Measure Theory

We list here definitions and results from basic measure theory. These can be found in any good book on measure theory, *e.g.,* the ones by S. J. Taylor or D. L. Cohn.

1 Measures

1.1 A *measure space* is a triple $(\Omega, \mathcal{F}, \mu)$, where

(i) Ω is a set;

(ii) $\mathcal{F}$ is a *σ-field on* Ω, *i.e.,* $\mathcal{F} \subset \mathcal{P}(\Omega)$ such that

- (a) $\emptyset \in \mathcal{F}$,
- (b) if $A \in \mathcal{F}$ then $\Omega \setminus A \in \mathcal{F}$,
- (c) if $A_n \in \mathcal{F}$ for all $n \in \mathbb{N}$, then $\displaystyle\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$;

(iii) $\mu\colon \mathcal{F} \to [0, \infty]$ is a *measure on* $\mathcal{F}$:

- (a) $\mu(\emptyset) = 0$,
- (b) $\displaystyle\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$ for pairwise disjoint sets A_n, $n \in \mathbb{N}$, in $\mathcal{F}$.

Here we use the usual conventions regarding ∞. *E.g.,* $x + \infty = \infty + x = \infty$ for all $x \in \mathbb{R}$.

1.2 A measure μ is *finite* if $\mu(\Omega) < \infty$. In this case $\mu(A) < \infty$ for all $A \in \mathcal{F}$.

1.3 $N \in \mathcal{F}$ is called a *null set* (or *μ-null set*) if $\mu(N) = 0$.

1.4 If $\mathcal{A}$ is an arbitrary family of subsets of a set Ω, then there is a (unique) smallest σ-field on Ω containing $\mathcal{A}$, which is the intersection of all σ-fields on Ω that contain $\mathcal{A}$. It is called the σ-field *generated by* $\mathcal{A}$.

1.5 Let X be a topological space. We denote by $\mathcal{G}$ the family of open subsets of X. The *Borel σ-field on* X is the σ-field $\mathcal{B}$ generated by $\mathcal{G}$. Elements of $\mathcal{B}$ are called *Borel sets.* A *Borel measure on* X is a measure on $\mathcal{B}$.

2 Outer measures

2.1 Given a set Ω, an *outer measure on* Ω is a function $\mu^*\colon \mathcal{P}(\Omega) \to [0, \infty]$ such that

(i) $\mu^*(\emptyset) = 0$

(ii) $\mu^*(A) \leqslant \mu^*(B)$ whenever $A \subset B$

(iii) $\displaystyle\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leqslant \sum_{n=1}^{\infty} \mu^*(A_n)$ for arbitrary subsets A_n of Ω.

2.2 $A \subset \Omega$ is called *μ^*-measurable* if

$$\mu^*(B) = \mu^*(B \cap A) + \mu^*(B \setminus A)$$

holds for all $B \subset \Omega$.

2.3 Theorem The family $\mathcal{M}$ of μ^*-measurable subsets of Ω is a σ-field on Ω, and the restriction μ of μ^* to $\mathcal{M}$ is a measure on $\mathcal{M}$.

3 Measurable functions

3.1 Let Ω be a set and $\mathcal{F}$ be a σ-field on Ω. A function $f\colon \Omega \to \mathbb{R}$ (or $\mathbb{C}$) is *measurable* if $f^{-1}(B) \in \mathcal{F}$ for every Borel set $B \subset \mathbb{R}$ (respectively, $\mathbb{C}$).

3.2 Examples

(i) If Ω is a topological space, $\mathcal{F}$ is the Borel σ-field on Ω, and f is a continuous scalar-valued function on Ω, then f is measurable.

(ii) In general, any *simple function*, *i.e.,* a function of the form $\sum_{k=1}^n a_k \mathbf{1}_{A_k}$ where $A_k \in \mathcal{F}$ and a_k is a scalar for all $1 \leqslant k \leqslant n$, is measurable.

