ON SOME ℓ-ADIC REPRESENTATIONS OF $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ ATTACHED TO NONCONGRUENCE SUBGROUPS.

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ABSTRACT. The ℓ-adic parabolic cohomology groups attached to noncongruence subgroups of $SL_2(\mathbb{Z})$ are finite-dimensional ℓ-adic representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ for some number field K. We exhibit examples (with $K = \mathbb{Q}$) for which the primitive parts give Galois representations whose images are open subgroups of the full group of symplectic similitudes (of arbitrary dimension). The determination of the image of the Galois group relies on Katz's classification theorem for semisimple subalgebras of $\mathfrak{sl}_n$ containing a principal nilpotent element, for which we give a short conceptual proof, suggested by I. Grojnowski.

1. INTRODUCTION

Let $\Gamma \subset PSL_2(\mathbb{Z})$ be a subgroup of finite index. In the papers [8, 9, 11] we studied ℓ-adic Galois representations attached to cusp forms on Γ. Attached to Γ is a certain field K_Γ and, for each even integer $k \geq 0$, a compatible system of ℓ-adic representations

$$\rho_\ell = \rho_{\ell,k,\Gamma} \colon \mathrm{Gal}(\overline{\mathbb{Q}}/K_\Gamma) \to GL_{2d}(\mathbb{Q}_\ell)$$

where $d = d_{k+2}$ is the dimension of the space $S_{k+2}(\Gamma)$ of cusp forms on Γ of weight $k + 2$. These representations are defined using ℓ-adic parabolic cohomology, and are a mild generalisation of the ℓ-adic representations of Deligne [3]. If Γ' is the smallest congruence subgroup of $SL_2(\mathbb{Z})$ containing Γ then $\rho_{\ell,k,\Gamma}$ contains as an invariant subspace the restriction of $\rho_{\ell,k,\Gamma'}$ to $\mathrm{Gal}(\overline{\mathbb{Q}}/K_\Gamma)$. The representation we are concerned with here is the quotient, which we denote $\rho_{\ell,k,\Gamma}^{\mathrm{prim}}$.

For any Γ the representations $\rho_{\ell,k,\Gamma}$ are the ℓ-adic realisations of a certain motive (in the sense of Grothendieck) $M_{k,\Gamma}$ defined over K_Γ. (For congruence subgroups this was shown in [10], and the trivial generalisation to other groups was explained in [11].) The Hodge type of $M_{k,\Gamma}$ is of the form $(k+1, 0)^d + (0, k+1)^d$, and so the representations $\rho_{\ell,k,\Gamma}$ are (by Faltings [5]) Hodge-Tate of the same type. Moreover by Deligne's proof of the Weil conjectures, they are pure of weight $k + 1$. As a final general remark, there is a perfect pairing

$$M_{k,\Gamma} \otimes M_{k,\Gamma} \to \mathbb{Q}(-k-1)$$

which is alternating (since k is even) and so the image of $\rho_{\ell,k,\Gamma}$ is (after suitable conjugation) contained in $GSp_{2d}(\mathbb{Q}_\ell)$, the group of symplectic similitudes. The

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same statements hold for the quotient $\rho_{\ell,k,\Gamma}^{\mathrm{prim}}$ (since it is the kernel of an algebraic projector, given by the trace from Γ' to Γ).

We considered in [11] the following three subgroups of $PSL_2(\mathbb{Z})$. Write Γ_{43} for the subgroup generated by the matrices

$$\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}$$

and Γ_{52} for the subgroup generated by

$$\begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}.$$

Thus Γ_{43} and Γ_{52} both have index 7 and two cusps, of widths 4 and 3 (5 and 2, respectively). Also let Γ_{711} be the subgroup of index 9 generated by

$$\begin{pmatrix} 1 & 7 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & -4 \\ 1 & 3 \end{pmatrix}$$

which has a cusp of width 7 and two cusps of width 1.

If Γ is one of these three groups then it can be shown (cf. [11], §4.9) that $K_\Gamma = \mathbb{Q}$. By applying standard formulae for the dimensions of spaces of modular forms, we find that in each case $\dim \rho_{\ell,k,\Gamma}^{\mathrm{prim}} = k$.

