STRONG STATEMENTS OF ANALYSIS

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Abstract Examples are discussed of natural statements about irrational numbers that are equivalent, provably in *ZFC*, to strong set-theoretical hypotheses, and of apparently classical statements provable in *ZFC* of which the only known proofs use strong set-theoretical concepts.

Introduction Opponents of a full-blooded set-theoretic account of the foundations of mathematics usually take their stand at one of three ditches:

Ditch X: they may hold, with Mac Lane, that the weak set theory *MAC* suffices for "all important mathematics";

Ditch Y: they may hold, with the early Bourbachistes, that *ZC* suffices for "all important mathematics";

Ditch Z: they may accept the axioms of *ZFC*, but deny the relevance of large cardinals to ordinary mathematics.

The main challenge of this paper is directed to those at Ditch Z, but the author's earlier sallies against Ditches X and Y will be reviewed, and in the penultimate section a new point against them will be made.

MAC is the system, discussed in Mac Lane [12], that suffices for "local" mathematics; among its axioms are the Axiom of Choice, the Power Set Axiom — which states that for each set x in the set-theoretic universe V, the power set $\mathcal{P}(x) =_{\mathrm{df}} \{y \mid y \subseteq x\}$ is also a member of V — the Axiom of Infinity and certain instances of the separation scheme

$$a \cap \{x \mid \mathfrak{A}(x)\} \in V, \tag{0·0}$$

which says that if a is a set, the class of those elements x of a which have the property $\mathfrak{A}$ is a set.

Some words of explanation here: the word "scheme" warns that $\mathfrak{A}$ might be any one formula of a family to be specified. In Zermelo's system, indeed, $\mathfrak{A}$ might be any formula of the language of set theory; whereas in *MAC* the formula $a \cap \{x \mid \mathfrak{A}(x)\} \in V$ is to be an axiom only when $\mathfrak{A}$ is a Δ_0 formula, where

0·1 Definition a formula is Δ_0 if its only quantifiers are *restricted* ones of the form $\forall x{:}\in u$ or $\exists y{:}\in v$, meaning "for all x in the set u" or "there is a y in the set v".

Here the letter x must be a distinct letter from the letter u, and the letter y from the letter v: expressions such as $\forall x{:}\in x$ are forbidden in Δ_0 formulæ.

Very many set-theoretical concepts are expressible in Δ_0 form. For example, "a is a subset of b" is $\forall x{:}\in a\ x \in b$; "$c$ is the union of the sets in d", in symbols, $c = \bigcup d$, is expressible as

$$\forall x{:}\in d\ \forall y{:}\in x\ y \in c\ \&\ \forall y{:}\in c\ \exists x{:}\in d\ y \in x;$$

"e has exactly one element" is $\exists x{:}\in e\ \forall y{:}\in e\ x = y$; if, as set-theorists do, we define the ordered pair (p, q) of two objects to be $\{\{p\}, \{p, q\}\}$, "f is a ordered pair" proves to be Δ_0; then "g is a relation" says that $\forall x{:}\in g\ x$ is an ordered pair, and is thus also Δ_0; "h is a function and $h(a) = b$" is Δ_0, and so on.

An important example of a concept **not** expressible in Δ_0 form is that of the power set of a set; to say $a = \mathcal{P}(b)$, we must say not only that every element of a is a subset of b, which *is* Δ_0, but also that every subset of b is a member of a, that is $\forall x(x \subseteq b \implies x \in a)$, and the quantifier $\forall x$ here is not restricted.

Thus quantifying over the members of a set is a restricted quantification, but quantifying over the subsets of a set is not. So an instance of the Δ_0 separation scheme would be "given a set a, the class of members of a which are linear orderings is a set", but the corresponding principle for well-orderings, "given a set a, the class of members of a which are well-orderings is a set" would be an axiom of Zermelo set theory, but not an axiom of Mac Lane set theory.

0·2 Remark Though not an *axiom*, the proposition "given a set a, the class of members of a which are well-orderings is a set" is actually a *theorem* of Mac Lane's system, by an astute application of the power set axiom to build a set containing every subset of every element of a. We digress to give an example of an instance of the full separation scheme which is not provable in Mac Lane set theory.

Write ω for the class of natural numbers 0, 1, 2, ...; ω is a set by the Axiom of Infinity. Define a *type sequence* to be a sequence s such that

$$Dom(s) \in \omega \,\&\, s(0) = \omega \,\&\, \forall k{:}\in\omega \,\big(k+1 \in Dom(s) \Longrightarrow s(k+1) = \mathcal{P}(s(k))\big).$$

We write $TS(n)$ for the formula asserting that there is a type sequence of length $n+1$.

Then the formula $\omega \cap \{n \mid TS(n)\} \in V$ is an instance of the separation scheme and therefore is an axiom of Zermelo set theory, (which indeed proves that the set in question equals ω), but, assuming *MAC* to be consistent, that formula is not provable in *MAC* since it implies, in the system *MAC*, the consistency of *MAC*. For details see section 9 of [17].

