An introduction to Kato's Euler systems

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Contents

Reprinted from 'Galois Representations in Arithmetic Arithmetic Geometry',
edited by A. J. Scholl & R. L. Taylor.

Introduction

In the conference there was a series of talks devoted to Kato's work on the Iwasawa theory of Galois representations attached to modular forms. The present notes are mainly devoted to explaining the key ingredient, which is the Euler system constructed by Kato, first in the K_2-groups of modular curves, and then using the Chern class map, in Galois cohomology. This material is based mainly on the talks given by Kato and the author at the symposium, as well as a series of lectures by Kato in Cambridge in 1993. In a companion paper [29] Rubin explains how, given enough information about an Euler system, one can prove very general finiteness theorems for Selmer groups whenever the appropriate L-function is non-zero (see §8 of his paper for precise results for elliptic curves).

Partly because of space, and partly because of the author's lack of understanding, the scope of these notes is limited. There are two particular restrictions. First, we only prove the key reciprocity law (Theorem 3.2.3 below), which allows one to compute the image of the Euler system under the dual exponential map, in the case of a prime p of good reduction (actually, for stupid reasons explained at the end of §2.1, we also must assume p is odd). Secondly, we say nothing about the case of Galois representations attached to forms of weight greater than 2. For the most general results, the reader will need to consult the preprint [17] and Kato's future papers.

Kato's K_2 Euler system has its origins in the work of Beilinson [1] (see also [30] for a beginner's treatment). Beilinson used cup-products of modular units to construct elements of K_2 of modular curves. He was able to compute the regulators of these elements by the Rankin-Selberg method and relate them to the L-function of the modular curve at $s = 2$, in partial confirmation of his general conjectures [1; 27] relating regulators and values of L-functions.

Kato discovered that, by using explicit modular units, one obtained norm-compatible families of elements of K_2. These modular units are the values, at torsion points, of what are called here *Kato-Siegel functions*. These are *canonical* (no indeterminate constant) functions on an elliptic curve (over any base scheme) with prescribed divisors, which are norm-compatible with respect to isogenies. Such functions were, over $\mathbb{C}$, first discovered by Siegel — the associated modular units were studied in depth by Kubert and Lang [19]. Over $\mathbb{C}$ generalisations of these functions were found by Robert [28]. It was Kato [16] who first found their elegant algebraic characterisation. In §1 I have given an "arithmetic" modular construction of these functions, which is more complicated than Kato's but at least reveals the key fact behind their existence — namely, the triviality of the 12th power of the sheaf $\boldsymbol{\omega}$ on the modular stack. (The Picard group of the modular stack was computed by Mumford [25] many years ago.)

In §2 we turn to K-theory, and give a fairly general construction of the Euler system in K_2 of modular curves, and the norm relations. It is relatively formal to pass from this to an Euler system in Galois cohomology of (say) a modular elliptic curve. The hard part is to show that the cohomology classes one gets are non-trivial if the appropriate L-value is nonzero. In his 1993 Cambridge lectures, Kato explained how this can be regarded as a consequence of a huge generalisation of the explicit reciprocity laws (Artin, Hasse, Iwasawa, Wiles ...) to local fields with imperfect residue field. This is the subject of the preprint [17]. At the Durham conference he sketched a slightly different proof, using the Fontaine-Hyodo-Faltings approach to p-adic Hodge theory. In §3 we give a stripped-down proof of a weak version of one of the reciprocity laws in [17] in the case of good reduction, using a minimal amount of p-adic Hodge theory.

In §4 we explain how Kato uses the Rankin-Selberg integral (very much as Beilinson did) to compute the projection of the the image of the dual exponential into a Hecke eigenspace. Finally in §5 we tie everything together for a modular elliptic curve.

The appendix to §2 (which is the author's only original contribution to this work) is an attempt to extend Kato's methods to other situations. We construct an Euler system in the higher K-groups of (a suitable open part of) Kuga-Sato varieties. This is a precise version of the construction used in [32] (see also §5 of [9] for a summary) to relate archimedean regulators of modular form motives and L-functions. The p-adic applications of these elements remain to be found.

I have many people to thank for their help in the preparation of this paper. Particular mention is due to Jan Nekovář. He encouraged me to think about norm relations in 1994, although in the end that work was overtaken by events, and all that remains of it is the appendix to §2. It is only because of his insistence that §§3–5 exist at all, and his careful reading of much of the paper eliminated many errors (although he is not to be held responsible for those that remain). I am also grateful to Amnon Besser, Spencer Bloch, Kevin Buzzard, John Coates, Ofer Gabber, Henri Gillet, Erasmus Landvogt and Christophe Soulé for useful discussions. Karl Rubin read the original draft of the manuscript and made invaluable suggestions. Above all, it is a great pleasure to thank Kato, the creator of this beautiful and powerful mathematics, for encouraging me to publish this account of his work and for pointing out some blunders in an earlier draft.

