Recurrent points and hyperarithmetic sets

A. R. D. Mathias

Centre de Recerca Matemàtica de l'Institut d'Estudis Catalans

Abstract We give an example of an iteration with recursive data which stabilises exactly at the first non-recursive ordinal. We characterise the points in the final set as those attacked by recurrent points, and use that characterisation to show that recurrent points must exist for any iteration with recursive data which does not stabilise at a recursive ordinal.

0: Introduction.

This paper is the third of a series that studies the closure ordinal $\theta(a, f)$ of an iteration defined using a continuous function f from some Polish space $\mathcal{X}$ to itself, starting from a point $a \in \mathcal{X}$. We summarise the definitions:

0·0 Definition $\omega_f(x)$ is the set of those points $y \in \mathcal{X}$ such that each neighbourhood of y contains for each n a point $f^m(x)$ for some $m > n$. It is not excluded that $f^m(x) = f^p(x)$ for some $p \leqslant n$; thus periodic points of the form $f^m(x)$ are counted as belonging to $\omega_f(x)$. We write $x \curvearrowright_f y$ if $y \in \omega_f(x)$, and omit the subscript f in discussions for which f has been fixed. We read $x \curvearrowright y$ as "*x attacks y*".

0·1 Proposition *(i) $\omega_f(x)$ is a closed subset of $\mathcal{X}$.*
(ii) If $x \curvearrowright y$ and $y \curvearrowright z$ then $x \curvearrowright z$.

0·2 Definition For $A \subseteq \mathcal{X}$, set $\Gamma(A) =_{\mathrm{df}} \bigcup\{\omega_f(x) \mid x \in A\}$. Then using this operator and given a point $a \in \mathcal{X}$, define recursively sets $A^\nu(a) = A^\nu(a, f)$:

$$\begin{aligned} A^0(a) &= \omega_f(a) \\ A^{\beta+1}(a) &= \Gamma(A^\beta(a)) \\ A^\lambda(a) &= \bigcap_{\nu<\lambda} A^\nu(a) && \text{for } \lambda \text{ a limit ordinal} \end{aligned}$$

0·3 Proposition *(i) for all ordinals α, β, $\alpha < \beta \implies A^\alpha(a) \supseteq A^\beta(a)$.*
(ii) $x \in A^\nu(a) \implies f(x) \in A^\nu(a)$.
(iii) For each ordinal μ, $A^\mu(a) = \omega_f(a) \cap \bigcap_{\nu<\mu} A^{\nu+1}(a)$.

0·4 Definition $\theta(a, f) =_{\mathrm{df}}$ the least ordinal θ with $A^\theta(a) = A^{\theta+1}(a)$; $A^\infty(a, f) =_{\mathrm{df}} A^{\theta(a,f)}$; $E(a, f) =_{\mathrm{df}} \omega_f(a) \smallsetminus A^\infty(a, f)$. Points in $A^\infty(a, f)$ are said to *abide*, and points in $E(a, f)$ are said to *escape.*

In the first paper* we linked escape to well-foundedness by establishing the equivalence, of a kind familiar to students of monotone inductive definitions, that a point $z \in \omega_f(a)$ abides if and only if there is a sequence z_i $(i < \omega)$ such that $z_0 = z$ and for each i $a \curvearrowright z_{i+1} \curvearrowright z_i$; and we used that link to show that for any $\mathcal{X}$, f and a, $\theta(a, f)$ was at most the first uncountable ordinal. In the second paper† we gave a method of placing points at the nodes of a well-founded tree, where the trees in question were all subtrees of $\mathcal{S}$, the set of finite strictly increasing sequences of odd prime numbers (excluding 1), including $\emptyset$, the empty sequence; and we showed, using the well-foundedness of the tree, that for a particular f the method would permit the construction of a point a with $\theta(a, f)$ equalling the rank of that tree, which might be any pre-assigned countable ordinal.

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* A.R.D.Mathias, *An application of descriptive set theory to a problem in dynamics*, CRM Preprint núm. 308, Octubre 1995

† A.R.D.Mathias, *Long delays in dynamics,* CRM Preprint núm. 334, Maig 1996

Two spaces were studied: the Baire space $\mathcal{N} = {}^{\omega}\omega$ of all infinite sequences of natural numbers, and the Cantor space ${}^{\omega}2$: in the first the function f considered was the (backward) *shift function* $\mathfrak{s}$ defined by $\mathfrak{s}(b)(n) = b(n+1)$ for $b : \omega \longrightarrow \omega$, and in the second it was $\mathfrak{s}^3$ defined by $\mathfrak{s}^3(b)(n) = b(n+3)$ for $b : \omega \longrightarrow \{0,1\}$.

But the method of assignment of points was independent of the tree under consideration, and in particular did not rely on the trees being well-founded; and in the present paper we apply it to ill-founded trees and obtain this result:

0·5 THEOREM *There is a recursive point a in Baire space $\mathcal{N}$ such that $\theta(a, \mathfrak{s}) = \omega_1^{CK}$, the first non-recursive ordinal.*

We now review this method: for each $s \in \mathcal{S}$ we shall define a point x_s in Baire space, by induction on the length of s, so that

$$s \prec t \implies x_s \curvearrowright x_t.$$

We start by setting

$$x_\emptyset = 0, 4, 8, \ldots$$

so that $x_\emptyset$ attacks nothing.

Suppose now that for some $s \in \mathcal{S}$, we have defined x_s, and that t is an immediate extension of s, so that $t = s^\frown\langle k\rangle$, where k is an odd prime exceeding all those occurring in s. Write π_t for the product of all the primes occurring in t, so that π_t is a square-free number of which k is a factor. For each natural number n we write $\pi_{t,n}$ for the number $(\pi_t)^{n+1}$. Note that for non-empty r_1 and r_2 in T, π_{r_1} divides π_{r_2} iff $r_2 \preccurlyeq r_1$. We then define

$$x_t =_{\mathrm{df}} \langle\pi_{t,0}\rangle^\frown(x_s \restriction n_{t,0})^\frown\langle\pi_{t,1}\rangle^\frown(x_s \restriction n_{t,1})^\frown\langle\pi_{t,2}\rangle^\frown(x_s \restriction n_{t,2})^\frown \ldots$$

where the positive integers $n_{t,i}$ are chosen strictly increasing and such that the predecessors of each occurrence of a power of π_t in x_t (after the first) form a strictly increasing sequence of even numbers: to enable that choice to be made we maintain inductively the property that the x_s have infinitely many even numbers in their range.

There is considerable freedom in the choice of the integers $n_{t,i}$, and a specific recursive choice is given in *Long Delays*. To prove our main theorem any recursive choice that guarantees the truth of the various lemmata in §1 of *Long Delays* will do: as before, when we are discussing the shift function $\mathfrak{s}$ acting on Baire space $\mathcal{N}$ we write $b \rhd c$ to mean that for some non-negative integer n, $b = \mathfrak{s}^n(c)$. We shall, though, see that a more careful choice of $n_{t,i}$ will give an example with rather sharper details.

Now let $T \subseteq \mathcal{S}$ be a tree which contains the empty sequence $\emptyset$ and which is closed under shortening. By a process recursive in T we define a point x_T of Baire space with the property that

$$s \in T \implies x_T \curvearrowright x_s.$$

The definition differs slightly from that in *Long Delays*: there we listed all bottom points of a well-founded tree; here, where we consider trees that might have no bottom points, we must list all points of the tree. In fact, in the interest of uniformity, we list all members of $\mathcal{S}$ recursively as $\langle s_i \mid i \in \omega\rangle$ so that each occurs infinitely often, and then proceed as in *Long Delays* to define

$$x_T = (x_{t_0} \restriction n_{T,0})^\frown\langle 2\rangle^\frown(x_{t_1} \restriction n_{T,1})^\frown\langle 6\rangle^\frown(x_{t_2} \restriction n_{T,2})^\frown\langle 10\rangle \ldots$$

where the integers $n_{T,i}$ are chosen strictly increasing, and such that the immediate predecessors of the occurrences of numbers $n \equiv 2 \pmod 4$ are distinct positive multiples of 4, and t_i is the first s_j in sequence after previous t_k's to be a member of T: so, in effect, we always check to see whether $s_j \in T$, and if it is not, we do nothing at that stage but proceed to the next.