0·3 Remark *MAC* is equiconsistent with two other well-established systems, the simple theory of types and Lawvere's system *ETCS* of the elementary theory of the category of sets; one may say that these three systems have the same conceptual strength.

Zermelo's system *ZC* is essentially *MAC* afforced by the separation scheme for *every* formula $\mathfrak{A}$; the Zermelo–Fraenkel system *ZFC* adds to *ZC* the scheme of collection,

$$\forall x \exists y\, \mathfrak{A} \implies \forall u \exists v \forall x{:}\in u\, \exists y{:}\in v\, \mathfrak{A}, \tag{0·4}$$

for every formula $\mathfrak{A}$ in which the variables u and v have no occurrence. If the axiom of choice be omitted from those systems, the results are called *M, Z* and *ZF*. The details of these axiomatic systems are given in [17], which contains further references.

About Ditch X: criticism of Mac Lane's views may be found in [15], to which Saunders Mac Lane, with whom it is a pleasure to debate, has published a courteous reply [13] in the same volume [11]. Though the conjunction of those essays makes it clear why Mac Lane and I disagree, they are at cross-purposes in that to some extent Mac Lane is discussing style whereas I am discussing content. For a detailed discussion of Mac Lane's set theory, see the long paper [17], where it is shown that *MAC* is weak when it comes to recursive definitions, and the formula $TS(n)$ defined above is shown to yield a failure in *MAC* of the principle of proof by induction, for both $TS(0)$ and $TS(n) \implies TS(n+1)$ are provable in *MAC* whilst $\forall n{:}\in\omega\, TS(n)$ is not.

About Ditch Y: criticism of the foundational stance of the Bourbaki group is given in [14]. My paper [16] shows that while *Z* can justify all instances of proof by induction (unlike *MAC* for which I have exhibited a failure) it retains the inability of *MAC* to carry out even rather simple recursive constructions. Later in the present paper I shall draw attention to a certain theorem of *MAC* and challenge the defenders of Ditches X and Y to prove it without adopting the *ZFC* mindset.

For a detailed appraisal and comparison of the aims of Mac Lane and his co-workers, of Bourbaki and of others, see the recent treatise of Corry [1].

Now for Ditch Z: to rebut the view that large cardinals are irrelevant to ordinary mathematics, I exhibit four "natural" assertions, **A, B, C, D**, about certain sets of real numbers, each of which assertions is provably equivalent (in *ZFC*) to an assertion that certain ordinals possess large-cardinal properties in certain inner models. It will follow trivially that the statement $\mathbf{C} \Longrightarrow \mathbf{B}$ is provable in *ZFC*: my first challenge to the defenders of Ditch Z will be to find a proof of that that does not involve flirting with large cardinals.

1: Four statements about sets of real numbers

Actually we shall work not with the real line but with *Baire space*, $\mathcal{N}$, which is known to be homeomorphic to the subspace of the irrational points on the real line. The points of Baire space are the functions $\alpha : \omega \longrightarrow \omega$, where $\omega = \{0, 1, 2, \ldots\}$. We write ${}^{<\omega}\omega$ for the set of finite sequences of natural numbers. Each $s \in {}^{<\omega}\omega$ has a *length*, denoted by $\ell h(s)$, which is a natural number, and indeed, remembering the set-theoretical custom of identifying each natural number $n \in \omega$ with the set $\{m \mid m < n\}$, we have $s : \ell h(s) \longrightarrow \omega$. When α is a function and a is a subset of its domain, $\alpha \restriction a$ denotes the restriction of α to a, namely the function β with domain a such that $\forall x{:}\in a\ \beta(x) = \alpha(x)$. Using that notation we may define for $s \in {}^{<\omega}\omega$

$$N_s \ =_{\mathrm{df}}\ \{\alpha \in \mathcal{N} \mid \alpha \restriction \ell h(s) = s\},$$

the set of those functions α of which s is an initial segment. The collection $\{N_s \mid s \in {}^{<\omega}\omega\}$ forms a basis for a topology, with which we endow $\mathcal{N}$.

Both $\omega \times \mathcal{N}$ and $\mathcal{N} \times \mathcal{N}$ are homeomorphic to $\mathcal{N}$: in particular Euclidean dimension has disappeared. For further detail see Moschovakis [18] or Kechris [8], or for a classical treatment see Kuratowski [9].

Review of the projective hierarchy

We wish to study those subsets of $\mathcal{N}^k$ for $k \geqslant 1$ which are generated from the open sets by the two methods of taking complements and of forming projections. We write $\mathcal{X}$ for an arbitrary product space $\mathcal{N}^k$. The classical terms for certain of these sets are recalled in Remarks 1·1 and 1·3; we use the more modern notation $\mathbf{\Sigma^0_n}$, $\mathbf{\Pi^0_n}$, $\mathbf{\Delta^0_k}$, $\mathbf{\Sigma^1_n}$, $\mathbf{\Pi^1_n}$, $\mathbf{\Delta^1_k}$ for the classes we get, as follows.