This paper was begun while the author was visiting the University of Münster in winter 1996 as a guest of Christopher Deninger, and completed during a stay at the Isaac Newton Institute in 1998. It is a pleasure to thank them for their hospitality.

Notation

If G is a commutative group (or group scheme) and $n \in \mathbb{Z}$ then $[\times n]_G \colon G \to G$ is the endomorphism "multiplication by n", written simply $[\times n]$ if no confusion can occur. We also write ${}_nG$ and G/n for the kernel and cokernel of $[\times n]$, respectively.

Throughout this paper we use the geometric Frobenius, and normalise the reciprocity laws of class field theory accordingly (see §3.1 below for precise conventions, as well as the remarks following Theorem 5.2.1).

The symbol "=" is used to denote equality or canonical isomorphism. We use the usual notation ":=" to indicate that the right-hand expression is the definition of that on the left (and "=:" for the reflected relation).

1 Kato-Siegel functions and modular units

1.1 Review of modular forms and elliptic curves

We review some well-known facts about the moduli of elliptic curves. See for example [7; 8; 18, Chapter 2]. For any elliptic curve $f \colon E \to S$, with zero-section e, we have the standard invertible sheaf

$$\boldsymbol{\omega}_{E/S} := f_*\Omega^1_{E/S} = e^*\Omega^1_{E/S}.$$

From the second description (as the conormal bundle of the zero-section of E/S) we have the isomorphism $\boldsymbol{\omega}_{E/S} = e^*\mathcal{O}_E(-e)$. Because $\Omega^1_{E/S}$ is free along the fibres of f, in fact $\boldsymbol{\omega}_{E/S} = x^*\Omega^1_{E/S}$ for any section $x \in E(S)$.

The formation of $\boldsymbol{\omega}_{E/S}$ is compatible with basechange — in fancy language, $\boldsymbol{\omega}$ is a sheaf on the *modular stack* $\mathcal{M}$ of elliptic curves.

A (meromorphic) modular form of weight k is a rule which assigns to each E/S a section of $\boldsymbol{\omega}_{E/S}^{\otimes k}$, compatible with basechange. By definition this is the same as an element of $\Gamma(\mathcal{M}, \boldsymbol{\omega}^{\otimes k})$. The discriminant $\Delta(E/S)$ is a nowhere-vanishing section of $\boldsymbol{\omega}_{E/S}^{\otimes 12}$ compatible with basechange, and it defines an invertible modular form Δ of weight 12. From this it follows in particular that

- The set of nowhere-vanishing sections of $\boldsymbol{\omega}^{\otimes 12d}$ is $\{\pm\Delta^d\}$, for any integer d.

Let $N \geq 1$ be an integer. The modular stack $\mathcal{M}_{\Gamma_0(N)}$ classifies pairs $(E/S, \alpha)$ where $\alpha\colon E \to E'$ is a cyclic isogeny of degree N of elliptic curves over S. (When N is not invertible on S the definition of cyclic can be found in [18, §3.4].) The functor $(E, \alpha) \mapsto E$ defines a morphism $c\colon \mathcal{M}_{\Gamma_0(N)} \to \mathcal{M}$. A (meromorphic) modular form on $\Gamma_0(N)$ of weight k is a section of $c^*\boldsymbol{\omega}^{\otimes k}$ over $\mathcal{M}_{\Gamma_0(N)}$. Equivalently, it is a rule which associates to each cyclic N-isogeny $\alpha\colon E \to E'$ of elliptic curves over S a section of $\boldsymbol{\omega}_{E/S}^{\otimes k}$, compatible with arbitrary basechange $S' \to S$. As well as Δ, one has the modular form $\Delta^{(N)}$ of weight 12, defined by

$$\Delta^{(N)}(E \xrightarrow{\alpha} E') = \alpha^*\Delta(E').$$

It is invertible exactly where α is étale. In particular, it is invertible on $S \otimes \mathbb{Z}[1/N]$.

Suppose $N = p$ is prime. The reduction of $\mathcal{M}_{\Gamma_0(p)}$ mod p has two irreducible components, one of which parameterises pairs $(E/S, \alpha)$ where α is Frobenius, and the other those pairs where α is Verschiebung. On the first component $\Delta^{(p)}$ vanishes, and on the second it does not.