Using methods from algebraic geometry, and in particular the theory of vanishing cycles, we obtained in [11] a criterion for the image of ρ_ℓ to contain a unipotent element with a "long" Jordan block. In particular, in §4 of [11] the following result is proved:

Theorem 1. *Let Γ be one of Γ_{52}, Γ_{43}, Γ_{711}. Let $p = 7$, 7 or 2 respectively, and let $\ell \neq p$. Let $k \geq 2$ be even. Then the image under $\rho_{\ell,k,\Gamma}^{\mathrm{prim}}$ of an inertia subgroup at p contains a unipotent element X such that $(X - 1)^{k-1} \neq 0$.*

We now fix once and for all a prime ℓ different from the prime p of Theorem 1, and write C for the completion of the algebraic closure of $\mathbb{Q}_\ell$. Let $G_{k,\Gamma} \subset GSp_{k/C}$ be the connected component of the identity in the Zariski closure of the image of $\rho_{\ell,k,\Gamma}^{\mathrm{prim}}$. It is a connected algebraic group over C. In this paper we use Theorem 1 to prove:

Theorem 2. *Let Γ be as in Theorem 1, and $k \geq 2$ an even integer. Then $G_{k,\Gamma} = GSp_{k/C}$.*

By Bogomolov's theorem [1] it follows that the image of $\rho_{k,\Gamma}^{\mathrm{prim}}$ is an open subgroup of $GSp_k(\mathbb{Q}_\ell)$.

Apart from showing that the motives associated to non-congruence subgroups can in some sense be as general as possible, Theorem 2 also gives an explicit construction, for every even k and every prime ℓ, of an ℓ-adic representation

$\rho\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GSp_k(\mathbb{Q}_\ell)$ with open image, which occurs in the ℓ-adic cohomology of a smooth projective variety over $\mathbb{Q}$. It does not seem easy to produce examples of such representations by other methods.

These methods apply also to the case of k odd (although there is some ambiguity in the notion of field of definition for odd weight — see [8, Remark 5.10(iii)] for a discussion) and, although we have not checked all the details, it seems likely that one will obtain odd-dimensional representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ whose image is open in a group of orthogonal similitudes (except perhaps in the case $k = 7$, where a group of type G_2 might conceivably occur).

2. Number-theoretic part

In this section we reduce Theorem 2 to a Lie-theoretic statement. It is convenient to axiomatise the properties of $\rho^{\mathrm{prim}}_{\ell,k,\Gamma}$ we use. Assume that we have a $\mathbb{Q}_\ell$-vector space V of dimension $k \geq 2$ and a continuous representation $\rho\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Aut}(V)$. Let $G \subset GL(V)$ be the connected component of the identity in the Zariski closure of the image of ρ. Consider the following conditions on (ρ, V):

(H1) ρ is pure of some weight $w \in \mathbb{Z}$;

(H2) The restriction of ρ to $\mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$ is Hodge-Tate, with exactly two Hodge-Tate weights;

(H3) For some $p \neq \ell$ there is an open subgroup $I' \subset I_p$ of the inertia group at p such that the restriction of ρ to I' is unipotent and indecomposable.

We remind the reader: (H1) means that ρ is unramified outside a finite set S of primes, and that for all $p \notin S \cup \{\ell\}$ the eigenvalues of a geometric Frobenius at p are algebraic numbers, all of whose conjugates have absolute value $p^{w/2}$. As for (H2), write σ for the unique continuous action of $\mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$ on C extending the Galois action on $\overline{\mathbb{Q}}_\ell$. Let $\chi_{\mathrm{cycl}}\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{Z}_\ell^*$ be the cyclotomic character, so that for any ℓ^n-th root of unity $\eta \in \overline{\mathbb{Q}}$ and $g \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, $g(\eta) = \eta^{\chi_{\mathrm{cycl}}(g)}$, and set

$$V(i) = \left\{ v \in V \otimes_{\mathbb{Q}_\ell} C \mid (\rho \otimes \sigma)(g)v = \chi^i_{\mathrm{cycl}}(g)v \text{ for all } g \in \mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell) \right\}.$$

Then V is Hodge-Tate if the natural map $\bigoplus V(i) \otimes_{\mathbb{Q}_\ell} C \to V \otimes_{\mathbb{Q}_\ell} C$ is an isomorphism, and its Hodge-Tate weights are those i for which $V(i) \neq 0$. Finally, by the structure of the tame inertia group, (H3) is equivalent to the existence of some unipotent $X \in \rho(I_p)$ whose Jordan form has a single block.

In the case $\rho = \rho^{\mathrm{prim}}_{k,\Gamma}$, as explained in the Introduction, (H1) is satisfied with $w = k + 1$ and (H2) with weights $\{0, -k-1\}$. Condition (H3) is the content of Theorem 1.