1·0 Definition The class $\mathbf{\Sigma^0_1}$ comprises all open subsets of each $\mathcal{N}^k$. A subset A of some $\mathcal{X}$ is in $\mathbf{\Pi^0_n}$ if $\mathcal{X} \smallsetminus A$ is in $\mathbf{\Sigma^0_n}$. A subset B of some $\mathcal{X}$ is in $\mathbf{\Sigma^0_{n+1}}$ if there is a set $C \subseteq \omega \times \mathcal{X}$ which is in $\mathbf{\Pi^0_n}$ such that

$$B = \{x \in \mathcal{X} \mid \exists k (k, x) \in C\}.$$

A set is in $\mathbf{\Delta^0_k}$ if it is in both $\mathbf{\Sigma^0_k}$ and $\mathbf{\Pi^0_k}$.

1·1 Remark The above classes form the finite levels of the Borel hierarchy, for example a set is in $\mathbf{\Pi^0_1}$ iff it is closed, $\mathbf{\Sigma^0_2}$ is the class of F_σ sets, and $\mathbf{\Pi^0_2}$ the class of G_δ sets.

Now for the *projective hierarchy*.

1·2 Definition We define $\mathbf{\Sigma^1_0}$ to be $\mathbf{\Sigma^0_1}$. A subset A of some $\mathcal{X}$ is in $\mathbf{\Pi^1_n}$ if $\mathcal{X} \smallsetminus A$ is in $\mathbf{\Sigma^1_n}$. A subset B of some $\mathcal{X}$ is in $\mathbf{\Sigma^1_{n+1}}$ if there is a set $C \subseteq \mathcal{N} \times \mathcal{X}$ which is in $\mathbf{\Pi^1_n}$ such that

$$B = \{x \in \mathcal{X} \mid \exists \beta (\beta, x) \in C\}.$$

A set is in $\mathbf{\Delta^1_k}$ if it is in both $\mathbf{\Sigma^1_k}$ and $\mathbf{\Pi^1_k}$.

1·3 Remark The sets in $\mathbf{\Sigma^1_1}$ were classically called the *analytic* sets, or the $\mathcal{A}$-sets, the sets in $\mathbf{\Pi^1_1}$ the *co-analytic* or $\mathcal{CA}$ sets, and those in $\mathbf{\Sigma^1_2}$ the $\mathcal{PCA}$ sets. This nomenclature, attractive at low levels, becomes cumbersome at higher.

1·4 Definition The Borel subsets of $\mathcal{X}$ are those in the smallest family containing the open sets and closed under complements and countable unions.

1·5 Remark Using a mild form of the Axiom of Choice it can be seen that all Borel sets are $\mathbf{\Delta^1_1}$. That all sets in $\mathbf{\Delta^1_1}$ are Borel is true, and was announced by Souslin in 1917. The proof uses a characterisation of $\mathbf{\Pi^1_1}$ sets in terms of countable well-orderings.

1·6 Remark Similar hierarchies may be defined in arbitrary Polish spaces, such as $C[0, 1]$ or the Euclidean plane: for many natural examples of projective sets in such spaces see Becker's survey paper [0].

We are going to give four statements about sets of points in Baire space. All the sets considered will be in $\mathbf{\Pi^1_1}$ or at worst $\mathbf{\Sigma^1_2}$, and thus are definable by formulæ using only variables ranging over Baire space.

Perfect sets

For the first we turn to a concept that goes back to Cantor:

1·7 Definition A subset of $\mathcal{X}$ is called *perfect* if it is closed and has no isolated points.

Our first statement is this:

A : *every uncountable co-analytic set has a non-empty perfect subset.*

Background: Cantor proved that every uncountable closed set has a non-empty perfect subset: indeed this result was bound up with his discovery of the ordinals. He showed that all non-empty perfect sets are of cardinality the continuum, which was evidence in favour of his continuum hypothesis. In 1917 Souslin proved that every uncountable analytic set has a non-empty perfect subset: Lusin tackled statement **A** but pronounced the problem impossible.

Games of infinite length

For our second and fourth statements, we turn to some ideas about games of infinite length between two players, whom we might whimsically call ∀dam and ∃ve. Why games are important in logic is that strings of alternating quantifiers ∀∃∀∃ may be construed as saying that the ∃ player has an answer to everything that the ∀ player does.

We imagine games of infinite length between these two players, who alternate in choosing natural numbers. Adam starts, and chooses a_0, then Eve plays e_0, then Adam a_1, and so on; at the end of time they have generated an infinite sequence $\gamma \in \mathcal{N}$, where

$$\gamma = (a_0, e_0, a_1, e_1, a_2, \ldots).$$

Let $C \subseteq \mathcal{N}$: Adam wins that run of the game $\mathcal{G}(C)$ associated with C if $\gamma \in C$, and Eve wins if $\gamma \notin C$.

