfand topology* and Φ_A with its Gelfand topology is the *spectrum of* A or the *character space* or the *maximal ideal space* of A.

For $x \in A$ define the *Gelfand transform* $\hat{x} \colon \Phi_A \to \mathbb{C}$ *of* x by $\hat{x}(\varphi) = \varphi(x)$. Thus $\hat{x}$ is the restriction to Φ_A of the image of x in the second dual A^{**} under the canonical embedding. Thus $\hat{x} \in C(\Phi_A)$. The map $x \mapsto \hat{x} \colon A \to C(\Phi_A)$ is called the *Gelfand map.*

Theorem 14. (The Gelfand Representation Theorem) Let A be a commutative unital Banach algebra. Then the Gelfand map $A \to C(\Phi_A)$ is a continuous unital homomorphism. For $x \in A$ we have

(i) $\|\hat{x}\|_\infty = r_A(x) \leqslant \|x\|$

(ii) $\sigma_{C(\Phi_A)}(\hat{x}) = \sigma_A(x)$

(iii) $x \in G(A)$ if and only if $\hat{x} \in G\big(C(\Phi_A)\big)$.

Note. In general, the Gelfand map need not be injective or surjective. Its kernel is

$$\{x \in A : \sigma_A(x) = \{0\}\} = \{x \in A : \lim \|x^n\|^{1/n} = 0\} = \bigcap_{\varphi \in \Phi_A} \ker \varphi = \bigcap_{M \in \mathcal{M}_A} M \ .$$

An element $x \in A$ with $\lim \|x^n\|^{1/n} = 0$ is called *quasi-nilpotent.* The intersection $\bigcap_{M\in\mathcal{M}_A} M$ is called the *Jacobson radical of* A and is denoted by $J(A)$. We say A is *semisimple* if $J(A) = \{0\}$, *i.e.,* precisely when the Gelfand map is injective.

6 Holomorphic functional calculus

Recall that for a non-empty open subset U of $\mathbb{C}$ we denote by $\mathcal{O}(U)$ the locally convex space of complex-valued holomorphic functions on U with the topology of local uniform convergence. The topology is induced by the family of seminorms $f \mapsto \|f\|_K = \sup_{z\in K}|f(z)|$ defined for each non-empty compact subset K of U. Note that $\mathcal{O}(U)$ is also an algebra under pointwise multiplication, which is continuous with respect to the topology of $\mathcal{O}(U)$. This is an example of a *Fréchet algebra*: a complete metrizable locally convex space which is also an algebra with a continuous multiplication.

Notation. We define elements e and u of $\mathcal{O}(U)$ by $e(z) = 1$ and $u(z) = z$ for all $z \in U$. Note that $\mathcal{O}(U)$ is a unital algebra with unit e.

Theorem 1. (Holomorphic Functional Calculus) Let A be a commutative, unital Banach algebra, let $x \in A$ and U be an open subset of $\mathbb{C}$ with $\sigma_A(x) \subset U$. Then there exists a unique continuous unital homomorphism $\Theta_x \colon \mathcal{O}(U) \to A$ such that $\Theta_x(u) = x$. Moreover, $\varphi\big(\Theta_x(f)\big) = f(\varphi(x))$ for all $\varphi \in \Phi_A$ and $f \in \mathcal{O}(U)$. It follows that $\sigma_A\big(\Theta_x(f)\big) = \{f(\lambda) : \lambda \in \sigma_A(x)\}$ for all $f \in \mathcal{O}(U)$.

Remark. Think of Θ_x as "evaluation at x" and write $f(x)$ for $\Theta_x(f)$. Then the conclusions above can be briefly expressed as $\varphi(f(x)) = f(\varphi(x))$ and $\sigma_A(f(x)) = f(\sigma_A(x))$. The requirement that $e(x) = \mathbf{1}$ and $u(x) = x$ implies that for a complex polynomial $p = \sum_{k=0}^n a_k z^k = \sum_{k=0}^n a_k u^k$ we have $p(x) = \sum_{k=0}^n a_k x^k$ as defined in Lemma 5.5. Thus, Holomorphic Functional Calculus can be thought of as a far-reaching generalization of the Polynomial Spectral Mapping Theorem.