Let m be the denominator of $(p-1)/12$. Then $\Delta^{(p)} \cdot \Delta^{-1}$ is the m^{th} power of a modular function $u_p \in \Gamma(\mathcal{M}_{\Gamma_0(p)}, \mathcal{O})$, which is invertible away from characteristic p by the previous remarks. It is a classical fact [26] that

$$\Gamma(\mathcal{M}_{\Gamma_0(p)} \otimes \mathbb{Q}, \mathcal{O}^*) = \langle \mathbb{Q}^*, u_p \rangle.$$

and therefore

$$\Gamma(\mathcal{M}_{\Gamma_0(p)}, \mathcal{O}^*) = \{\pm 1\}.$$

Recall the Kodaira-Spencer map (see e.g. [18, 10.13.10]); if E/S is an elliptic curve and S is smooth over T, one has an $\mathcal{O}_S$-linear map

$$KS = KS_{E/S}\colon \boldsymbol{\omega}_{E/S}^{\otimes 2} \to \Omega^1_{S/T}.$$

If $T = \operatorname{Spec} \mathbb{Q}$ and E/S is the universal elliptic curve over the modular curve $Y(N)$, $N \geq 3$ (the definition is recalled in §2.2 below), then KS is an isomorphism.

If $S \hookrightarrow \overline{S}$, $E \hookrightarrow \overline{E}$ is an extension of E to a curve $\overline{E}/\overline{S}$ of genus 1 (not necessarily smooth), and the identity section $e \in E(S)$ extends to a a section $e\colon \overline{S} \to \overline{E}$ whose image is contained in the smooth part, then $\boldsymbol{\omega}_{\overline{E}/\overline{S}} := e^*\Omega_{\overline{E}/\overline{S}}$ is an invertible sheaf on $\overline{S}$ extending $\boldsymbol{\omega}_{E/S}$. If $\overline{S}$ is smooth over the base scheme T, and S is the complement in $\overline{S}$ of a divisor S^∞ with relative normal crossings, the Kodaira-Spencer map extends to a homomorphism

$$KS_{\overline{E}/\overline{S}}\colon \boldsymbol{\omega}_{\overline{E}/\overline{S}}^{\otimes 2} \to \Omega^1_{\overline{S}/T}(\log S^\infty). \tag{1.1.1}$$

If $\overline{S} = X(N)_{/\mathbb{Q}}$ for $N \geq 3$ and $\overline{E}$ is the regular minimal model of the universal elliptic curve, then (1.1.1) is an isomorphism.

1.2 Kato-Siegel functions

If $\mathfrak{D}$ is a principal divisor on an elliptic curve over (say) a field, there is in general no 'canonical' function with divisor $\mathfrak{D}$. For certain special divisors, such canonical functions do exist. In their analytic construction they have been used extensively in the theory of elliptic units. Kato observed that they have a completely algebraic characterisation. Here we give a slightly more general, modular, description of such a class of functions.

Theorem 1.2.1. *Let D be an integer with $(6, D) = 1$. There is one and only one rule ϑ_D which associates to each elliptic curve $E \to S$ over an arbitrary base a section $\vartheta_D^{(E/S)} \in \mathcal{O}^*(E - \ker[\times D])$ such that:—*

(i) as a rational function on E, $\vartheta_D^{(E/S)}$ has divisor $D^2(e) - \ker[\times D]$;

(ii) if $S' \to S$ is any morphism, and $g\colon E' = E \times_S S' \to E$ is the basechange, then $g^*\vartheta_D^{(E/S)} = \vartheta_D^{(E'/S')}$;

(iii) if $\alpha\colon E \to E'$ is an isogeny of elliptic curves over a connected base S whose degree is prime to D, then

$$\alpha_*\vartheta_D^{(E/S)} = \vartheta_D^{(E'/S)}$$

(iv) $\vartheta_{-D} = \vartheta_D$ and $\vartheta_1 = 1$. If $D = MC$ with M, $C \geq 1$ then

$$[\times M]_*\vartheta_D = \vartheta_C^{M^2} \text{ and } \vartheta_C \circ [\times M] = \vartheta_D/\vartheta_M^{C^2}.$$

In particular, $[\times D]_*\vartheta_D = 1$.

(v) if $\tau \in \mathbb{C}$ with $\mathrm{Im}(\tau) > 0$ and $E_\tau/\mathbb{C}$ is the elliptic curve whose points are $\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$, then $\vartheta_D^{(E_\tau/\mathbb{C})}$ is the function

$$(-1)^{\frac{D-1}{2}}\Theta(u,\tau)^{D^2}\Theta(Du,\tau)^{-1}$$

where

$$\Theta(u,\tau) = q^{\frac{1}{12}}(t^{\frac{1}{2}} - t^{-\frac{1}{2}})\prod_{n>0}(1 - q^n t)(1 - q^n t^{-1})$$

and $q = e^{2\pi i\tau}$, $t = e^{2\pi i u}$.