1·8 Definition We call C *determined* if one of the players has a winning strategy in the associated game $\mathcal{G}(C)$.

We could also study games of a more general kind, in which the moves a_i, e_i are not restricted to being natural numbers but might be other objects, as in the Banach–Mazur game mentioned below; we shall, when emphasis or clarity requires, refer to the games described above as *number-games*.

In the more general context, the natural setting is to consider the game as taking place in a tree of finite sequences of objects, and the outcome of a play of the game will then be an infinite path through the tree. We may topologise the set of such paths by taking a basic open set to be the set of paths with a given finite sequence as initial segment, and then we may speak of Borel games to mean games $\mathcal{G}(C)$ where C is a Borel set in that topology.

We shall write *BD* for the assertion that all Borel number-games are determined. The *Axiom of Determinacy*, *AD*, is the assertion that every number-game is determined.

Our second statement is this:

B : *every analytic number-game is determined.*

Background: In a game proposed by Mazur and analysed by Banach, a subset A of a non-empty real interval I is given and the two players choose non-empty intervals I_n in turn with $I \supseteq I_0 \supseteq I_1 \ldots$. The second player *wins* if $A \cap \bigcap_n I_n = \varnothing$. The second player proves to have a winning strategy if and only if A is meagre, and the first if and only if A is co-meagre in some non-empty interval, which the first player would be well advised to play as his first move. For details, see Oxtoby [19], page 27. The discussion is presented in a more general setting by Kechris [8], page 51.

Gale and Stewart proved in the 50's that every open game is determined. That was later extended to low finite levels of the Borel hierarchy. In 1968 D. A. Martin proved that statement **B** follows from the assumption of the existence of a measurable cardinal, to be defined below. In 1968 Harvey Friedman proved that statement **B** is false assuming *ZFC* + Gödel's axiom $V = L$ — which axiom will be explained in the discussion following Remark 2·5 below — and that *BD* is not a theorem of *ZC*. In 1974 D. A. Martin proved that *BD* is a theorem of *ZFC*.

Universally Baire sets

Let us call a topological space *regularly based* if it has a basis consisting of regular open sets, where an open set is termed *regular* if it is the interior of its closure.

1·9 DEFINITION A subset D of $\mathcal{N}$ will be called *universally Baire* if whenever $\mathcal{Z}$ is regularly based and $f : \mathcal{Z} \longrightarrow \mathcal{N}$ is continuous, the pre-image $f^{-1}[D]$ differs from an open subset of $\mathcal{Z}$ by some set of first category.

Our third statement is

$$\boxed{\mathbf{C}: \textit{ every } \mathcal{PCA} \textit{ set is universally Baire.}}$$

Background: it is provable in *ZFC* that every $\mathbf{\Sigma^1_1}$ or $\mathbf{\Pi^1_1}$ set is universally Baire. The paper [2] of Feng, Magidor and Woodin develops this notion, which turns out to have important connections with an abstract version of Martin's proof of $\mathbf{\Pi^1_1}$ determinacy from a measurable cardinal.

Our fourth statement will be this strengthening of our second:

$$\boxed{\mathbf{D}: \textit{ every } \mathcal{PCA} \textit{ number-game is determined.}}$$

Note that statements **A**, **B** and **D** are statements solely about members of Baire space. **C** is not, since it involves topological spaces of all cardinalities; nevertheless it is a statement that would be intelligible to classical workers in point set topology, such as Kuratowski.

2: Set-theoretic notions

Each of our four statements makes an assertion about subsets of $\mathcal{N}$ at a certain level of the projective hierarchy, where we know that the corresponding statement for earlier levels of the hierarchy is provable in *ZFC*. To analyze the difficulty of these statements as given we turn to some concepts from set theory: we work in *ZFC*, but shall draw attention to places where we apparently exceed the grammatical bounds of that system.

2·0 DEFINITION A set or class x is *transitive* if $y \in z \in x \implies y \in x$.

2·1 DEFINITION We write ε for the membership relation $\{(x, y) \mid x \in y\}$.

2·2 DEFINITION An *ordinal* is a transitive set u well–ordered by the (membership) relation $\varepsilon \cap u \times u$.

In *ZF* one can prove that every well-ordering is isomorphic to exactly one ordinal.

2·3 DEFINITION We write ω for the set of finite ordinals, ω_1 for the set of finite or countable ordinals and ON for the class of all ordinals.

2·4 DEFINITION An *inner model* is a class M which is transitive, which contains all ordinals, and such that all the axioms of *ZFC* hold when relativised to M: here the relativisation Φ^M of a formula of set theory to a class M means the result of replacing all quantifiers $\forall x \ldots$ by $\forall x(x \in M \implies \ldots)$ and $\exists x \ldots$ by $\exists x(x \in M \;\&\; \ldots)$. We shall often say "$\Phi$ holds in M of x" rather than "$\Phi(x)^M$".