Theorem 2. (Runge's Approximation Theorem) Let K be a non-empty compact subset of $\mathbb{C}$. Then $\mathcal{O}(K) = \mathcal{R}(K)$, *i.e.,* if f is a holomorphic function on some open set containing K, then for all $\varepsilon > 0$ there is a rational function r without poles in K such that $\|f - r\|_K < \varepsilon$. More precisely, given any set Λ containing exactly one point from each bounded component of $\mathbb{C} \setminus K$, if f is a holomorphic function on some open set containing K, then for all $\varepsilon > 0$ there is a rational function r whose poles lie in Λ such that $\|f - r\|_K < \varepsilon$.

Note. If $\mathbb{C} \setminus K$ is connected, then $\Lambda = \emptyset$ which yields the polynomial approximation theorem $\mathcal{O}(K) = \mathcal{P}(K)$.

Vector-valued integration. Let $[a, b]$ be a closed bounded interval in $\mathbb{R}$, let X be a Banach space and $f \colon [a, b] \to X$ a continuous function. We define the integral $\int_a^b f(t)\,\mathrm{d}t$ as follows. Let $\mathcal{D}_n : a = t_0^{(n)} < t_1^{(n)} < \cdots < t_{k_n}^{(n)} = b$, $n \in \mathbb{N}$, be a sequence of dissections of $[a, b]$ with

$$|\mathcal{D}_n| = \max_{1\leqslant j\leqslant k_n} |t_j^{(n)} - t_{j-1}^{(n)}| \to 0 \qquad \text{as} \qquad n \to \infty\ .$$

It follows from uniform continuity of f that the limit

$$\lim_{n\to\infty} \sum_{j=1}^{k_n} f(t_j^{(n)})(t_j^{(n)} - t_{j-1}^{(n)})$$

exists in X and is independent of the sequence of dissections chosen. The integral

$\int_a^b f(t)\,\mathrm{d}t$ is defined to be this limit. It follows easily from the definition that

$$\varphi\left(\int_a^b f(t)\,\mathrm{d}t\right) = \int_a^b \varphi(f(t))\,\mathrm{d}t \qquad \text{for all } \varphi \in X^* .$$

Applying the above with a norming functional φ at the element $\int_a^b f(t)\,\mathrm{d}t$ of X, we obtain

$$\left\| \int_a^b f(t)\,\mathrm{d}t \right\| \leqslant \int_a^b \|f(t)\|\,\mathrm{d}t .$$

Now let $\gamma\colon [a,b] \to \mathbb{C}$ be a path, and $f\colon [\gamma] \to X$ be a continuous function. We then define the *integral of f along γ* by

$$\int_\gamma f(z)\,\mathrm{d}z = \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t .$$

If Γ is a chain $(\gamma_1, \dots, \gamma_n)$, then the integral along Γ of a continuous function $f\colon [\Gamma] \to \mathbb{C}$ is defined to be the sum of the integrals along the paths γ_j, *i.e.*, $\int_\Gamma f(z)\,\mathrm{d}z = \sum_{j=1}^n \int_{\gamma_j} f(z)\,\mathrm{d}z$.

Using the results about Banach space valued integrals over an interval, we immediately obtain the following results about Banach space valued integrals along a chain Γ:

$$\varphi\left(\int_\Gamma f(z)\,\mathrm{d}z\right) = \int_\Gamma \varphi(f(z))\,\mathrm{d}z \qquad \text{for all } \varphi \in X^*$$

and

$$\left\| \int_\Gamma f(z)\,\mathrm{d}z \right\| \leqslant \ell(\Gamma) \cdot \sup_{z\in[\Gamma]} \|f(z)\|$$

where $\ell(\Gamma)$ denotes the length of Γ. These properties together with the Hahn–Banach theorem allow us to deduce Banach space valued versions of classical scalar theorems.

Vector-valued Cauchy's theorem. Let U be a non-empty open subset of $\mathbb{C}$, and let Γ be a cycle in U such that $n(\Gamma, w) = 0$ for all $w \notin U$ (Γ does not wind round any point outside U). If X is a Banach space and $f\colon U \to X$ is holomorphic, then

$$\int_\Gamma f(z)\,\mathrm{d}z = 0 .$$

This follows by applying a norming functional to the integral and by applying the scalar Cauchy's theorem.