Remarks. (i) We do not require that D be invertible on S.

(ii) Locally for the Zariski topology, any elliptic curve may be obtained by basechange from an elliptic curve over a reduced base. It is therefore enough to restrict to reduced base schemes S.

(iii) Properties (i) and (iii) alone already determine $\vartheta_D^{(E/S)}$ uniquely; any other function with the same divisor is of the form $u\vartheta$, for some $u \in \mathcal{O}^*(S)$, and applying (ii) for the isogenies $[\times 2]$, $[\times 3]$ would give $u^4 = u = u^9$, whence $u = 1$.

(iv) In down-to-earth terms, if $S = \operatorname{Spec} k$ for an algebraically closed field k then for a separable isogeny $\alpha\colon E \to E'$, the property (iii) is just the *distribution relation*

$$\prod_{\substack{x \in E(k) \\ \alpha(x) = y}} \vartheta_D^{(E/k)}(x) = \vartheta_D^{(E'/k)}(y), \quad \text{for any } y \in E'(k).$$

(v) Over $\mathbb{C}$ this theorem was obtained by Robert [28], who proves rather more: he shows that for any elliptic curve $E/\mathbb{C}$ and any finite subgroup $P \subset E$ of order prime to 6, there is a certain canonical function with divisor $\#P(e) - P$ and properties generalising those of ϑ_D. One can prove his more general result in a manner similar to the proof of 1.2.1; in place of the modular form Δ^{D^2-1} one should use $\Delta(E)^{\#P}/\beta^*\Delta(E/P)$, where $\beta\colon E \to E/P$ is the quotient map.

Proof. We begin with the first two conditions. First observe that if S is a spectrum of a field, then the divisor $\ker[\times D] - D^2(e)$ is principal (because D is odd, the sum of $\ker[\times D] - D^2(e)$ in the Jacobian is zero). To give a rule ϑ_D satisfying (i) and (ii) is equivalent to giving, for any elliptic curve E/S, an isomorphism of line bundles on E

$$\mathcal{O}_E(\ker[\times D]) \xrightarrow{\sim} \mathcal{O}_E(D^2 e) \tag{1.2.2}$$

compatible with basechange. We have just observed that the line bundles are isomorphic when restricted to any fibre of E/S. Since we can assume (by remark (ii) above) that S is reduced, the seesaw theorem tells us that to give an isomorphism (1.2.2) is equivalent to giving an isomorphism of the restriction of the bundles to the zero-section. In other words, the existence of ϑ_D is equivalent to finding, for each E/S, a trivialisation of the bundle

$$\begin{aligned} e^*\mathcal{O}_E(\ker[\times D]) \otimes e^*\mathcal{O}_E(D^2 e)^\vee &= e^*[\times D]^*\mathcal{O}_E(e) \otimes e^*\mathcal{O}_E(-D^2 e) \\ &= e^*\mathcal{O}_E(e)^{\otimes(1-D^2)} \\ &= \boldsymbol{\omega}_{E/S}^{\otimes(D^2-1)} \end{aligned}$$

compatible with base-change. Note that $(6, D) = 1$ implies $D^2 \equiv 1 \pmod{12}$. There are then exactly two non-vanishing sections of $\boldsymbol{\omega}_{E/S}^{\otimes(D^2-1)}$ compatible with arbitrary basechange, namely $\pm\Delta(E/S)^{(D^2-1)/12}$. Choose one of them, and let $\phi^{(E/S)}$ be the corresponding function on $E - \ker[\times D]$. So the rule

($\phi\colon E/S \mapsto \phi^{(E/S)}$) satisfies properties (i) and (ii). In a moment we will see that exactly one of $\pm\phi$ satisfies (iii). (See also remark 1.2.3 below).

By the basechange compatibility (ii), we are free to make any faithfully flat basechange in order to check (iii). There exists such a basechange over which α factors as a product of isogenies of prime degree. It is therefore enough to verify (iii) when $\deg\alpha = p$ is prime. The quotient

$$g_p(E/S,\alpha) = \alpha_*\phi^{(E/S)}(\phi^{(E'/S)})^{-1} \in \mathcal{O}^*(S)$$

is compatible with basechange. It therefore defines a modular unit $g_p \in \Gamma(\mathcal{M}_{\Gamma_0(p)}, \mathcal{O}^*)$, and so $g_p(E/S,\alpha) \in \{\pm 1\}$ for every $(E/S,\alpha)$. Moreover the sign depends only on p.