Thus if T is recursive so is the point x_T and the sequence $\langle x_s \mid s \in T\rangle$.

We consider what happens when we have an infinite path through T, and our principal technical lemma, Proposition 3·0, which is proved for the case of a general Polish space and an arbitrary continuous function, states, roughly, that at the end of each such path a point may always be found which attacks every point along it.

In §1 we review some familiar material on recurrent and minimal points. In §§2 and 3 we use arguments from analysis to prove our main lemma and some corollaries, and in §4 we establish the existence of points at the end of paths. In §5, we review some material from the theory of hyperarithmetic sets, in §6 we establish similar results for our dynamical context, and in §7 we complete the proof of the principal theorem.

1: **Minimal points.**

1·0 Definition A *recurrent* point is a b such that $b \curvearrowright b$.

It has long been known that the existence of recurrent points is neither certain nor impossible:

1·1 Example Let $\mathcal{X} = \mathbb{R}$, and $f(x) \equiv x + 1$. Then f has no recurrent points.

1·2 Theorem (*AC*) *Let $\mathcal{X}$ be a compact Polish space and $f : \mathcal{X} \longrightarrow \mathcal{X}$ continuous. Then recurrent points exist.*

1·3 Remark The above use of *AC* could be reduced to an application of *DC* by working in $L[a, f]$ and appealing to Shoenfield's absoluteness theorem.

We may use the following lemma since in a metric space second countability and separability are equivalent conditions.

1·4 Lemma (*AC*) *In a second countable space $\mathcal{X}$ there can exist neither a strictly descending sequence $\langle C_\nu \mid \nu < \omega_1 \rangle$ nor a strictly ascending sequence $\langle D_\nu \mid \nu < \omega_1 \rangle$ of non-empty closed subsets of $\mathcal{X}$.*

Proof: given a descending counter-example in a space with countable basis $\{N_s \mid s \in \omega\}$, pick $p_\nu \in C_\nu \setminus C_{\nu+1}$, and $s_\nu \in \omega$ with $p_\nu \in N_{s_\nu}$ and $N_{s_\nu} \cap C_{\nu+1}$ empty. There will be $\nu < \delta < \omega_1$ with $s_\nu = s_\delta$. But then $p_\delta \in C_\delta \cap N_{s_\delta} \subseteq C_{\nu+1} \cap N_{s_\nu} = \varnothing$, a contradiction.

In the ascending case, pick $p_\nu \in D_{\nu+1} \setminus D_\nu$, and $s_\nu \in \omega$ with $p_\nu \in N_{s_\nu}$ and $N_{s_\nu} \cap D_\nu$ empty. Again there will be $\nu < \delta < \omega_1$ with $s_\nu = s_\delta$. But then $p_\nu \in D_{\nu+1} \cap N_{s_\nu} \subseteq D_\delta \cap N_{s_\delta} = \varnothing$, another contradiction. ⊣ (1·4)

1·5 Remark Hausdorff in §27 of his book *Mengenlehre* proves with a beautiful argument that, more generally, there cannot be an uncountable sequence, whether strictly increasing or strictly decreasing, of sets that are simultaneously F_σ and G_δ. We shall see in §7 that this result has consequences for the present problem.

Proof of 1·2: We know that each $\omega_f(x)$ is a closed set, which, by sequential compactness is non-empty, and that if $y \in \omega_f(x)$ then $\omega_f(y) \subseteq \omega_f(x)$. Start from x, and set $C_0 = \omega_f(x)$. We shall define a shrinking sequence of closed sets all of the form $\omega_f(z)$.

If $C_\xi = \omega_f(x_\xi)$ ask if there is a $y \in C_\xi$ such that $\omega_f(y)$ is a proper subset of C_ξ: if not, then x_ξ is recurrent (in a strong sense, indeed). If there is, pick some such and call it $x_{\xi+1}$, and take $C_{\xi+1} = \omega_f(x_{\xi+1})$.

At limit stages, take the intersection, call it C'_λ: by compactness it will be non-empty. Pick x_λ in it. Then for each $\nu < \lambda$ $x_\nu \curvearrowright x_\lambda$; so $\omega_f(x_\lambda) \subseteq C'_\lambda$. Set $C_\lambda = \omega_f(x_\lambda)$ and continue.

By the Lemma this process breaks down before stage ω_1: when it does, we have reached a z such that $\forall w :\in \omega_f(z)\ w \curvearrowright z$: since $\omega_f(z)$ is non-empty, such a z is evidently recurrent. ⊣ (1·2)

1·6 Remark In such a case the set $\omega_f(z)$, or the point z, is called *minimal*.

The minimal sets are pairwise disjoint closed sets, which might, but need not, partition the set of recurrent points. Here are some examples.

1·7 Example Let $\mathcal{X} = [0, 1]$ and $f(x) \equiv x^2$: then $A^\infty = \{0, 1\}$, although f is onto and 1-1 and $\mathcal{X}$ is compact. The minimal sets in this case are $\{0\}$ and $\{1\}$.

1·8 Example Let $\mathcal{X} = \{x \in \mathbb{R}^2 \mid d(x, 0) \leqslant 1\}$. Let f be rotation by an irrational multiple of 2π. The minimal sets are the concentric circles with centre the origin, so they partition the space, in which every point is recurrent.

1·9 Example of a case where a non-minimal but recurrent y attacks two inequivalent minimal points. Let $y \in 4^\omega$ attack everything in that space, let a have only 0's and 1's, and let b have only 2's and 3's. By compactness, a and b, if not minimal themselves, will attack minimal points. Neither a nor b can attack y.

1·10 REMARK In a compact space, the set of all points in some minimal set is $\{x \mid \forall y\, x \curvearrowright y \implies y \curvearrowright x\}$, so it is Π^1_1. Hence by a theorem of Burgess either there are at most $\aleph_1$ minimal sets or there is a perfect set of inequivalent minimal points, where points x and y are considered equivalent if $x \curvearrowright y \curvearrowright x$. To see that, let $P(x, f)$ be a Π^1_1 formula saying that x is a minimal point with respect to the continuous function f. Define an equivalence relation $\equiv$ on the whole space by

$$x \equiv y \iff_{\mathrm{df}} \big[(P(x,f) \text{ or } P(y,f)) \implies (x \curvearrowright y \;\&\; y \curvearrowright x)\big] :$$

under this relation, the set of non-minimal points forms a single equivalence class, and two minimal points are equivalent iff they attack each other. From its definition, $\equiv$ is $\Sigma^1_1(f)$ and is defined on the whole space, and so Burgess' theorem applies.