3.3 The set of all measurable functions on Ω is an algebra under pointwise operations. If $f\colon \Omega \to \mathbb{C}$ is measurable, then so are $|f|$, the real part $\mathcal{R}(f)$ of f, and the imaginary part $\mathcal{I}(f)$ of f. If $f, g\colon \Omega \to \mathbb{R}$ are measurable, then so are their maximum $f \vee g$, and their minimum $f \wedge g$. Finally, if (f_n) is a sequence of measurable functions that converges pointwise to a function f, then f is measurable.

4 Integration

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. We define $\int_\Omega f \,\mathrm{d}\mu$ for *certain* scalar-valued, measurable functions on Ω.

4.1 If $f \geqslant 0$ is a simple function, *i.e.,* $f = \sum_{k=1}^n a_k \mathbf{1}_{A_k}$ where $A_k \in \mathcal{F}$ and $a_k \geqslant 0$ for all $1 \leqslant k \leqslant n$, then we define

$$\int_\Omega f \,\mathrm{d}\mu = \sum_{k=1}^n a_k \mu(A_k)$$

which is a number in $[0, \infty]$. We use the convention $0 \cdot \infty = \infty \cdot 0 = 0$.

4.2 If $f \geqslant 0$ is measurable, then we let

$$\int_\Omega f \,\mathrm{d}\mu = \sup \left\{ \int_\Omega g \,\mathrm{d}\mu : \, 0 \leqslant g \leqslant f, \; g \text{ a simple function} \right\}$$

which is again a number in $[0, \infty]$.

4.3 $f\colon \Omega \to \mathbb{R}$ is called *integrable* if it is measurable and $\int_\Omega |f| \,\mathrm{d}\mu$ is finite. We then set

$$\int_\Omega f \,\mathrm{d}\mu = \int_\Omega f^+ \,\mathrm{d}\mu - \int_\Omega f^- \,\mathrm{d}\mu$$

where $f^+ = f \vee 0$ and $f^- = (-f) \vee 0$.

4.4 $f\colon \Omega \to \mathbb{C}$ is called *integrable* if it is measurable and $\int_\Omega |f| \,\mathrm{d}\mu$ is finite. We then set

$$\int_\Omega f \,\mathrm{d}\mu = \int_\Omega \mathcal{R}(f) \,\mathrm{d}\mu + \mathrm{i} \cdot \int_\Omega \mathcal{I}(f) \,\mathrm{d}\mu$$

where $\mathcal{R}(f)$ and $\mathcal{I}(f)$ are the real and imaginary parts of f, respectively.

4.5 Properties

(i) Linearity:

(a) If $f \geqslant 0,\ g \geqslant 0$ are measurable, and $\alpha \geqslant 0,\ \beta \geqslant 0$ are real numbers, then

$$\int_\Omega (\alpha f + \beta g) \,\mathrm{d}\mu = \alpha \cdot \int_\Omega f \,\mathrm{d}\mu + \beta \cdot \int_\Omega g \,\mathrm{d}\mu \ .$$

(b) If f, g are integrable functions and α, β are scalars, then $\alpha f + \beta g$ is integrable and

$$\int_\Omega (\alpha f + \beta g) \,\mathrm{d}\mu = \alpha \cdot \int_\Omega f \,\mathrm{d}\mu + \beta \cdot \int_\Omega g \,\mathrm{d}\mu \ .$$

(ii) Monotone convergence: if $0 \leqslant f_n \nearrow f$ pointwise a.e. (almost everywhere), then $\int_\Omega f_n \,\mathrm{d}\mu \nearrow \int_\Omega f \,\mathrm{d}\mu$.

(iii) Fatou's lemma: if (f_n) is a sequence of measurable functions such that $f_n \geqslant g$ for all $n \in \mathbb{N}$ for some integrable function g, then

$$\int_\Omega \liminf f_n \,\mathrm{d}\mu \leqslant \liminf \int_\Omega f_n \,\mathrm{d}\mu \ .$$

(iv) Dominated convergence: if f_n $(n \in \mathbb{N})$, f and g are measurable functions such that $|f_n| \leqslant g$ for all $n \in \mathbb{N}$, $f_n \to f$ pointwise a.e., and g is integrable, then f is integrable and $\int_\Omega f_n \,\mathrm{d}\mu \to \int_\Omega f \,\mathrm{d}\mu$.

