Winding numbers, Cauchy's Theorem and Cauchy's Integral Formula

We recall a number of definitions and results from complex analysis. Different versions of these exist, so the purpose here is to spell out the ones used in lectures.

1. Paths

A **path** is a continuously differentiable function $\gamma\colon [a,b] \to \mathbb{C}$ defined on some closed bounded interval of $\mathbb{R}$.

The **track** of γ is $[\gamma] = \{\gamma(t) : t \in [a,b]\}$ (*i.e.*, simply the image of γ).

Given $S \subset \mathbb{C}$, we say γ is a **path in *S*** if $[\gamma] \subset S$.

The **length** of γ is $\ell(\gamma) = \displaystyle\int_a^b |\gamma'(t)|\,\mathrm{d}t$.

Integral along $\boldsymbol{\gamma}$: $\displaystyle\int_\gamma f(z)\,\mathrm{d}z = \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t$ for $f\colon [\gamma] \to \mathbb{C}$ continuous.

2. Chains and cycles

A **chain** is a finite sequence $\Gamma = (\gamma_1, \dots, \gamma_n)$ of paths $\gamma_j\colon [a_j, b_j] \to \mathbb{C}$.

The **track** of Γ is $[\Gamma] = \displaystyle\bigcup_{j=1}^{n} [\gamma_j]$.

Given $S \subset \mathbb{C}$, we say Γ is a **chain in *S*** if $[\Gamma] \subset S$.

The **length** of Γ is $\ell(\Gamma) = \displaystyle\sum_{j=1}^{n} \ell(\gamma_j)$.

Integral along $\boldsymbol{\Gamma}$: $\displaystyle\int_\Gamma f(z)\,\mathrm{d}z = \sum_{j=1}^{n} \int_{\gamma_j} f(z)\,\mathrm{d}z$ for $f\colon [\Gamma] \to \mathbb{C}$ continuous.

Γ is a **cycle** if there is a permutation $\rho \in S_n$ such that $\gamma_j(b_j) = \gamma_{\rho(j)}(a_{\rho(j)})$ for $j = 1, \dots, n$.

3. Winding numbers

Let Γ be a cycle and $w \in \mathbb{C} \setminus [\Gamma]$. The **winding number of $\boldsymbol{\Gamma}$ round $\boldsymbol{w}$** is

$$n(\Gamma, w) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{\mathrm{d}z}{z - w}$$

which is always an integer.

4. Lemma

Assume we have $\emptyset \neq K \subset U \subset \mathbb{C}$, where K is compact and U is open. Then there is a cycle Γ in $U \setminus K$ such that

$$n(\Gamma, w) = \begin{cases} 1 & \text{if } w \in K \\ 0 & \text{if } w \notin U \end{cases}$$

5. Cauchy's Theorem

Let U be a non-empty open subset of $\mathbb{C}$, and let Γ be a cycle in U such that $n(\Gamma, w) = 0$ for all $w \notin U$ (Γ does not wind round any point outside U). Then

$$\int_{\Gamma} f(z)\,\mathrm{d}z = 0$$

for any holomorphic function $f\colon U \to \mathbb{C}$.

6. Cauchy's Integral Formula

Let U be a non-empty open subset of $\mathbb{C}$, and let Γ be a cycle in U such that $n(\Gamma, w) = 0$ for all $w \notin U$ (Γ does not wind round any point outside U). Let $f\colon U \to \mathbb{C}$ be holomorphic. Then

$$n(\Gamma, a) f(a) = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma} \frac{f(z)}{z - a}\,\mathrm{d}z$$

for any $a \in U \setminus [\Gamma]$.