1·11 PROPOSITION *Suppose no image of x is recurrent. Then $\omega_f(x)$ is nowhere dense.*

Proof : Each point of $\omega_f(x)$ is a limit of points (namely the $f^k(x)$, for $k \in \omega$) not in that closed set, hence its interior is empty. $\dashv$ (1·11)

1·12 COROLLARY *Let r be Cohen generic over $L[f, x]$ and suppose that no image of x is recurrent. Then r is not attacked by x.*

1·13 LEMMA *Suppose that $C = \omega_f(b)$ is compact. Then $f \restriction C$ maps C onto C.*

Proof : C is closed under f. Let $u \in C$. Then $b \curvearrowright u$, so let $(k_i)_i$ be an increasing sequence of positive integers such that $\lim_{i\to\infty} f^{k_i}(b) = u$. Let $(m_j)_j$ be a subsequence of the sequence $(k_i - 1)_i$ such that the sequence $f^{m_j}(b)$ is convergent, to w say. $w \in C$, as C is closed, and $f(w) = u$. $\dashv$ (1·13)

1·14 PROPOSITION *Let $\mathcal{X}$ be connected. Suppose that f is 1-1 and that C is a compact minimal set with non-empty interior. Then $C = \mathcal{X}$.*

Proof : Deny, let D be the closure of $\mathcal{X} \setminus C$, let $t \in C \cap D$, which will be non-empty by the connectedness of $\mathcal{X}$, and let v be in the interior of C. Pick $\varepsilon > 0$ so that all points distant less than 2ε from v are in C.

$t \curvearrowright v$, by the minimal character of C: let $k > 0$ be such that $d(f^k(t), u) < \varepsilon$. Choose $\delta > 0$ so that $d(f^k(t), (f^k(v)) < \varepsilon$ whenever $d(t, v) < \delta$, and choose $w \notin C$ with $d(t, w) < \delta$. Then $f^k(w) \in C$, so for some i with $0 \leqslant i < k$, $x =_{\mathrm{df}} f^i(w) \notin C$ and $f(x) \in C$. But by the lemma, $f(x) = f(y)$ for some $y \in C$, contradicting the 1-1 character of f. $\dashv$ (1·14)

2: Recurrent points

The following result shows that provided not every point in $\omega_f(a)$ escapes, recurrent points exist. We emphasize that the space is not assumed to be compact. The apparent use of the Axiom of Choice is avoidable.

2·0 THEOREM *Let $\mathcal{X}$ be a complete separable metric space, $f : \mathcal{X} \longrightarrow \mathcal{X}$ a continuous map, and a, x arbitrary points in $\mathcal{X}$. Then*

$$x \in A^\infty(a, f) \iff \exists b \; a \curvearrowright b \curvearrowright b \curvearrowright x.$$

Proof : if $b \curvearrowright b \curvearrowright x$, the point x is in A^∞, as we could take $x_i = b$ for $i > 0$. In particular every recurrent point is in A^∞.

Suppose therefore that for each $i < \omega$, $a \curvearrowright x_{i+1} \curvearrowright x_i$. Our task is to build a recurrent b with $a \curvearrowright b \curvearrowright x_0$.

We shall define a sequence of points y_i starting with $y_0 = x_0$ and converging to a point b, such that for each i, $a \curvearrowright y_i$ and $b \curvearrowright y_i$. Since $\omega_f(a)$ and $\omega_f(b)$ are closed, that will give $a \curvearrowright b$ and $b \curvearrowright b$, so b is a recurrent point with $a \curvearrowright b \curvearrowright x_0$.

To define the sequence y_i we shall define various sequences of positive reals tending monotonically to 0, and we shall define various strictly increasing sequences of positive integers.

More specifically, for each $i < \omega$ we shall define a sequence $(\varepsilon^i_k)_{k<\omega}$ of positive reals tending monotonically to 0, and for $0 < i < \omega$ a strictly increasing sequence $(\ell^i_k)_{k<\omega}$ of natural numbers. Further we shall define a decreasing sequence $(\eta_i)_{i<\omega}$ of positive reals tending to 0, and we shall define a strictly increasing sequence of positive integers $(m_i)_{1 \leqslant i < \omega}$.

Our definition takes place in infinitely many rounds. In Round 0, we shall define the point y_0, the sequence (ε^0_k) and the positive real η_0. In Round 1, we shall define m_1, y_1, the sequences (ℓ^1_k) and (ε^1_k) and the positive real η_1. For $n > 1$, we shall by the end of Round $n-1$ have defined m_{n-1}, y_{n-1}, ℓ^{n-1}_k and ε^{n-1}_k for each k, and η_{n-1}, and in Round n we shall define m_n, y_n, ℓ^n_k, ε^n_k and η_n.

Let $\Psi(i, n, \gamma, k)$ be the statement that

$$|\gamma - y_n| < \varepsilon^n_k \implies |f^{\ell^n_k + \ell^{n-1}_k + \ldots + \ell^i_k}(\gamma) - y_{i-1}| < \varepsilon^{i-1}_k$$

We shall verify in Round n, for $n \geqslant 1$, that

$$\forall k \forall \gamma \forall i \big((k \in \omega \,\&\, \gamma \in \mathcal{X} \,\&\, 1 \leqslant i \leqslant n) \implies \Psi(i, n, \gamma, k) \big).$$

In fact, for each γ and k, $\Psi(n, n, \gamma, k)$ will follow from our choice of ε^n_k and ℓ^n_k; and then the other cases will be covered by the following

2·1 LEMMA *If ε^n_k and ℓ^n_k have been defined, then for $i < n$,*

$$\big(\Psi(n, n, \gamma, k) \;\&\; \Psi(i, n-1, f^{\ell^n_k}(\gamma), k)\big) \implies \Psi(i, n, \gamma, k).$$

Proof : Let $|\gamma - y_n| < \varepsilon^n_k$. By $\Psi(n, n, \gamma, k)$, $|f^{\ell^n_k}(\gamma) - y_{n-1}| < \varepsilon^{n-1}_k$: so we may apply $\Psi(i, n-1, f^{\ell^n_k}(\gamma), k)$ and use the fact that

$$f^{\ell^{n-1}_k + \ldots + \ell^i_k}\big(f^{\ell^n_k}(\gamma)\big) = f^{\ell^n_k + \ell^{n-1}_k + \ldots + \ell^i_k}(\gamma) \qquad \dashv \; (2\cdot 1)$$

We are now ready to begin our construction. In it, we shall use without further comment the general lemma that if $c \curvearrowright d$ then $f(c) \curvearrowright d$ and $c \curvearrowright f(d)$.

Round 0. Put $y_0 = x_0$, choose an arbitrary sequence ε^0_k of positive reals tending monotonically to 0 as $k \longrightarrow \infty$, and set $\eta_0 = \frac{1}{4}\varepsilon^0_0$.

Round 1. Pick m_1 such that $|f^{m_1}(x_1) - y_0| < \eta_0$: that is possible as $x_1 \curvearrowright y_0 = x_0$. Put $y_1 = f^{m_1}(x_1)$. Choose a sequence $\ell^1_0 < \ell^1_1 < \ell^1_2 < \ldots$ such that $\forall k \; |f^{\ell^1_k}(y_1) - y_0| < \frac{1}{2}\varepsilon^0_k$: that can be done as $y_1 \curvearrowright y_0$.

Choose a sequence ε^1_k tending to 0 monotonically from above such that

$$\forall k \forall \gamma {:\in} \mathcal{X} \; \big(|\gamma - y_1| < \varepsilon^1_k \implies |f^{\ell^1_k}(\gamma) - f^{\ell^1_k}(y_1)| < \tfrac{1}{2}\varepsilon^0_k\big) :$$

that can be done as $f^{\ell^1_k}$ is continuous at y_1.

That implies that for all $k \in \omega$ *and all* $\gamma \in \mathcal{X}$,

$$|\gamma - y_1| < \varepsilon^1_k \implies |f^{\ell^1_k}(\gamma) - y_0| < \varepsilon^0_k$$

which is the statement $\Psi(1, 1, \gamma, k)$. Set $\eta_1 = \min(\frac{1}{8}\varepsilon^0_0, \frac{1}{4}\varepsilon^1_1) = \min(\frac{1}{2}\eta_0, \frac{1}{4}\varepsilon^1_1)$.