2·5 REMARK I am cheating with this definition as *ZFC* comprises an infinite set of axioms, and is known not to be finitely axiomatisable. However there is a single theorem $\mathfrak{A}_0$ of *ZFC* such that for any other theorem $\mathfrak{B}$ of *ZFC*, and for any class M, it is provable in *ZFC* that if M is transitive and contains all ordinals and $\mathfrak{A}_0^M$ then $\mathfrak{B}^M$: as $\mathfrak{B}$ varies, different axioms of ZFC will be involved in the proof. We therefore take the formal definition of "M is an inner model of ZFC" to be the conjunction "M is transitive, every ordinal is in M, and $\mathfrak{A}_0^M$".

Gödel defined a minimal inner model which he called L, the *constructible universe*. His definition is absolute in the following sense. Let $\Psi(x)$ be the formula such that $x \in L \iff \Psi(x)$: actually that formula is of the form "there is an ordinal number ζ such that $\ldots$". Let M be any transitive class; we extend the notion of relativisation by writing L^M for the class of those members x of M such that $\Psi(x)^M$. Then if M is an inner model, $L^M = L$.

In particular, $L^L = L$; as for any inner model $V^M = M$, where V is the class $\{x \mid x = x\}$ of all sets, the *axiom of constructibility* $V = L$ holds in L.

Gödel showed that both the Axiom of Choice, AC, and the Continuum Hypothesis CH are theorems of the system $ZF + V = L$.

2·6 Remark In this paper I wish to avoid discussing models where AC fails. I should, though, salve my conscience by emphasizing, first, that one could drop the requirement that $(AC)^M$ from the definition of *inner model* and obtain a perfectly coherent concept; and, second, that Gödel made no use of AC in defining L or in proving that $(AC)^L$.

We may in ZFC carry out the following definition by transfinite induction on ζ:

$$\begin{aligned} V_0 &= \varnothing \\ V_{\zeta+1} &= \mathcal{P}(V_\zeta) \\ V_\lambda &= \bigcup_{\nu<\lambda} V_\nu \end{aligned}$$

$\mathcal{P}(x)$ is the power set of x, the set of all subsets of x. Each V_ζ is a transitive set, and $V_\zeta \subseteq V_{\zeta+1}$. By the axiom of foundation, it is provable in ZFC that every set in the universe V will be a member of some V_ζ, and so

$$V = \bigcup_{\nu \in ON} V_\nu$$

Suppose we modify the above definition by using instead of the function $x \mapsto \mathcal{P}(x)$, the function $x \mapsto \mathrm{Def}(x)$, where for a set x we define $\mathrm{Def}(x)$ to be the set of all subsets of x which are definable over the structure $(x, \varepsilon \cap (x \times x))$ by a formula of the language of set theory, permitting members of x to occur as parameters in the definition. Then if x is transitive, so is $\mathrm{Def}(x)$, and $x \subseteq \mathrm{Def}(x)$. The iteration of this operation gives the *constructible hierarchy*:

$$\begin{aligned} L_0 &= \varnothing \\ L_{\zeta+1} &= \mathrm{Def}(L_\zeta) \\ L_\lambda &= \bigcup_{\nu<\lambda} L_\nu \end{aligned}$$

We now *define*

$$L =_{\mathrm{df}} \bigcup_{\nu \in ON} L_\nu.$$

That is the inner model called the constructible universe. We may modify that definition in various ways, to obtain for example for any $\alpha \in \mathcal{N}$, the inner model $L[\alpha]$ which will be an inner model of which α is a member, and indeed the intersection of all such. Similarly if A is a set of ordinals we may define $L[A]$.

Measurable cardinals.

As our source of appropriate large cardinal concepts we turn to the notion of a measurable cardinal. This concept arose from a question in measure theory in response to Vitali's construction of a set of real numbers that is not Lebesgue measurable.

2·7 Definition (*provisional*) An ordinal κ is measurable if there is a function $\mu : \mathcal{P}(\kappa) \longrightarrow \{0, 1\}$ which is a countably additive non-trivial two-valued measure: that is, that these conditions hold (remembering the set-theoretic fact that each ordinal is the set of all smaller ordinals):

> $\mu(\kappa) = 1$; for each $\nu < \kappa$, $\mu(\{\nu\}) = 0$; every subset of a set of μ-measure 0 is of μ-measure 0; for $X \subseteq \kappa$, $\mu(\kappa \smallsetminus X) = 1 - \mu(X)$; the union of countably many sets of μ-measure 0 is again of μ-measure 0; μ is defined for all subsets of κ.

It transpires that the least κ satisfying this definition will be uncountable and have the further property that the union of fewer than κ sets of μ-measure 0 is of measure 0; thus the official definition will be this:

2·8 Definition κ is *measurable* if κ is an uncountable cardinal carrying a two-valued measure that gives κ measure 1 and each singleton $\{\nu\}$ measure 0, and which is $< \kappa$-additive in the sense that the union of fewer than κ sets of measure 0 is of measure 0.