Proposition 3. *Let* $\rho\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Aut}(V)$ *be a representation satisfying (H1) and (H3), and whose restriction to* $\mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$ *is Hodge-Tate. Then the restriction of* ρ *to any open subgroup of* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *is absolutely irreducible.*

Proof. Let $E/\mathbb{Q}_p$ be a finite extension such that I' is the inertia subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/E)$, let q be the order of the residue field of E, and let $\mathrm{Frob}_q \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/E)$ be a geometric Frobenius — that is, the inverse of any element lifting the q-power Frobenius on the residue field. In particular, $\chi_{\mathrm{cycl}}(\mathrm{Frob}_q) = q^{-1}$.

Hypothesis (iii) says that the Jordan normal form of $\rho|_{I'}$ has one block, so the invariants $V^{I'}$ form a 1-dimensional subspace of V, on which Frob_q acts as a scalar $\alpha \in \mathbb{Q}_\ell^*$. Picking $X \in \rho(I') - \{1\}$, set $N = \log X \colon V \to V$ (the "logarithm of monodromy at p" operator [4]), so that N is a nilpotent endomorphism of V satisfying $\rho(g)N = \chi_{\mathrm{cycl}}(g)N\rho(g)$ for any $g \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. Then $V^{I'} = \ker N$, and $\ker N^{j+1}/\ker N^j \simeq V \otimes \chi_{\mathrm{cycl}}^{-j}$ for $0 \leq j < k$. Therefore the complete set of eigenvalues of $\rho(\mathrm{Frob}_q)$ is $\{\alpha q^j \mid 0 \leq j \leq k-1\}$.

Recall that if $\chi\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{Q}_\ell^*$ is a continuous homomorphism whose restriction to the decomposition group at ℓ is Hodge-Tate, then χ is the product of an integral power of χ_{cycl} and a character of finite order. So there exists an integer m and a character ϵ of finite order such that $\det \rho = \chi_{\mathrm{cycl}}^{-m}\epsilon$. By hypothesis (i), $m = wk/2$. Then

$$\det \rho(\mathrm{Frob}_q) = \prod_{j=0}^{k-1} \alpha q^j = q^m \epsilon(\mathrm{Frob}_q)$$

and so α is the product of $q^{(w-k+1)/2}$ and a root of unity.

Now if $V' \subset V$ is a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant subspace of dimension $k' > 0$ then V and V' have the same space of I'-invariants (since $V^{I'}$ is 1-dimensional) and V' satisfies the hypotheses of the Proposition. Therefore the previous argument applied to V' gives

$$|\alpha| = q^{(w-k'+1)/2}, \qquad i.e. \quad k' = k.$$

So ρ is irreducible. Finally, let U be any subspace of the space of ρ which is invariant under some open subgroup $H \subset \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, and let $g \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Then U and $\rho(g)U$ are both invariant under the open subgroup $I'' = H \cap H^g \cap I'$ of I_p. But since the action of I'' also has one Jordan block, it has a unique invariant subspace of each dimension. So $\rho(g)U = U$, hence U is invariant under $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. So as ρ is irreducible, its restriction to H is also irreducible.

Finally, the same argument carries through if we replace $\mathbb{Q}_\ell$ by a finite extension, so the restriction of ρ to any open subgroup is absolutely irreducible. $\square$

As a consequence, since G contains the image of an open subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, it acts (absolutely) irreducibly on V, and therefore (being connected by definition) it is reductive. In particular, for $k = \dim(V) = 2$ we have $G = GL_2$. Henceforth we assume that (ρ, V) satisfies hypotheses (H1)–(H3) above, and that $k > 2$.

Let $\mathfrak{g} = \mathrm{Lie}\, G \cap \mathfrak{sl}(V \otimes C)$; since V is an irreducible G-module, the centre of $\mathrm{Lie}\, G$ has dimension at most one, hence $\mathfrak{g}$ is a semisimple Lie algebra over C. By (H3) there exists a unipotent element of G whose Jordan decomposition has one

block; let $x \in \mathfrak{g}$ be its logarithm. Then x is a nilpotent element of $\mathfrak{g}$ which has just one Jordan block, viewed as an endomorphism of $V \otimes C$.