Round 2. Pick $m_2 > m_1$ such that $|f^{m_2}(x_2) - y_1| < \eta_1$ — possible as $x_2 \curvearrowright y_1$ — and put $y_2 = f^{m_2}(x_2)$. Choose a sequence $\ell^2_0 < \ell^2_1 < \ell^2_2 < \ldots$ such that $\forall k\, |f^{\ell^2_k}(y_2) - y_1| < \frac{1}{2}\varepsilon^1_k$: that can be done as $y_2 \curvearrowright y_1$.

Choose a sequence ε^2_k tending to 0 monotonically from above such that

$$\forall k \forall \gamma{:}\in \mathcal{X} \left(|\gamma - y_2| < \varepsilon^2_k \implies |f^{\ell^2_k}(\gamma) - f^{\ell^2_k}(y_2)| < \tfrac{1}{2}\varepsilon^1_k\right) :$$

that can be done as $f^{\ell^2_k}$ is continuous at y_2.

That implies that for all $k \in \omega$ *and for all* $\gamma \in \mathcal{X}$,

$$|\gamma - y_2| < \varepsilon^2_k \implies |f^{\ell^2_k}(\gamma) - y_1| < \varepsilon^1_k$$

and therefore

$$|\gamma - y_2| < \varepsilon^2_k \implies |f^{\ell^2_k + \ell^1_k}(\gamma) - y_0| < \varepsilon^0_k$$

which are the statements $\Psi(2, 2, \gamma, k)$ *and* $\Psi(1, 2, \gamma, k)$ *respectively.* Set $\eta_2 = \min(\frac{1}{2}\eta_1, \frac{1}{4}\varepsilon^2_2)$ and continue to the next round.

Round n, for $n > 2$. Pick $m_n > m_{n-1}$ such that $|f^{m_n}(x_n) - y_{n-1}| < \eta_{n-1}$, and put $y_n = f^{m_n}(x_n)$. Choose a sequence $\ell^n_1 < \ell^n_2 < \ldots$ such that $\forall k\, |f^{\ell^n_k}(y_n) - y_{n-1}| < \frac{1}{2}\varepsilon^{n-1}_k$: that can be done as $y_n \curvearrowright y_{n-1}$.

Choose a sequence ε^n_k tending to 0 monotonically from above such that

$$\forall k \forall \gamma{:}\in \mathcal{X} \Big(|\gamma - y_n| < \varepsilon^n_k \implies |f^{\ell^n_k}(\gamma) - f^{\ell^n_k}(y_n)| < \tfrac{1}{2}\varepsilon^{n-1}_k\Big) :$$

that can be done as $f^{\ell^n_k}$ is continuous at y_n.

That implies that for all $k \in \omega$ *and for all* $\gamma \in \mathcal{X}$

$$|\gamma - y_n| < \varepsilon^n_k \implies |f^{\ell^n_k}(\gamma) - y_{n-1}| < \varepsilon^{n-1}_k$$

which is the statement $\Psi(n, n, \gamma, k)$*; we have seen that it follows from statements established in previous rounds that for* $n > i \geqslant 1$,

$$|\gamma - y_n| < \varepsilon^n_k \implies |f^{\ell^n_k + \ell^{n-1}_k + \ldots + \ell^i_k}(\gamma) - y_{i-1}| < \varepsilon^{i-1}_k$$

which is $\Psi(i, n, \gamma, k)$. Set $\eta_n = \min(\frac{1}{2}\eta_{n-1}, \frac{1}{4}\varepsilon^n_n)$.

Once all the rounds have been completed, we shall have defined a sequence y_i such that for each i, $|y_{i+1} - y_i| < \eta_i$. By definition $\eta_{i+1} = \min(\frac{1}{2}\eta_i, \frac{1}{4}\varepsilon^{i+1}_{i+1})$, so in particular $\eta_{i+1} \leqslant \frac{1}{2}\eta_i$, and so $\sum_{i<\omega} \eta_i$ is convergent. Hence (y_i) is a Cauchy sequence, and hence by the completeness of the space $\mathcal{X}$ is convergent. Let b be its limit.

2·2 Lemma *For each* k, $|b - y_k| < \varepsilon^k_k$.

Proof : For each k, $\eta_k \leqslant \frac{1}{4}\varepsilon^k_k$, $\eta_{k+1} \leqslant \frac{1}{2}\eta_k \leqslant \frac{1}{8}\varepsilon^k_k$, and so for each $j \geqslant k$, $\eta_j \leqslant 2^{k-2-j}\varepsilon^k_k$; we know that for each i, $|y_{i+1} - y_i| < \eta_i$, and hence for $k < j$, $|y_j - y_k| < \eta_k + \ldots + \eta_{j-1}$; thus

$$|b - y_k| \leqslant \sum_{j \geqslant k} \eta_j \leqslant (\tfrac{1}{4} + \tfrac{1}{8} + \ldots)\, \varepsilon^k_k = \tfrac{1}{2}\varepsilon^k_k. \qquad \dashv (2{\cdot}2)$$

Fix i. We assert that $b \curvearrowright y_i$. Thus, we must show that

$$\forall \varepsilon > 0 \exists n\, |f^n(b) - y_i| < \varepsilon;$$

moreover that n may be chosen arbitrarily large.

Fix $\varepsilon > 0$. Pick $k > i$ such that $\varepsilon^i_k < \varepsilon$. By the lemma, $|b - y_k| < \varepsilon^k_k$, and so applying $\Psi(i{+}1, k, b, k)$,

$$|f^{\ell^k_k + \ell^{k-1}_k + \ldots + \ell^{i+1}_k}(b) - y_i| < \varepsilon^i_k < \varepsilon,$$

as required.

Note finally that as k can be chosen arbitrarily large, the power of f applied to b, which is at least ℓ^{i+1}_k, can also be made arbitarily large.

Our theorem is proved. $\dashv$ (2·1)

3: Maximal recurrent points.

In fact our proof of 2·0 has established the following statement:

3·0 Proposition *Given $\mathcal{X}$, f, and a, suppose that for all i $a \curvearrowright z_{i+1} \curvearrowright z_i \curvearrowright \ldots \curvearrowright z_0$. Then there are natural numbers $m_0 < m_1 < \ldots$ such that setting $y_i = f^{m_i}(z_i)$, the sequence (y_i) is convergent with limit b, say, and $b \curvearrowright y_i$ for each i. It follows that b is recurrent, and that for all i $a \curvearrowright b \curvearrowright z_i$ and $\omega_f(z_i) = \omega_f(y_i)$.*

3·1 Remark Note that if the points z'_i form a second set satisfying the hypothesis of the Proposition, with $\forall i\ z_i \curvearrowright z'_i \curvearrowright z_i$, and y'_i, b' are the outcome of repeating the argument, then

$$\forall i\ b \curvearrowright z_{i+1} \curvearrowright z'_{i+1} \curvearrowright y'_i \ \&\ b' \curvearrowright z'_{i+1} \curvearrowright z_{i+1} \curvearrowright y_i$$

and so $b \curvearrowright b' \curvearrowright b$.

3·2 Proposition *In these circumstances, $\omega_f(b)$ is the closure of $\bigcup_i \omega_f(z_i)$.*

Proof: write C_b for $\omega_f(b)$, C_i for $\omega_f(z_i)$, and C for the closure of $\bigcup_i C_i$. Each C_i is closed topologically and also under the action of f, hence so is C. C_b is a closed set containing $\bigcup_i C_i$, and therefore $C_b \supseteq C$. But $b \in C$, being the limit of the sequence y_i, so each $f^k(b)$ is in C, and therefore each point of C_b is in C. $\dashv$ (3·2)

3·3 Definition Call a point b *maximal recurrent in* $\omega_f(a)$ if $a \curvearrowright b \curvearrowright b$ and whenever $a \curvearrowright c \curvearrowright c \curvearrowright b$, then $b \curvearrowright c$.