Let us see first that the real line cannot be measurable in this sense:

2·9 Proposition $2^{\aleph_0}$ *is not a measurable cardinal.*

Proof : we consider a countably additive non-trivial measure defined on the closed unit interval $[0, 1]$. The whole interval is of measure 1. Consider the two half intervals $[0, 1/2]$, $[1/2, 1]$. Precisely one of those must be of measure 1, as $\{1/2\}$ is of measure 0. Choose the subinterval of measure 1, and repeat. After ω steps, we shall have built a descending sequence of intervals each of measure 1, with intersection a single point, which by countable additivity must also be of measure 1, and thus the measure is trivial after all. $\dashv$ (2·9)

That argument generalises to show that if $\lambda < \kappa$ then $2^\lambda < \kappa$. Further a measurable cardinal must be *regular* in the sense that it is not the union of fewer than κ sets each of smaller cardinality. An uncountable cardinal κ which enjoys the two properties just given is called *strongly inaccessible.*

For further information on large cardinals see, *e.g.*, [6] or [10].

3: Strongly inaccessible cardinals and Statement A

Strongly inaccessible cardinals cannot be proved to exist in ZFC, assuming that system consistent, for this reason: suppose that κ is strongly inaccessible. Then V_κ is a model of all the axioms of ZFC. Hence if κ is the least strongly inaccessible, V_κ will be a model of ZFC + "there are no strongly inaccessible cardinals".

3·0 Remark An important point is this: let M be an inner model, for example $M = L$. M contains all ordinals. It might well be the case that a particular ordinal η, say, is strongly inaccessible in M, but not in V. Indeed η might even be countable in V.

3·1 Exercise Show that if η is regular in V then it is regular in M, and if it is strongly inaccessible in V then it is strongly inaccessible in M.

The following result, of which one direction is due to Gödel and the other to Solovay, explains Lusin's difficulties with Statement **A**.

3·2 Theorem *Statement* **A** *holds if and only if for each* $\alpha \in \mathcal{N}$, *the true* ω_1 *is a strongly inaccessible cardinal in* $L[\alpha]$.

3·3 Remark This result emphasises the importance of speaking not of large cardinals but of large-cardinal properties, and of recognising that while it is rash to suppose that any ordinal possesses a large-cardinal property in the set-theoretical universe, it may well be that that certain ordinals will in certain inner models possess large-cardinal properties. No statement of analysis could imply the truth of a large-cardinal hypothesis in V, but we have just seen one that implies the truth in many inner models of the existence of strongly inaccessible cardinals. We shall shortly see three more.

4: More on measurable cardinals

Dana Scott in 1960 proved the following very beautiful result, the proof of which has led to much mathematics, notably from Gaifman and Kunen.

4·0 THEOREM (Scott) *If $V = L$ there are no measurable cardinals.*

His argument was roughly this. Let μ be a $< \kappa$-additive non-trivial measure on κ. Let $U = \{x \subseteq \kappa \mid \mu(x) = 1\}$. Then U is a (countably complete) non-principal ultrafilter on κ. Use that in an adaptation of the model-theoretic ultrapower construction to form an ultrapower of the universe V. That will lead to a map $j : V \longrightarrow M$ where M is some inner model, and j is an *elementary embedding* in the following sense: whenever $\mathfrak{A}(\mathfrak{x}_1, \ldots, \mathfrak{x}_n)$ is a formula with the free variables shown and a_1, …, a_n are sets,

$$\mathfrak{A}(a_1, \ldots, a_n) \Longleftrightarrow \big(\mathfrak{A}(j(a_1), \ldots, j(a_n))\big)^M.$$

κ will be the *critical point* $\mathrm{crit}(j)$ of j in the sense that $j(\kappa) > \kappa$ while for $\nu < \kappa$, $j(\nu) = \nu$. So if κ were the least measurable cardinal in V, $j(\kappa)$ would be the least measurable cardinal in M, and thus $M \neq V$, and so $V \neq L$, yielding Scott's theorem.

Alternatively, it may be shown (using AC but with no assumption on the minimality of κ) that $\mu \notin M$, and so $M \neq V$. Thus V has an inner model distinct from itself, and therefore cannot be L. ⊣ (4·0)

4·1 REMARK Under AC, any elementary embedding k which is not the identity must move some ordinal, and therefore will have a critical point, λ, say. λ will be measurable, for we may define a measure μ_k by setting

$$\mu_k(X) = 1 \Longleftrightarrow \lambda \in k(X).$$

Suppose that we have an elementary embedding $j : V \longrightarrow M$ for some M. Since L is uniformly definable inside every inner model, the restriction of j to L gives an elementary embedding from L into itself.

This situation has been considerably illuminated by Silver's study, in his Berkeley doctoral dissertation of 1966, of the theory of indiscernibles for the inner model L, and Kunen's development, originating in his Stanford doctoral dissertation of 1968, of Gaifman's theory of iterated ultrapowers of inner models; with the remarkable outcome that (in ZFC) the existence of an elementary embedding of L into itself is equivalent to the existence of a certain real number, called $0^\sharp$. More generally, if A is a set of ordinals, the existence of an elementary embedding of $L[A]$ into itself is equivalent to the existence of another set of ordinals, of the same size as A, called $A^\sharp$.