Lemma 3. Let A, x, U be as in Theorem 1. Set $K = \sigma_A(x)$ and fix a cycle Γ in $U \setminus K$ such that

$$n(\Gamma, w) = \begin{cases} 1 & \text{if } w \in K \\ 0 & \text{if } w \notin U \end{cases}$$

Define $\Theta_x\colon \mathcal{O}(U) \to A$ by letting

$$\Theta_x(f) = \frac{1}{2\pi\mathrm{i}} \int_\Gamma f(z)(z\mathbf{1} - x)^{-1}\,\mathrm{d}z$$

for $f \in \mathcal{O}(U)$. Then

(i) Θ_x is well-defined, linear and continuous.

(ii) For a rational function r without poles in U we have $\Theta_x(r) = r(x)$ in the usual sense.

(iii) $\varphi\big(\Theta_x(f)\big) = f(\varphi(x))$ for all $\varphi \in \Phi_A$ and $f \in \mathcal{O}(U)$, and hence

$$\sigma_A\big(\Theta_x(f)\big) = \{f(\lambda) : \lambda \in \sigma_A(x)\} \ .$$

Note. Lemma 3 can be viewed as a Banach algebra valued version of Cauchy's Integral Formula. It comes close to proving Theorem 1. What is missing is that Θ_x is multiplicative (which in fact follows from Lemma 3 for semisimple algebras) and that Θ_x is unique. Our strategy is to deduce Runge's theorem (Theorem 2) from Lemma 3 and then use it to complete the proof of Theorem 1 by first showing the following corollary of Runge's theorem.

Corollary 4. Let U be a non-empty open subset of $\mathbb{C}$, and let $\mathcal{R}(U)$ be the set of rational functions without poles in U. Then $\mathcal{R}(U)$ is dense in $\mathcal{O}(U)$ in the topology of local uniform convergence.

7 C^*-algebras

A $*$-*algebra* is a complex algebra A with an *involution*: a map $x \mapsto x^* \colon A \to A$ such that

$$\text{(i)}\ \ (\lambda x + \mu y)^* = \bar{\lambda} x^* + \bar{\mu} y^* \qquad \text{(ii)}\ \ (xy)^* = y^* x^* \qquad \text{(iii)}\ \ x^{**} = x$$

for all $x, y \in A$ and $\lambda, \mu \in \mathbb{C}$. Note that if A is unital with unit $\mathbf{1}$, then $\mathbf{1}^* = \mathbf{1}$.

A C^*-*algebra* is a Banach algebra A with an involution that satisfies the C^*-*equation*:

$$\|x^* x\| = \|x\|^2 \qquad \text{for all } x \in A \ .$$

A complete algebra norm on a $*$-algebra that satisfies the C^*-equation is called a C^*-*norm*. Thus, a C^*-algebra is a $*$-algebra with a C^*-norm.

Remarks. 1. If A is a C^*-algebra, then $\|x^*\| = \|x\|$ for all $x \in A$. It follows that the involution is continuous. A *Banach* $*$-*algebra* is a Banach algebra A with an involution such that $\|x^*\| = \|x\|$ for all $x \in A$. Thus, every C^*-algebra is a Banach $*$-algebra.
2. A C^*-algebra which has a multiplicative identity $\mathbf{1} \neq 0$, is automatically a unital C^*-algebra, *i.e.,* $\|\mathbf{1}\| = 1$.

Definitions. A subalgebra B of a $*$-algebra A is a $*$-*subalgebra of* A if $x^* \in B$ for every $x \in B$. A closed $*$-subalgebra (called a C^*-*subalgebra*) of a C^*-algebra is a C^*-algebra. The closure of a $*$-subalgebra of a C^*-algebra A is a $*$-subalgebra of A, and hence a C^*-subalgebra of A. A homomorphism $\theta \colon A \to B$ between $*$-algebras is a $*$-*homomorphism* if $\theta(x^*) = \theta(x)^*$ for all $x \in A$. A $*$-*isomorphism* is a bijective $*$-homomorphism.

Examples. 1. Let K be a compact Hausdorff space. Then $C(K)$ is a commutative unital C^*-algebra with the uniform norm and with involution $f \mapsto f^*$, where $f^*(k) = \overline{f(k)}$ for $k \in K$, $f \in C(K)$.

2. Let H be a Hilbert space. Then the algebra $\mathcal{B}(H)$ of bounded linear operators on H is a C^*-algebra in the operator norm and involution $T \mapsto T^*$, where T^* is the adjoint of T defined by the equation $\langle Tx, y\rangle = \langle x, T^*y\rangle$, $x, y \in H$.
3. Every C^*-subalgebra of $\mathcal{B}(H)$ is a C^*-algebra.