With the help of the axiom of choice the above proposition yields the following

3·4 Corollary (AC) *If d is a recurrent point in $\omega_f(a)$, then there is a point b which is maximal recurrent in $\omega_f(a)$ with $a \curvearrowright b \curvearrowright d$.*

Proof: set $d_0 = d$. If d_0 is not maximal in $\omega_f(a)$, pick d_1 with $a \curvearrowright d_1 \curvearrowright d_1 \curvearrowright d_0 \not\curvearrowright d_1$; if d_1 is not maximal, continue. The proposition tells us that our construction can be continued at countable limit ordinals. If we never encounter a maximal recurrent point, then our construction will yield for every countable ordinal ν a recurrent point d_ν with $a \curvearrowright d_\zeta \curvearrowright d_\nu \not\curvearrowright d_\zeta$ for $\nu < \zeta < \omega_1$. But then the sequence $\langle \omega_f(d_\nu) \mid \nu < \omega_1 \rangle$ will form a strictly increasing sequence of closed sets of order type ω_1, contradicting Lemma 1·4. $\dashv$ (3·4)

3·5 Remark Again, that use of AC could be reduced to an application of DC by working in $L[a, f]$ and appealing to Shoenfield's absoluteness theorem.

3·6 Remark We could also formulate the notion of a *maximal recurrent* point in the space $\mathcal{X}$ as a whole, without reference to a particular point a; the same argument will prove that if recurrent points exist, so do maximal ones. In a case such as the shift function acting on Baire space, the maximal recurrent points will be simply be those whose orbit is dense in the whole space.

4: Points at the end of paths

We apply the results of §3 in the context of §0. We are in Baire space and consider the shift function $\mathfrak{s}$. We have a tree $T \subseteq \mathcal{S}$ which is closed under shortening and we have defined points x_s for $s \in T$, and a point x_T.

4·0 First, suppose that we have an infinite path p through T.

Set $a_i = x_{p\restriction i}$. Then by construction $a_{i+1} \curvearrowright a_i$; as in §3 we may find integers m_i such that if we set $y_i = f^{m_i}(a_i)$, the sequence (y_i) will be convergent to a point we shall call x_p. There is likely to be freedom in our choice of integers m_i, so that we do not know that x_p is uniquely determined by the path p. However, by 3·2, $\omega_f(x_p)$ is uniquely determined, and we know that for each i,

$$x_T \curvearrowright x_p \curvearrowright x_p \curvearrowright x_{p\restriction i}$$

and that for any point z,

$$\text{if } \forall i\ x_T \curvearrowright z \curvearrowright x_{p\restriction i} \text{ then } \forall i\ z \curvearrowright y_i \text{ and hence } z \curvearrowright x_p.$$

4·1 Proposition *We may adjust our choices of the various integers employed so that x_p is always recursive in p.*

Proof: by inspection of the argument of §2. $\dashv$ (4·1)

Now we may use lemmata from *Long Delays* to prove:

4·2 Proposition *If $x_T \curvearrowright \beta$ then* **either** *there is a uniquely determined infinite path p through T such that $x_T \curvearrowright x_p \curvearrowright x_p \curvearrowright \beta$,* **or** *there is an $s \in T$ with $\beta \triangleright x_s$.*

Proof: Suppose that $x_T \curvearrowright \beta$. Any odd number that occurs in β is of the form $\pi_{t,n}$. Suppose that $\beta(i) = \pi_{t,n}$ and $\beta(j) = \pi_{s,m}$ are odd numbers occurring in β, then, taking $k > \max\{i, j\}$, and applying LD 1·10 to a $v \in T$ with $\beta \restriction k \sqsubset x_v$, we see that both s and t must be initial segments of v. Hence the t's such that a power of π_t occurs in β define a path through T which may be empty, or finite, or infinite.

If that path is empty, or, in other words, if no odd number occurs in β, then $\beta \triangleright x_\emptyset$, by LD 1·12 and the fact that no number occurs twice in $x_\emptyset$. If that path is non-empty but finite, then there is a longest s such that some power of π_s occurs, and then by LD 1·13 $\beta \triangleright x_s$. If the path is infinite, let us call it p_β. Then $x_{p_\beta} \curvearrowright \beta$, since given k there is an ℓ such that $\beta \restriction k \sqsubset x_{p\restriction \ell}$, and $x_p \curvearrowright x_{p\restriction \ell}$. $\dashv$ (4·2)

4·3 Remark Thus, in this context, if β defines p_β, $x_p \leqslant_{\text{Turing}} p_\beta \leqslant_{\text{Turing}} \beta$.

The above proposition, coupled with some facts about hyperarithmetic sets, is enough for the proof of the main theorem of the paper. We pause to prove a refinement.

4·4 Proposition *Given any subtree T of $\mathcal{S}$ closed under shortening, the numbers $n_{t,k}$ (for $t \in T$) may be chosen so that whenever $x_T \curvearrowright \beta$ and β defines an infinite path p_β through T, $\beta \curvearrowright x_s$ for each $s \in p_\beta$, and therefore $\beta \curvearrowright x_{p_\beta}$, and β is recurrent.*

Proof: we ensure that whenever $t = s^\frown\langle k\rangle$, the integers $n_{t,\ell}$ are chosen so that for each $s' \not\succcurlyeq s$, $x_{s'} \restriction \ell h(t) \sqsubset x_s \restriction n_{t,\ell}$, in addition to our earlier requirement, set out in *Long Delays*, that the numbers $x_s(n_{t,\ell} - 1)$ immediately preceding each power of π_t in x_t should form a strictly increasing sequence of multiples of 4, and the (new) requirement that $x_s(n_{t,\ell})$ will always be a power of π_s.

If we have done that, then we may show that for $\bar{s} \in p_\beta$, $\beta \curvearrowright x_{\bar{s}}$. For let N be given, and pick t of length at least $\max(N, \ell h(\bar{s}) + 1)$ for which some power of π_t equals $\beta(a)$, where $a > N$. Let c be the least integer exceeding a such that $\beta(c)$ is a power of π_u for some u with $u \prec t$. Let b be the largest number less than c for which $\beta(b)$ is a power of π_t, so that $a \leqslant b < c$. Let $s = t \restriction (\ell h(t) - 1)$. Then $\beta \restriction c + 1 \sqsubset x_w$ say, and so the segment $\beta \restriction [b+1, c)$ equals $x_s \restriction n_{t,m}$ for some m, and hence $x_{\bar{s}} \restriction N$ is a segment of $\beta \restriction [b+1, c)$ and thus is a segment of β starting after stage N. $\dashv$ (4·4)

4·5 Remark Generally, by §2,

$$A^\infty = \{\beta \mid \exists \gamma\ x_T \curvearrowright \gamma \curvearrowright \gamma \curvearrowright \beta\}.$$

In our context, for given β not near or attacked by any x_s, the recurrent $\gamma = x_{p_\beta}$ that attacks β is recursive in β. This gives us an easy way of showing that $\theta(x_T, \mathfrak{s})$ is countable.