We cannot say more in this short essay about the exact definition of the sharp operator $^\sharp$, beyond saying that the model-theoretic theory of indiscernibles plays a rôle, and remarking that the following subset of $\mathcal{N} \times \mathcal{N}$ is Π^1_2:

$$\{(\beta, \alpha) \mid \beta = \alpha^\sharp\}$$

5: Sharps, Statements B and C, and a challenge for those at Ditch Z

The theory of sharps is one of the most attractive innovations in set theory in the past half-century; the assumptions that every real has a sharp or that every set of ordinals has a sharp are now standard hypotheses among set theorists. Curiously they are intimately linked to two of our four statements:

5·0 THEOREM (Martin; Harrington) *Statement* **B** *is equivalent to the assertion that every real has a sharp.*

5·1 THEOREM (Feng, Magidor, Woodin [2]) *Statement* **C** *is equivalent to the assertion that every set of ordinals has a sharp.*

5·2 CHALLENGE It follows that $\mathbf{C} \Longrightarrow \mathbf{B}$ is a theorem of ZFC, and it is an assertion of point-set topology, of a kind that Kuratowski could have understood. But the only known proof uses a deep set-theoretical result called Jensen's covering lemma. **I therefore challenge those who wish to deny a rôle in mathematics to set theory to find a proof of this implication using only arguments of ordinary mathematics.**

6: Woodin cardinals and Statement D

The elementary embeddings arising from measurable cardinals are of a restricted kind. Here are some concepts involving more general ones.

6·0 Definition κ is λ-*supercompact* if there is an elementary embedding $j : V \longrightarrow M$ with $\mathrm{crit}(j) = \kappa$ and ${}^{\lambda}M \subseteq M$, meaning that M is closed under formation of sequences of length λ of its elements. κ is *supercompact* if it is λ-supercompact for every λ.

In the 1960's Solovay proved that the existence of measurable cardinals implied the Lebesgue measurability of all $\mathbf{\Sigma^1_2}$ sets of reals, and conjectured that the existence of supercompact cardinals would have much stronger consequences for the real line, such as the Lebesgue measurability or even the determinacy of all projective sets. His prescience has been triumphantly vindicated.

We shall require a notion intermediate in strength between supercompact and measurable, which since its formulation in 1984 has proved to be of great importance in the study of determinacy.

6·1 Definition A *Woodin cardinal* is an uncountable ordinal θ such that

$$\forall H{:}\subseteq\theta\ \exists\delta{:}<\theta\ \forall\lambda\Big[\delta<\lambda<\theta \implies \exists j : V \longrightarrow M\big[\mathrm{crit}(j)=\delta\ \&\ j(\delta)>\lambda\ \&\ H\cap\lambda = j(H\cap\delta)\cap\lambda\big]\Big]$$

6·2 Remark The reader will have noticed that that definition contains a quantifier ranging over elementary embeddings of the universe: such embeddings are proper classes and therefore not properly the range of a quantifier in ZFC. However, just as the existence of the simplest kind of elementary embedding is equivalent to the existence of a measure, so the existence of these more delicate embeddings is equivalent to the existence of certain sets called *extenders* which, broadly, are directed systems of measures.

6·3 Theorem (Woodin) *Statement* **D** *is equivalent to the following assertion:*

$$\forall\alpha{:}\in\mathcal{N}\ \exists\zeta{:}<\omega_1\ \exists S{:}\subseteq\zeta\ \ \alpha\in L[S]\ \&\ (\zeta \textit{ is a Woodin cardinal})^{L[S]}.$$

That statement is more striking than that equivalent to Statement A, as it is countable ordinals this time which are behaving as large cardinals in certain inner models, rather than just ω_1.

So small ordinals might have large cardinal properties in certain inner models. Note that in all four theorems, an *equivalence* is being proved not just an *equiconsistency*; and that in all except Case **C**, one of the statements is a simple statement purely about real numbers. Thus the above examples show the necessity of introducing large cardinal properties into mathematics.

7: A challenge for those at Ditches X and Y

We now discuss the hypothesis *BD* that all Borel number-games are determined. We suppose known the concept of one member, α, of $\mathcal{N}$ being *recursive in* another, β, which we write as $\alpha \leqslant_T \beta$: that means that there is a Turing machine equipped with an oracle for β which will compute the values of α in succession. We write $\alpha =_T \beta$ if α and β are each recursive in the other. We call a subset C of $\mathcal{N}$ *Turing closed* if $\alpha =_T \beta \in C \implies \alpha \in C$.

We write *TBD* for the assertion that every Borel subset of $\mathcal{N}$ which is Turing closed is determined. Trivially *BD* implies *TBD*. It is far from trivial, but follows from the results stated above of H. Friedman and D.A.Martin concerning *BD*, that the implication *TBD* $\Longrightarrow$ *BD* is provable in *MAC*.