Now recall the 1-dimensional Hodge-Tate torus associated to ρ (as a representation of the local Galois group). Let $H_\ell \subset GL(V \otimes_{\mathbb{Q}_\ell} C)$ be the Zariski closure of the the image of $\mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$ by ρ. Since ρ is Hodge-Tate, there is a unique homomorphism $\zeta \colon \mathbb{G}_m \to H_\ell$ for which $V(i) \otimes C$ is the eigenspace of the character $t \mapsto t^i$ of $\mathbb{G}_m$. (See [12], §1.4, where ζ is denoted h_V.) Passing to the Lie algebra, there is a unique semisimple element $z_{HT} = d\zeta \in \mathrm{Lie}\, H_\ell \subset \mathrm{Lie}\, G$ such that $V(i) \otimes C = \ker(z_{HT} - i) \subset V \otimes C$.

We now appeal to the following result of Katz (the Classification Theorem 9.10 in [6]):

Proposition 4. *Let V be a finite-dimensional vector space of dimension k over an algebraically closed field of characteristic zero, and $\mathfrak{g}$ a semisimple Lie subalgebra of $\mathfrak{sl}(V)$. Assume that $\mathfrak{g}$ contains a nilpotent element x which as an endomorphism of V has only one Jordan block. Then one of the following holds:*

  (i) *$\mathfrak{g} \simeq \mathfrak{sl}_2$ with $V \simeq \mathrm{Sym}^{k-1}$.*
  (ii) *$\mathfrak{g} = \mathfrak{sl}(V)$;*
  (iii) *k is even, $\mathfrak{g} = \mathfrak{sp}(V)$ for a nondegenerate alternating form on V;*
  (iv) *k is odd, $\mathfrak{g} = \mathfrak{so}(V)$ for a nondegenerate symmetric form on V;*
  (v) *$k = 7$ and $\mathfrak{g}$ is of type G_2.*

The hypothesis (H2) enables us to eliminate the case (i) unless $k = 2$. Indeed, the Hodge-Tate element $z_{HT} \in \mathrm{Lie}\, G$ then has exactly 2 eigenvalues, namely the (integral) Hodge-Tate weights of ρ. Therefore for some $a, b \in \mathbb{Z}$, $az_{HT} + b$ is a semisimple element of $\mathfrak{g}$ with exactly 2 eigenvalues. However in the representation Sym^{k-1} of $\mathfrak{sl}_2$, every non-zero semisimple element has k distinct eigenvalues.

This completes the proof of Theorem 2, since in that case, k is even and V has a $\mathfrak{g}$-invariant symplectic form, so we must be in case (iii).

In an earlier version of this paper we gave an ugly proof of the only case of Proposition 4 needed here (k even, $\mathfrak{g} \subset \mathfrak{sp}(V)$), involving a detailed case-by-case analysis of minuscule representations of $\mathfrak{g}$. Subsequently Laumon pointed out to me that this was a special case of Katz's result, whose proof also depends on a (longer) case-by-case analysis. I am grateful to Ian Grojnowski for suggesting a short proof of Katz's general result along the lines given in the next section.

3. Lie-theoretic part

Assume $\mathfrak{g}$ satisfies the hypotheses of Proposition 4. By the Jacobson-Morosov theorem, there is a homomorphism $\lambda_x \colon \mathfrak{sl}_2 \to \mathfrak{g}$ such

$$\lambda_x \colon \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto x.$$

Write $\mathfrak{a}_x \subset \mathfrak{g}$ for the image of λ_x, and

$$y = \lambda_x \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad h = \lambda_x \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(In the terminology of [2], VIII.11.1, (x, h, y) is an $\mathfrak{sl}_2$-triplet.)

We first observe:

Lemma 5. $\mathfrak{g}$ *is simple.*

Proof. Suppose that $\mathfrak{g} = \mathfrak{g}_1 \times \mathfrak{g}_2$ with $\mathfrak{g}_i$ nonzero. As V is an irreducible $\mathfrak{g}$-module, it factorises as a tensor product of irreducible $\mathfrak{g}_i$-modules V_i. But since x has maximal rank, the restriction of V to $\mathfrak{a}_x$ is an irreducible representation of $\mathfrak{sl}_2$, and the tensor product of two non-trivial representations of $\mathfrak{sl}_2$ is never irreducible, by the Clebsch-Gordan formula. $\square$

The triple (x, h, y) is a principal $\mathfrak{sl}_2$-triplet in $\mathfrak{sl}(V)$, since x has maximal rank, and so is also a principal $\mathfrak{sl}_2$-triplet in $\mathfrak{g}$ ([2] VIII.11.4). Let n be the rank of $\mathfrak{g}$ and $1 \leq r_1 \leq r_2 \leq \cdots \leq r_n$ be the exponents of its root system. Then one knows ([7], or see for example [2] VIII.11, exercise 11) that under the adjoint action of $\mathfrak{a}_x \simeq \mathfrak{sl}_2$, $\mathfrak{g}$ decomposes as the direct sum of the irreducible representations Sym^{2r_i}.