Let $\mathfrak{A}$ be the set of nodes s such that the tree below s is ill-founded. $\mathfrak{A}$ is of course a countable set of finite sequences, and therefore codable by a single real, $\mathfrak{a}$. Thus we have

$$A^\infty = \{\beta \mid \exists s{:}{\in}\mathfrak{A}\ \beta \triangleright x_s\} \cup \{\beta \mid \exists\gamma{\leqslant}_{\mathrm{Turing}}\ \beta\ x_T \curvearrowright \gamma \curvearrowright \gamma \curvearrowright \beta\}$$

which is arithmetical in $\mathfrak{a}$. A^∞ is thus Borel and the ordinal $\theta(x_T, \mathfrak{s})$ countable.

5: Hyperarithmetic points and closed sets

5·0 There is a countable family of functions from ω to ω called the hyperarithmetic functions (or HYP for short) which are the trouble-makers when it comes to inductive definitions.

The paper *The next admissible set* by Barwise, Gandy and Moschovakis lists nine equivalent definitions of this family, of which we state four. The reader will find much more information than we can give here in the treatises of Barwise, of Mansfield and Weitkamp, of Moschovakis, of Hartley Rogers, Jr., and of Shoenfield.

(5·0·0) *A function $\alpha : \omega \to \omega$ is HYP iff its graph $\{\langle m, n\rangle \mid \alpha(m) = n\}$ is Δ^1_1;*

Since for a total function $\alpha : \omega \to \omega$, $\alpha(m) = n \iff \forall k(k \neq n \implies \alpha(m) \neq k)$, it is sufficient to require that the graph be Σ^1_1.

(5·0·1) *A function $\alpha : \omega \to \omega$ is HYP if and only if it is a member of every ω-model of analysis;*

From that definition it is plain that $\{\alpha \mid \alpha \text{ is HYP}\}$ is Π^1_1.

(5·0·2) *A function $\alpha : \omega \to \omega$ is HYP if and only if it is recursive in some H_e, for e a recursive well-ordering;*

Here H_e is the hierarchy defined by Kleene proceeding by Turing jump and effective union.

(5·0·3) *A function $\alpha : \omega \to \omega$ is HYP if and only if it is a member of the smallest admissible set containing* $\omega + 1$.

Some comments on that last definition. Let us write $\mathfrak{M}$ for the smallest transitive model of Kripke-Platek set theory including the axiom of infinity. $\mathfrak{M} = L_{\omega_1^{CK}}$, the initial segment of the Gödel constructible hierarchy up to the Church–Kleene ordinal, the first non-recursive ordinal, ω_1^{CK}.

$\mathfrak{M}$ is the collection of all sets coded by a HYP well-founded extensional relation on ω. It is well-known that $\mathfrak{M}$ is not a β-model; that is, that there are linear orderings in $\mathfrak{M}$ which are not well-orderings but which $\mathfrak{M}$ believes to be well-orderings. Such linear orderings we shall call, following Harrison, *pseudo-well-orderings.*

5·1 In particular it follows from the Kleene Boundedness theorem that there exist recursive pseudo-well-orderings: for by that theorem the set of indices e of recursive well-orderings is a Π^1_1 set, but not Σ^1_1; the set of those indices e of recursive linear-orderings with no HYP descending chains may be seen to be Σ^1_1 by using definition (5·0·1); and so there must be an e in the second set but not in the first.

5·2 The proof of the celebrated Cantor–Bendixson theorem, that every closed set is the union of a perfect set and a countable set, starts from a closed set C and proceeds by iterating the operation of taking the derived set; the sequences of sets C^ν is a descending sequence of closed sets and so by Lemma 1·4 stops at some countable ordinal, called the *closure ordinal* of the construction. Call C^∞ the final set: it equals its derived set (otherwise the sequence would continue to shrink) and so C^∞ is a perfect set, that is, is closed and has no isolated points. If C is countable, C^∞ will be empty, otherwise C^∞ will be of cardinality the continuum. Each $C^\nu \setminus C^{\nu+1}$ is countable (or finite). So $C \setminus C^\infty$ is countable, and we have shown that every closed set is the union of a perfect set and a countable set.

That proof was analysed by Lorenzen and by Kreisel [5] in the context of $\mathfrak{M}$. For a precise statement of their results the reader should consult the review [10] by Moschovakis. For our purposes we note the following:

5·3 THEOREM *Let C be a recursive tree defining a closed set. Let C^ν denote the Cantor–Bendixson sequence, and C^∞ the perfect kernel. Let δ denote the first non-recursive ordinal.*

(5·3·0) If $\alpha \notin HYP$, $\alpha \in [C^\infty] \iff \alpha \in [C]$.

(5·3·1) If $\alpha \in HYP$, $\alpha \in [C^\infty] \iff \alpha \in [C^\delta]$.

(5·3·2) $C^\infty = C^\delta$.

Related to that is the following curious compactness phenomenon at δ, which holds even if the space we are thinking about is not compact:

5·4 PROPOSITION *If C is recursive and for each recursive ordinal C^ν is non-empty, then C^∞ is non-empty.*

Proof in brief: the hypothesis implies that for each $e \in WO$ there is an x and an e-CB-*frame* for x, namely an array of points that witnesses the survival of the Cantor–Bendixson process for $|e|$ steps. But that is a Σ^1_1 statement, which by Kleene must be true of some pseudo-well-ordering: so there is an x and a *pseudo-frame* for x. But that gives $x \in C^\infty$. ⊣ (5·4)

5·5 COROLLARY *Let C be a countable recursive closed set. Then there is a recursive ordinal ν such that $[C^\nu] = \varnothing$.*

Proof : all its members are HYP, so C^∞ is empty so some C^ν is empty for a recursive ν. ⊣ (5·5)

6: Proof of our main result.

Consider a non-empty recursive linear ordering which is ill-founded but which has no HYP descending paths: such exists by 5·1. Form the tree of all finite sequences of strictly decreasing sequences in that ordering, ordered under end-extension. As described in *Long Delays*, that may be copied to a tree $T \subseteq \mathcal{S}$. We work with that latter tree. It is recursive and ill-founded but has no HYP descending paths.

We build points x_s, for $s \in T$, and x_T.

6·0 PROPOSITION $\theta(x_T, \mathfrak{s}) = \omega_1^{CK}$.

Proof : As proved in Harrison's thesis, the said linear ordering will have a well-ordered initial segment of length exactly ω_1^{CK}. Hence the tree T will have points s below which it is well-founded, and the ranks of such s will be exactly the set of recursive ordinals. So by the arguments of *Long Delays*, $\theta(x_T, \mathfrak{s}) \geqslant \omega_1^{CK}$.

Now suppose $x_T \curvearrowright \beta$. If $\beta \triangleright x_s$ where $s \in T$, β will be HYP, indeed recursive, as each x_s is recursive. Given our choice of $x_\emptyset$, there will be no points attacked by some x_s that are not near some x_t with shorter t. If the tree below s is ill-founded, β will abide, otherwise, if the tree below s is well-founded, then β will escape, and the ordinal at which it escapes will be recursive.

So the only case remaining to be discussed is when β defines an infinite path p_β through the tree: in this case, since p_β is recursive in β, and, by choice of T, p_β cannot be HYP, β cannot be HYP. By §4, $x_{p_\beta} \curvearrowright \beta$: since x_{p_β} is recurrent, $\beta \in A^\infty$.

So every point that escapes does so at a recursive ordinal, yielding $\theta(x_T, \mathfrak{s}) \leqslant \omega_1^{CK}$. Hence $\theta(x_T, \mathfrak{s}) = \omega_1^{CK}$. ⊣ (6·0)

6·1 REMARK If we make the more refined choice of $n_{t,i}$'s sketched in §4, we shall have the following exact picture of $\omega_{\mathfrak{s}}(x_T)$: the points that escape are those near to, that is, are finite shifts of, the x_s with s in the well-founded part of the tree. All such points are recursive. The points that abide are those near to x_s with s in the ill-founded part, — those points again, individually, are recursive — and the recurrent points, which are exactly the points equivalent to the points x_p placed at the end of each infinite path p. All recurrent points are non-HYP. There are no minimal points, and all recurrent points are maximal.