The proof goes as follows. There are simple functions f, g, of countable ordinals such that for Martin's proof of the determinacy of Borel games of rank a countable ordinal ζ it is sufficient to have *ZC* plus the existence of $V_{f(\zeta)}$, and by Friedman's analysis of *BD*, for any ξ there is a Turing-closed game of rank $g(\xi)$ which suffices, reasoning in *MAC*, to build a "correct" model of *ZC* plus the existence of V_ξ. Here "correct" means that not only are the integers of that model standard integers, but also the model preserve the truth of such assertions as "I am a winning strategy for the first player in the game with the following Borel description". So given a Borel set B of rank ζ, we use a Turing-closed game of rank $g(f(\zeta))$ to build a correct model of *ZC* plus $\exists V_{f(\zeta)}$ containing a Borel description of the set B; within that model we run Martin's argument to create a winning strategy for one of the players in the game B that works inside the model; since the integers of the model are standard, that strategy will be a strategy in the real world; and we then appeal to the correctness of the model to argue that that strategy will be a winning strategy in the real world for the same player for whom it was a winning strategy in the model.

BD is one of many mathematically interesting assertions found by H. Friedman that are purely about real numbers and are provable in *ZFC* but not in *ZC*. Here is a new one: Kanovei in [7] answered a classical question of Lusin by proving that the constituents of an analytic but non-Borel set do not have bounded Borel rank. Hjorth [5] has shown that Kanovei's proof must use the axiom of replacement to the same extent that proving Borel determinacy does.

7·0 Challenge As for the implication *TBD* $\Longrightarrow$ *BD*, it appears to be psychologically necessary to accept the *ZFC* mindset in order to find a proof of this theorem of *MAC*, a point against those who say that they are not interested in theorems that need more than the axioms of *ZC* to prove, for if they refuse to assume, even as a hypothesis to be later discharged, some portion of *ZFC* lying outside *ZC* they deny themselves the chance of finding proofs of theorems of their chosen system. Therefore **I challenge the defenders of *MAC* and Ditch X, still more the defenders of *ZC* and Ditch Y, to find a proof of the implication *TBD* $\Longrightarrow$ *BD* not going outside the conceptual bounds of their chosen ditch.**

8: Some further challenges for those at Ditch Z

Within the study of projective games, many implications are known which are theorems of ZC or MAC or even of analysis, but whose only known proofs are set-theoretical. Here are some:

(Martin): *if every analytic game is determined, so is every game in the Boolean algebra generated by the analytic sets.*

(Kechris and Woodin): *if every PCA game is determined, so is every game in the Boolean algebra generated by the PCA sets.*

(Neeman and Woodin): *for each positive n, if every $\mathbf{\Sigma^1_{2n+1}}$ game is determined so is every game in the σ-algebra generated by the $\mathbf{\Sigma^1_{2n+1}}$ sets.*

8·0 CHALLENGE My challenge here is to **find direct game-theoretic proofs**.

Hauser [4] has proved in analysis but using set-theoretical techniques this result:

if all projective sets are Lebesgue measurable and have the property of Baire, and $\mathbf{\Pi^1_3}$ uniformisation holds, then all $\mathcal{PCA}$ games are determined.

Background: $\mathbf{\Pi^1_3}$ uniformisation is the hypothesis that whenever $A \subseteq \mathcal{N} \times \mathcal{N}$ is $\mathbf{\Pi^1_3}$ there is another $\mathbf{\Pi^1_3}$ set $B \subseteq A$ such that $\forall \alpha\beta\gamma(\langle\alpha,\, \beta\rangle \in B \;\&\; \langle\alpha,\, \gamma\rangle \in B \implies \beta = \gamma)$ and $\{\alpha \mid \exists\beta\langle\alpha,\, \beta\rangle \in A\} = \{\alpha \mid \exists\beta\langle\alpha,\, \beta\rangle \in B\}$. Such uniformisation principles were studied by Lusin and his school. All the hypotheses of Hauser's theorem are known to follow from the determinacy of all projective games; the question of the reversibility of this implication was raised by Woodin [22]; a counterexample to Woodin's broadest conjecture has been found by Steel [21].

8·1 CHALLENGE: **find a classical proof of Hauser's theorem**.

Harrington [3] and Steel [20] between them have proved that statement **B** is equivalent to the following:

$\mathbf{B}'$: *any two non-Borel analytic subsets of $\mathcal{N}$ are Borel isomorphic.*

8·2 CHALLENGE Hence the implication $\mathbf{C} \Longrightarrow \mathbf{B}'$ is another statement using only concepts from classical point-set topology and is a theorem of ZFC: so my last challenge to those at ditch Z is to **find a proof using only such concepts of this theorem:**

If every $\mathcal{PCA}$ set is universally Baire then any two non-Borel analytic sets are Borel isomorphic.

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