For the adjoint action of $\mathfrak{a}_x$ on $\mathfrak{sl}(V)$ the exponents are $\{1, 2, \ldots, k-1\}$ and one can write down the decomposition into irreducibles totally explicitly: consider the matrix powers $x^r \in \mathfrak{sl}(V)$ for $1 \leq r \leq k-1$. Let U_r be the $\mathfrak{a}_x$-submodule of $\mathfrak{sl}(V)$ generated by x^r. Obviously $\mathrm{ad}(x)x^r = 0$, and since $[h, x] = 2x$ one gets $\mathrm{ad}(h)x^r = 2rx^r$. Thus x^r is a highest weight vector in U_r, which is isomorphic to Sym^{2r}, and a basis for U_r is given by $\{\mathrm{ad}(y)^i x^r \mid 0 \leq i \leq 2r\}$. Thus

$$\mathfrak{sl}(V) = \bigoplus_{r=1}^{k-1} U_r.$$

Therefore $\mathfrak{g} = \bigoplus_{i=1}^{n} U_{r_i}$. In particular this proves part (i) of the following Lemma.

Lemma 6. *(i) The exponents of $\mathfrak{g}$ are distinct and satisfy $r_i \leq k-1$.*

(ii) If r and s are exponents of $\mathfrak{g}$ and $r + s \leq k$ then $r + s - 1$ is also an exponent of $\mathfrak{g}$.

of (ii). As the Lie bracket $\mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ is $\mathfrak{a}_x$-equivariant and $U_r \simeq \mathrm{Sym}^{2r}$, by the Clebsch-Gordan formula we see that if $r \geq s$ then

$$[U_r, U_s] = \bigoplus_{t \in T} U_t$$

for some subset $T \subset \{t \in \mathbb{Z} \mid r - s \leq t \leq \min(r+s, k-1)\}$. If $r + s \in T$ then the Lie bracket would give a non-zero pairing $U_r \otimes U_s \to U_{r+s}$, which would necessarily be non-zero on the tensor product of the highest weight vectors. But $[x^r, x^s] = 0$,

hence $r + s \notin T$. On the other hand, since x^s is a highest weight vector for U_s one has $\mathrm{ad}(x)\,\mathrm{ad}(y)x^s = 2sx^s$, and therefore

$$[x^r, \mathrm{ad}(y)x^s] = x[x^{r-1}, \mathrm{ad}(y)x^s] + [x, \mathrm{ad}(y)x^s]x^{r-1} = x[x^{r-1}, \mathrm{ad}(y)x^s] + 2sx^{r+s-1}$$

and so by induction one obtains

$$[x^r, \mathrm{ad}(y)x^s] = 2rsx^{r+s-1}.$$

Therefore $[U_r, U_s] \supset U_{r+s-1}$ if $r + s \leq k$. As $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{sl}(V)$, the Lemma follows. □

So to finish the proof, it suffices to determine those simple Lie algebras which admit a representation of dimension k and whose exponents satisfy the conditions of Lemma 6. From standard tables (for example [2], Chapters IV and VIII) one extracts the information contained in the table below.

$\mathfrak{g}$	exponents of $\mathfrak{g}$	least dimensions of representations
A_n	$1, 2, 3, \ldots, n$	$n + 1,\ n(n + 1)/2$
B_n,	$1, 3, 5, \ldots, 2n - 1$	$2n + 1,\ n(2n + 1)$
C_n,	$1, 3, 5, \ldots, 2n - 1$	$2n,\ n(2n - 1)$
D_n	$1, 3, 5, \ldots, 2n - 3,\ n - 1$	
E_6	$1, 4, 5, 7, 8, 11$	
E_7	$1, 5, 7, 9, 11, 13, 17$	
E_8	$1, 7, 11, 13, 17, 19, 23, 29$	
F_4	$1, 5, 7, 11$	
G_2	$1, 5$	$7,\ 14$

From this one sees that the only cases satisfying the hypotheses of Lemma 6 are: A_1 with k arbitrary; A_n with $k = n + 1$; B_n with $k = 2n + 1$; C_n with $k = 2n$; and G_2 with $k = 7$, which are precisely those cases listed in Proposition 4.

The author would like to thank D. Blasius, G. Harder, and J-P. Serre for useful discussions, G. Laumon for drawing his attention to [6], and especially I. Grojnowski for suggesting the proof of Proposition 4.

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