6·2 REMARK Note that in these examples, if $\gamma \triangleright x_s$ where s is in the ill-founded part of the tree, there is a sequence of points attacking γ, with each point of the sequence being recursive, but the sequence itself not even hyperarithmetic.

7: Open problems.

The present papers leave open the question whether there can be $\mathcal{X}$, f and a with $\theta(a, f) = \omega_1$: I would guess not, and that in all cases $\theta(a, f)$ is at most the least ordinal not recursive in a and (a code of) f.

We collect here some thoughts directed to that question. Our examples are all in Baire space with the shift function unless otherwise stated.

7·0 Proposition *There is a recursive* α *such that* $\alpha \curvearrowright \beta \implies \beta \notin HYP$*, but continuum many such* β *exist.*

Proof : let C be a recursive tree with no HYP paths but with a continuum of non-HYP paths. Let p_i enumerate, monotonically, the odd primes, so $p_0 = 3$. For $\emptyset \neq s \in C$ let $\pi(s) = \prod_{i<\ell h(s)} p_{s(i)}$, and let $\tau(s) = \langle \pi(s \restriction k) \mid 0 < k \leqslant \ell h(s)\rangle$. So $\tau(s)$ is a sequence as long as s of strictly increasing odd numbers, each dividing the next with quotient an odd prime. Let $\tau(\emptyset) = \emptyset$.

Let $D = \{\tau(s) \mid s \in C\}$: then D also defines a closed set, and D has no HYP paths, for recursive in any path through D is a path through C. Let α intersperse occurrences of $s \in D$ with occurrences of even numbers, such that no even number occurs more than once. Suppose that $\alpha \curvearrowright \beta$. Then no even number can occur in β, so each initial segment of β is a segment of something in D. So as in LD 1·13, β will be a finite shift of some path through D. β itself will attack nothing. Finally, every path through D, of which there are continuum many, is attacked by α. ⊣ (7·0)

7·1 Corollary *There is a recursive* α *and a non-HYP* γ *such that* $\alpha \curvearrowright \gamma$ *but there is no* β *with* $\alpha \curvearrowright \beta \curvearrowright \gamma$.

Proof : let α be as in the previous example, and γ anything attacked by α. Then γ is not HYP, and there can be no β with $\alpha \curvearrowright \beta \curvearrowright \gamma$, because anything attacked by α attacks nothing. ⊣ (7·1)

7·2 Remark That corollary shows the impossibility of proving that $A^\infty = A^\delta$ by imitating the Lorenzen–Kreisel proof that $C^\infty = C^\delta$.

7·3 Problem In that corollary, $A^1(\alpha, \mathfrak{s})$ is empty. Is there an example where A^1 is non-empty, and non-trivial, and A^2 is empty ?

An answer to that problem might perhaps be found by adapting the following instance of a case where A^1 is not closed. We work again in Baire space $\mathcal{N}$ with the shift function $\mathfrak{s}$. Let p_i enumerate the primes, so $p_0 = 2, p_1 = 3, p_2 = 5, \ldots$; let $q_i = p_{2i+1}$. Let $e_i = q_i + 1$, an even number. For i, k in ω, let $\pi_{i,k} = (p_{2i+4})^{k+1}$, so these numbers are distinct powers of distinct odd primes > 5.

We define a point z, points y_i, x_i for each $i < \omega$, and a point a, all in $\mathcal{N}$, thus:

$$z =_{\text{df}} q_0, q_1, q_2, q_3, \ldots;$$
$$y_i =_{\text{df}} q_0, q_1, \ldots, q_i, e_i, e_i, e_i, \ldots;$$
$$x_i =_{\text{df}} \langle \pi_{i,0}\rangle{}^\frown(y_i \restriction n_{i,0}){}^\frown\langle \pi_{i,1}\rangle{}^\frown(y_i \restriction n_{i,1}){}^\frown \ldots,$$

where the $n_{i,k}$ are chosen with $i + 2 \leqslant n_{i,0} < n_{i,1} < n_{i,2} < \ldots$; and

$$a =_{\text{df}} (x_0 \restriction m_o){}^\frown\langle 5\rangle{}^\frown(x_1 \restriction m_1){}^\frown\langle 5^2\rangle{}^\frown(x_0 \restriction m_2){}^\frown \ldots,$$

where the m's are strictly increasing and each x_i is visited infinitely often.

7·4 Proposition z *is not in* $A^1(a, \mathfrak{s})$ *but is in its closure.*

Proof : evidently $a \curvearrowright x_i \curvearrowright y_i$ for each i, so that each y_i is in A^1, and $\lim_i y_i = z$.

Suppose that $a \curvearrowright b \curvearrowright z$: we shall derive a contradiction and thereby prove the proposition.

Note that e_j occurs in x_k iff $k = j$, and that q_i has exactly one occurrence in $y_k \restriction n_{k,\ell}$ if $k \geqslant i$ and none if $k < i$, so that q_i occurs in x_k iff $k \geqslant i$. Further, no power of 5 can occur in b as each only occurs once in a, and so each finite segment $u \sqsubset b$ is a segment of some x_k.

Each q_i occurs infinitely often in b as $b \curvearrowright z$. Let u be a segment of b of length at least 2 that both begins and ends with q_i. Then $u \sqsubset x_k$ for some $k \geqslant i$; the two occurrences of q_i must come from different initial segments of y_k and so between them is an occurrence of e_k.

So for some k, e_k occurs in b; q_{k+1} occurs in b, as all q_i do; so some segment v of b contains occurrences of both e_k and q_{k+1}. But no such v can be a segment of any x_ℓ, since if e_k occurs in x_ℓ then $k = \ell$, and if q_{k+1} occurs in x_ℓ, $k + 1 \leqslant \ell$. ⊣ (7·4)

7·5 Hausdorff's criterion for a set to be $\mathbf{\Delta^0_2}$, mentioned in Remark 1·5, may enable us to find cases where A^1 is not $\mathbf{\Sigma^0_2}$. For a subset H of an appropriate topological space Hausdorff defines H_ρ to be $c\ell(H) \setminus H$, and H_ψ to be $H_{\rho\rho}$. It is easily checked that $H \supseteq H_\psi$. Hausdorff proves in §27 of his book *Mengenlehre* that H is simultaneously F_σ and G_δ if and only if the (possibly transfinite) sequence H, H_ψ, $H_{\psi\psi}$, …, $\bigcap_{n<\omega} H_{\psi^n}$, … eventually reaches the empty set. In particular if $H = H_\psi \neq \varnothing$, H cannot be $\mathbf{\Delta^0_2}$. Thus we obtain the following:

7·6 PROPOSITION *In the case of the shift function, both* $B = \{\beta \mid \beta \curvearrowright \beta\}$ *and* $B^\varepsilon = \{\beta \mid \beta \curvearrowright \varepsilon\}$ *are* Π^0_2, *but neither is* $\mathbf{\Sigma^0_2}$.

Proof: we prove that for $H = B$ or B^ε, $H = H_\psi$ in the notation of Hausdorff, *Mengenlehre*, §27. Unravelling the definition, we see that it suffices to show that each point in H is a limit of points in H_ρ.

$H = B^\varepsilon$: given $\beta \curvearrowright \varepsilon$, let $\gamma_n \restriction n = \beta_n$, let the rest of γ_n be something not attacking ε: possible provided $\{\gamma \mid \gamma \curvearrowright \varepsilon\}$ is not the whole space. Then let $\delta^m_n \restriction m = \gamma_n \restriction m$ and let the rest of δ^m_n be something attacking ε, e.g. β, or, even better, $\beta \restriction \omega \setminus m$.