4.6 A property of points of Ω is said to hold *almost everywhere* (or *a.e.* for short) if it holds for all $\omega \in \Omega \setminus N$ for some null set $N \in \mathcal{F}$. We sometimes use the term μ-*almost everywhere* (or μ-*a.e.* for short) to emphasize the measure μ.

5 L_p spaces

Throughout this section, $(\Omega, \mathcal{F}, \mu)$ is a measure space.

5.1 Let $1 \leqslant p < \infty$. We define $L_p(\Omega, \mathcal{F}, \mu)$ or simply $L_p(\mu)$, to be the real (or complex) vector space of all measurable functions $f\colon \Omega \to \mathbb{R}$ (respectively, $\mathbb{C}$) such that $\int_\Omega |f|^p \,\mathrm{d}\mu < \infty$.

5.2 Let $1 \leqslant p < \infty$. For $f \in L_p(\mu)$ we define its L_p-*norm* by

$$\|f\|_p = \left(\int_\Omega |f|^p \,\mathrm{d}\mu \right)^{\frac{1}{p}} \ .$$

5.3 A measurable function $f\colon \Omega \to \mathbb{R}$ (or $\mathbb{C}$) is *essentially bounded* if there is a μ-null set $N \in \mathcal{F}$ such that f is bounded on $\Omega \setminus N$.

5.4 We define $L_\infty(\Omega, \mathcal{F}, \mu)$, or simply $L_\infty(\mu)$, to be the real (or complex) vector space of all measurable, essentially bounded functions $f \colon \Omega \to \mathbb{R}$ (respectively, $\mathbb{C}$).

5.5 For $f \in L_\infty(\mu)$ we define its *essential sup norm* or L_∞*-norm* by

$$\|f\|_\infty = \operatorname{ess\,sup}|f| = \inf\big\{ \sup_{\Omega\setminus N}|f| : N \in \mathcal{F},\ \mu(N) = 0\big\} .$$

Note that the essential supremum is attained: there is a μ-null set $N \in \mathcal{F}$ such that $\operatorname{ess\,sup}|f| = \sup_{\Omega\setminus N}|f|$.

5.6 Theorem (Hölder) Let $1 \leqslant p, q \leqslant \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then for all $f \in L_p(\mu)$ and $g \in L_q(\mu)$ we have $fg \in L_1(\mu)$ and

$$\|fg\|_1 \leqslant \|f\|_p \cdot \|g\|_q .$$

5.7 Theorem (Minkowski) Let $1 \leqslant p \leqslant \infty$ and let $f, g \in L_p(\mu)$. Then $f + g \in L_p(\mu)$ and

$$\|f + g\|_p \leqslant \|f\|_p + \|g\|_p .$$

5.8 It follows from the above that for $1 \leqslant p \leqslant \infty$ the space $L_p(\mu)$ is a normed space in the L_p-norm provided we identify functions f and g if $f = g$ a.e. (almost everywhere), *i.e.,* when $\{\omega \in \Omega : f(\omega) \neq g(\omega)\}$ is a μ-null set (has μ-measure zero).

5.8.1 Remark Strictly speaking $\|\cdot\|_p$ is a seminorm on $L_p(\mu)$ for $1 \leqslant p \leqslant \infty$. In general, if $\|\cdot\|$ is a seminorm on a real or complex vector space X, then $N = \{z \in X : \|z\| = 0\}$ is a subspace of X, and $\|x + N\| = \|x\|$ defines a norm on the the quotient space X/N. However, we will not do this for $L_p(\mu)$. We prefer to think of elements of $L_p(\mu)$ as functions rather than equivalence classes of functions. One must remember that equality in $L_p(\mu)$ means a.e. equality.

5.9 Theorem For $1 \leqslant p \leqslant \infty$, the space $L_p(\mu)$ is complete in the L_p-norm.