Remark. The Gelfand–Naimark theorem states that for every C^*-algebra A there is a Hilbert space H and an isometric $*$-isomorphism between A and a C^*-subalgebra of $\mathcal{B}(H)$. We will prove only a commutative version of this (Theorem 4 below).

Definition. Let A be a C^*-algebra and $x \in A$. We say x is

(i) *hermitian* or *self-adjoint* if $x^* = x$;

(ii) *unitary* if (A is unital and) $x^*x = xx^* = \mathbf{1}$.

(iii) *normal* if $x^*x = xx^*$.

Examples. 1. The unit $\mathbf{1}$ is both hermitian and unitary. In general, hermitian and unitary elements are normal.
2. In $C(K)$ a function f is hermitian if and only if $f(K) \subset \mathbb{R}$ (*i.e.,* f has real spectrum), and f is unitary if and only if $f(K) \subset \mathbb{T}$. This will be generalized in Corollary 3 below.

Remarks. 1. For all $x \in A$ there exits unique hermitian elements h, k such that $x = h + \mathrm{i}k$. It follows that $x^* = h - \mathrm{i}k$, and that x is normal if and only if $hk = kh$.
2. If A is unital then $x \in G(A)$ if and only if $x^* \in G(A)$, in which case $(x^*)^{-1} = (x^{-1})^*$. It follows that $\sigma_A(x^*) = \{\bar{\lambda} : \lambda \in \sigma_A(x)\}$, and thus $r_A(x^*) = r_A(x)$.

Lemma 1. Let A be a C^*-algebra and $x \in A$ be normal. Then $r_A(x) = \|x\|$.

Lemma 2. Let A be a unital C^*-algebra. Then $\varphi(x^*) = \overline{\varphi(x)}$ for all $x \in A$ and for all $\varphi \in \Phi_A$.

Remark. The condition that A is unital is superfluous. However, unitization for a C^*-algebra is not quite straightforward (see Examples Sheet 4) and the above will suffice for us.

Corollary 3. Let A be a unital C^*-algebra.

(i) If $x \in A$ is hermitian then $\sigma_A(x) \subset \mathbb{R}$.

(ii) If $x \in A$ is unitary then $\sigma_A(x) \subset \mathbb{T}$.

If B is a unital C^*-subalgebra of A and $x \in B$ is normal, then $\sigma_A(x) = \sigma_B(x)$.

Remarks. Let H be a Hilbert space and $T \in \mathcal{B}(H)$. If T is hermitian or unitary, then $\sigma(T) = \partial\sigma(T)$. Recall that $\partial\sigma(T) \subset \sigma_{\mathrm{ap}}(T)$, the approximate point spectrum of T (which is true for bounded linear operators on a Banach space). It follows that $\sigma(T) = \sigma_{\mathrm{ap}}(T)$, *i.e.,* every element of the spectrum is an approximate eigenvalue. This holds more generally for any normal operator T.

Theorem 4. (Commutative Gelfand–Naimark) Let A be a commutative unital C^*-algebra. Then A is isometrically $*$-isomorphic to $C(K)$ for some compact Hausdorff space K. More precisely, the Gelfand map $x \mapsto \hat{x} \colon A \to C(\Phi_A)$ is an isometric $*$-isomorphism.

Applications. 1. Let A be a unital C^*-algebra. We say that an element $x \in A$ is *positive* if x is hermitian and $\sigma_A(x) \subset [0, \infty)$. Every positive element x of A has a unique positive square root: there exists a unique positive element $y \in A$ such that $y^2 = x$. The unique positive square root of x is denoted $x^{1/2}$.

This applies in particular to positive elements of $\mathcal{B}(H)$, H a Hilbert space. Recall that $T \in \mathcal{B}(H)$ is positive if and only if $\langle Tx, x\rangle \geqslant 0$ for all $x \in H$.

2. (Polar decomposition) Let T be an invertible operator on a Hilbert space H. Then there exist unique operators R and U such that R is positive, U is unitary and $T = RU$.

8 Borel functional calculus and spectral theory

Throughout H is a (non-zero) complex Hilbert space, $\mathcal{B}(H)$ is the C*-algebra of all bounded, linear operators on H, K is a compact Hausdorff space and $\mathcal{B}$ is the Borel σ-field on K.