$H = B$: given $\beta \curvearrowright \beta$, let $\gamma_n \restriction n = \beta_n$, let the rest of γ_n be something not attacking γ_n — note the self-reference. Then let $\delta^m_n \restriction m = \gamma_n \restriction m$ and let the rest of δ^m_n be something attacking δ^m_n, e.g. some Cohen real. $\dashv$ (7·6)

If we want to establish similar results for $\{\beta \mid a \curvearrowright \beta \curvearrowright \beta\}$ and $\{\beta \mid a \curvearrowright \beta \curvearrowright \gamma\}$, we have to build γ_n, δ^m_n according to the following matrices.

$$\begin{bmatrix} a & \curvearrowright & \delta^m_n & \curvearrowright & \varepsilon \\ & & \downarrow_m & & \\ a & \curvearrowright & \gamma_n & \not\curvearrowright & \varepsilon \\ & & \downarrow_n & & \\ a & \curvearrowright & \beta & \curvearrowright & \varepsilon \end{bmatrix}, \qquad \begin{bmatrix} a & \curvearrowright & \delta^m_n & \curvearrowright & \delta^m_n \\ & & \downarrow_m & & \\ a & \curvearrowright & \gamma_n & \not\curvearrowright & \gamma_n \\ & & \downarrow_n & & \\ a & \curvearrowright & \beta & \curvearrowright & \beta \end{bmatrix}$$

But note that the constructions of *Long Delays* easily provide examples of points a, ε where these sets are countable, finite or even empty. Hence they might be $\mathbf{\Sigma^0_2}$. So all we may hope to do here is give examples where they are not: but for such, any a with orbit dense in the space will do.

7·7 PROBLEM Where do the sets

$$\{a \mid \omega_f(a) \cap B \text{ is } F_\sigma\} \quad \text{and} \quad \{a \mid \omega_f(a) \cap B^\varepsilon \text{ is } F_\sigma\}$$

lie in the projective hierarchy ?

7·8 PROBLEM Find a case where $\{a \mid a \text{ minimal }\}$ is a complete $\Pi^1_1(f)$ set.

7·9 EXAMPLE Baire space ${}^\omega\omega$ is homœomorphic to the space $\mathcal{Q}$ of infinite sequences of rational numbers, *via* a homœomorphism that preserves the shift map $\mathfrak{s}$. Working in $\mathcal{Q}$ we may exhibit a large number of points recurrent under the shift map, as follows. For each real number λ let $\mathcal{Q}_\lambda$ be the space of all sequences of rationals less than λ, and let q_λ be a point of $\mathcal{Q}_\lambda$ the orbit of which under $\mathfrak{s}$ is dense in $\mathcal{Q}_\lambda$. For η rational let $\mathcal{R}_\eta$ be the space of all sequences of rationals less than or equal to η, and let r_η be a point of $\mathcal{R}_\eta$ with orbit under $\mathfrak{s}$ dense in $\mathcal{R}_\eta$.

Then for η rational and λ and μ real, with $\lambda < \eta < \mu$, we have

$$\mathcal{Q}_\lambda \subset \mathcal{Q}_\eta \subset \mathcal{R}_\eta \subset \mathcal{Q}_\mu,$$

all the inclusions being strict, and hence

$$q_\mu \curvearrowright q_\mu \curvearrowright r_\eta \curvearrowright r_\eta \curvearrowright q_\eta \curvearrowright q_\eta \curvearrowright q_\lambda \curvearrowright q_\lambda,$$

while $q_\lambda \not\curvearrowright q_\eta \not\curvearrowright r_\eta \not\curvearrowright q_\mu$.

Hence in Baire space there is a set of recurrent points strictly linearly ordered by the relation $\curvearrowright$ and with that ordering isomorphic to the real line with every rational point doubled.

7·10 PROBLEM Can we use the Gandy–Harrington topology to discuss minimal sets ?

7·11 PROBLEM Is there an example where A^1 is strictly Σ^1_1 ?

7·12 Theorem *Let a be the starting point, f the function. f is continuous and so is coded by a real which we also denote by f. Let $\delta = \delta(f, a, x)$ be the least ordinal not recursive in the triple (f, a, x). Then*

$$x \in A^\infty(a) \iff x \in A^\delta.$$

Proof : one direction is trivial, so suppose towards the other that $\forall\nu{:<}\delta\ x \in A^\nu$.

We consider trees $\mathcal{T} \subseteq \mathcal{S}$, the set of sequences of primes we have considered before, with a top point ω.

An *x-frame* on such a tree is a function attaching to each node s a point y_s, the top node must get point x, all other nodes must get a point y with $a \curvearrowright y \curvearrowright x$, and the attachment must be such that for $s \prec t$, $y_s \curvearrowright y_t$. So a frame with top point x provides evidence that x survives at least as long as the rank of the tree supporting the frame. We can construct frames by using the axiom of choice and

7·13 The Richness Lemma *Let $x \in A^\eta$ and let $\zeta < \eta$. Then $\exists y \in A^\zeta$ with $y \curvearrowright x$.*

Proof : by induction on η. The lemma is vacuous for $\eta = 0$; the induction is easy for η a limit. For $\eta = \xi + 1$, take two cases, $\zeta < \xi$ and $\zeta = \xi$. $\dashv$ (7·13)

We allude to the triple (f, a, x) as *the data*, and call something *data-recursive* if it is recursive in the data.

Proof of 7·11 continued: Now the collection of trees coded by a data-recursive relation on ω for which there exists an x-frame is $\Sigma^1_1(f, a, x)$, and contains all data-recursive well-founded trees, by our assumption on x, using the Richness Lemma to construct the requisite frames. Hence there is an ill-founded tree in the collection: we shall call the frame on it a *pseudo-frame* to emphasize its ill-founded character. Any descending infinite path through the tree will therefore yield a sequence of points $x = y_0, y_1, \ldots$ such that for each i, $a \curvearrowright y_{i+1} \curvearrowright y_i$ and $y_1 \curvearrowright x$, proving that $x \in A^\infty$. $\dashv$ (7·11)

7·14 Corollary *For any f, a, x, if $x \in A^\xi \setminus A^{\xi+1}$ then ξ is recursive in a, x, f.*

7·15 Corollary *Suppose that $\theta(a, f) = \omega_1$. Then $\bigcup\{\omega_1^b \mid b \in E\} = \omega_1$. Hence E is in this case a complete $\mathbf{\Pi^1_1}$ set.*

Similarly,

7·16 Proposition *Let a and f be recursive. Let δ be the least non-recursive ordinal. If for each recursive ν $A^\nu(a)$ is non-empty, then A^∞ is non-empty.*

Proof : For each e coding a recursive ordinal ν there is a b and a ν-frame for b. Hence there is a pseudo-frame for some b. $\dashv$ (7·16)

Finally, we recast the above argument in terms of non-standard models, in analogy to one approach to the Kreisel–Lorenzen result.

7·17 Proposition *Let x be in an ill-founded ω-model N containing a and f with an ill-founded ordinal c such that $N \models x \mathrel{\epsilon} A^c$ and $N \models$ The Richness Lemma. Then $x \in A^\infty$.*

Proof : externally to N choose a descending sequence c_i of ordinals of N starting from $c_0 = c$. Set $y_0 = x$, and repeatedly apply the richness lemma to pick $y_{i+1} \in N$ such that $N \models a \curvearrowright y_{i+1} \curvearrowright y_i$ and $N \models y_i \epsilon A^{c_i}$. Then this sequence is genuinely a descending attacking sequence and so each y_i is in A^∞. $\dashv$ (7·17)

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