Operator-valued measures. A *resolution of the identity of H over K* is a map $P \colon \mathcal{B} \to \mathcal{B}(H)$ such that

(i) $P(\emptyset) = 0$, $P(K) = I$;

(ii) $P(E)$ is an orthogonal projection for every $E \in \mathcal{B}$;

(iii) $P(E \cap F) = P(E)P(F)$ for all $E, F \in \mathcal{B}$;

(iv) if $E \cap F = \emptyset$, then $P(E \cup F) = P(E) + P(F)$;

(v) for every $x, y \in H$, the function $P_{x,y} \colon \mathcal{B} \to \mathbb{C}$ defined by

$$P_{x,y}(E) = \big\langle P(E)x, y\big\rangle$$

is a regular complex Borel measure on K.

Example. $H = L_2[0,1]$, $K = [0,1]$ and $P(E)f = \mathbf{1}_E \cdot f$.

Simple properties.

(i) Any two projections $P(E)$ and $P(F)$ commute.

(ii) if $E \cap F = \emptyset$, then $P(E)H \perp P(F)H$.

(iii) if $x \in H$, then $P_{x,x}$ is a positive measure of total mass $P_{x,x}(K) = \|x\|^2$.

(iv) P is finitely additive. Moreover, for each $x \in H$, the map $E \mapsto P(E)x$ is a countably additive H-valued function on $\mathcal{B}$.

(v) Although P need not be countably additive, we do have $P\big(\bigcup_n E_n\big) = 0$ whenever $P(E_n) = 0$ for all $n \in \mathbb{N}$.

Motivation. Having defined a notion of measure, our next step is to define a notion of integral. To motivate this, consider a compact hermitian operator T. We know (see the notes *Resumé on Hilbert spaces and Spectral Theory*) that $\sigma(T)$ is countable and every non-zero $\lambda \in \sigma(T)$ is an eigenvalue of T. For $\lambda \in \sigma(T)$ let P_λ be the orthogonal projection onto the eigenspace $E_\lambda = \ker(\lambda I - T)$ (which may be zero if $\lambda = 0$ and $\sigma(T)$ is infinite). Then the series $\sum_{\lambda\in\sigma(T)} \lambda P_\lambda$ converges in norm to T.

For $E \subset \sigma(T)$ define $P(E)$ to be the orthogonal projection onto the orthogonal direct sum $\bigoplus_{\lambda\in E} E_\lambda$ (so that $P(E)(x) = \sum_{\lambda\in E} P_\lambda(x)$ for all $x \in H$). It is straightforward to check that P is a resolution of the identity of H over $\sigma(T)$. Now the only sensible notion of integral on the countable set $\sigma(T)$ is given by summation. So in particular, $\int_{\sigma(T)} \lambda \, \mathrm{d}P = \sum_{\lambda\in\sigma(T)} \lambda P_\lambda = T$. The spectral theorem for normal operators (Theorem 4 below) is a far-reaching generalization of this.

***P*-essentially bounded functions.** Let P be a resolution of the identity of H over K. A Borel function $f\colon K \to \mathbb{C}$ is *P-essentially bounded* if there exists $E \in \mathcal{B}$ with $P(E) = 0$ such that f is bounded on $K \setminus E$. We then set

$$\|f\|_\infty = \inf\big\{\|f\|_{K\setminus E} : E \in \mathcal{B},\ P(E) = 0\big\}\ ,$$

where $\|f\|_{K\setminus E} = \sup_{z\in K\setminus E}|f(z)|$. As in the case of scalar measures, the infimum is attained: there is a Borel set E with $P(E) = 0$ such that $\|f\|_\infty = \|f\|_{K\setminus E}$. The set $L_\infty(P)$ of all P-essentially bounded Borel functions on K is a commutative, unital C*-algebra with pointwise operations and norm $\|\cdot\|_\infty$. (Technically, $\|\cdot\|_\infty$ is not a norm as $\|f\|_\infty = 0$ need not imply that $f = 0$. As usual in measure theory, we identify functions f and g if there is a Borel set E with $P(E) = 0$ such that f and g agree on $K \setminus E$, *i.e.,* when $f = g$ P-almost everywhere.)

Lemma 1. (Definition of $\int_K f \, \mathrm{d}P$.) Let P be a resolution of the identity of H over K. Then there is an isometric, unital $*$-isomorphism Φ of $L_\infty(P)$ onto a commutative, unital C*-subalgebra of $\mathcal{B}(H)$ such that

(i) $\big\langle\Phi(f)x, y\big\rangle = \int_K f \, \mathrm{d}P_{x,y}$ for every $f \in L_\infty(P),\ x, y \in H$;

(ii) $\|\Phi(f)x\|^2 = \int_K |f|^2 \, \mathrm{d}P_{x,x}$ for every $f \in L_\infty(P),\ x \in H$;

(iii) $S \in \mathcal{B}(H)$ commutes with every $\Phi(f)$ if and only if it commutes with every $P(E)$.

Note. The identity $\big\langle\Phi(f)x, y\big\rangle = \int_K f \, \mathrm{d}P_{x,y}$ uniquely defines $\Phi(f)$. We shall denote $\Phi(f)$ by $\int_K f \, \mathrm{d}P$.

Bounded Borel functions. We let $L_\infty(K)$ denote the set of all bounded Borel functions $f\colon K \to \mathbb{C}$. This is a commutative, unital C*-algebra equipped with the 'sup norm' $\|f\|_K$. Note that if P is a resolution of the identity of H over K, then $L_\infty(K) \subset L_\infty(P)$ and the inclusion is a norm-decreasing unital $*$-homomorphism.

Theorem 2. (Spectral theorem for commutative C*-algebras.) Let A be a commutative, unital C*-subalgebra of $\mathcal{B}(H)$ and let $K = \Phi_A$. Then there is a unique resolution of the identity P of H over K such that

$$T = \int_K \widehat{T}\,\mathrm{d}P \qquad \text{for every } T \in A\ ,$$

where $\widehat{T}$ is the Gelfand transform of T. Moreover,

(i) $P(U) \neq 0$ for every non-empty open subset U of K; and for $S \in \mathcal{B}(H)$,

(ii) S commutes with every $T \in A$ if and only if it commutes with every $P(E)$ $(E \in \mathcal{B})$.

Exponentials in Banach algebras. Let A be a unital Banach algebra. Then for $x \in A$ we define $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. The sum converges absolutely, and hence it converges in A. It is easy to check that if $x, y \in A$ commute then $e^{x+y} = e^x e^y$ (see Examples Sheet 4, question 8).

Lemma 3. (Fuglede-Putnam-Rosenblum) Let A be a unital C*-algebra and $x, y, z \in A$. Assume that x and y are normal and $xz = zy$. Then $x^*z = zy^*$. In particular, if z commutes with x, then it commutes with x^*.

Theorem 4. (Spectral theorem for normal operators.) Let $T \in \mathcal{B}(H)$ be a normal operator. Then there is a unique resolution P of the identity of H over $\sigma(T)$ such that

$$T = \int_{\sigma(T)} \lambda\,\mathrm{d}P\ .$$

Moreover, $S \in \mathcal{B}(H)$ commutes with every projection $P(E)$ $(E \in \mathcal{B})$ if and only if $ST = TS$.

This integral representation of T is called the *spectral decomposition* of T. The projections $P(E)$ $(E \in \mathcal{B})$ are called *spectral projections.*

Theorem 5. (Borel functional calculus for a normal operator) Let $T \in \mathcal{B}(H)$ be a normal operator and let $K = \sigma(T)$. For $f \in L_\infty(K)$ define

$$f(T) = \int_K f\,\mathrm{d}P\ ,$$

where P is the resolution of the identity over $K = \sigma(T)$ given by Theorem 4. The map $f \mapsto f(T)$ has the following properties:

(i) it is a unital $*$-homomorphism from $L_\infty(K)$ to $\mathcal{B}(H)$ with $z(T) = T$ (where $z(\lambda) = \lambda,\ \lambda \in K$);

(ii) $\|f(T)\| \leqslant \|f\|_K$ for all $f \in L_\infty(K)$ with equality for $f \in C(K)$;

(iii) if $S \in \mathcal{B}(H)$ and $ST = TS$, then $Sf(T) = f(T)S$ for all $f \in L_\infty(K)$.

(iv) $\sigma(f(T)) \subset \overline{f(K)}$ for all $f \in L_\infty(K)$.

Polar decomposition of normal operators. Let $T \in \mathcal{B}(H)$ be a normal operator. Then $T = RU$, where R is positive, U is unitary, and R, U, T pairwise commute.

Representation of unitary operators. Let $U \in \mathcal{B}(H)$ be a unitary operator. Then $U = e^{iQ}$ for some hermitian operator Q.

Connectedness of $G(\mathcal{B}(H))$. The group of all invertible operators in $\mathcal{B}(H)$ is connected. Moreover, every invertible operator is the product of two exponentials.