To determine the sign, evaluate $g_p(E/\mathbb{F}_p, F_E)$ for an elliptic curve over $\mathbb{F}_p$ and its Frobenius endomorphism. The norm map $F_{E*}\colon \kappa(E)^* \to \kappa(E)^*$ is then the identity map, so $g_p(E,F_E) = 1$, and therefore if p is odd we have $g_p = +1$. Notice that replacing $\phi^{(E/S)}$ by $-\phi^{(E/S)}$ does not change g_p for p odd, but replaces g_2 by $-g_2$. Therefore for exactly one choice $\vartheta_D = \pm\phi$ it will be the case that $g_2 = +1$, so exactly one of these choices satisfies (iii).

Now for property (iv). Evidently ϑ_{-D} also satisfies the characteristic properties (i) and (iii), hence $\vartheta_{-D} = \vartheta_D$. Also $\vartheta_1 = 1$ for the same reason. The function $[\times M]_*\vartheta_D$ has divisor $M^2(C^2(e) - \ker[\times C])$ and is compatible with base change, so we can write $[\times M]_*\vartheta_D = \varepsilon\vartheta_C^{M^2}$ for some $\varepsilon = \pm 1$. Now property (iii) gives

$$\begin{aligned}\varepsilon\vartheta_C^{M^2} &= [\times M]_*\vartheta_D = [\times M]_*[\times 2]_*\vartheta_D = \\ &= [\times 2]_*[\times M]_*\vartheta_D = [\times 2]_*(\varepsilon\vartheta_C^{M^2}) = \varepsilon^4\vartheta_C^{M^2} = \vartheta_C^{M^2}\end{aligned}$$

and so $\varepsilon = 1$. The same calculation works for $M = D$ by writing $\vartheta_1 = 1$.

If $D = MC$ then the functions $\vartheta_C \circ [\times M]$ and $\vartheta_D/\vartheta_M^{C^2}$ both have divisor

$$C^2 \ker[\times M] - \ker[\times D]$$

hence their ratio is a unit compatible with basechange. The norm compatibility (iii) then shows that this unit equals 1, as in Remark (iii) above. This proves property (iv).

Finally we check (v). Classical formulae (as can for example be found in [39] — the function Θ is essentially the same as the Jacobi theta function ϑ_1) show that

$$F(u,\tau) = \Theta(u,\tau)^{D^2}\Theta(Du,\tau)^{-1}$$

is a function on E_τ with divisor $D^2(e) - \ker[\times D]$, and is $SL_2(\mathbb{Z})$-invariant.[1] Hence $F(u,\tau)$ is a constant multiple (independent of τ) of $\vartheta_D^{(E_\tau/\mathbb{C})}$. As a

[1] The $SL_2(\mathbb{Z})$ action is: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} : (u,\tau) \mapsto \left(\dfrac{u}{c\tau+d}, \dfrac{a\tau+b}{c\tau+d}\right)$.

formal power series,

$$F = q^{(D^2-1)/12} t^{-D(D-1)/2} \prod_{n\geq 0} \frac{(1-q^n t)^{D^2}}{1-q^n t^D} \prod_{n>0} \frac{(1-q^n t^{-1})^{D^2}}{1-q^n t^{-D}}$$

is a unit in the ring of Laurent q-series with coefficients in $\mathbb{Z}[t, 1/t(1-t^D)]$. So by the q-expansion principle, the constant has to be ± 1. To determine the sign, consider any elliptic curve E_τ defined over $\mathbb{R}$ with 2 real connected components. For such a curve one can assume that $\mathrm{Re}(\tau) = 0$. (To be definite, take E to be the curve

$$Y^2 = X^3 - X$$

for which $\tau = i$.) The real components of E_τ are the images of the line segments $[0,1]$ and $[\tau/2, 1+\tau/2]$ in the complex plane. We compute $[\times 2]_* F(u,\tau)$ for such a curve. The explicit formula for $\Theta(u,\tau)$ shows that the first non-vanishing u-derivative of $F(u,\tau)$ at the origin is real and positive. On the interval $[0,1]$, $F(u,\tau)$ has simple poles at $u = k/D$ $(1 \leq k \leq D-1)$ and so by calculus $(-1)^{(D-1)/2} F(1/2,\tau) > 0$. On the segment $[\tau/2, 1+\tau/2]$, F is real, finite and non-zero, hence the product $F(\tau/2,\tau)F((1+\tau)/2,\tau)$ is positive. Therefore at the origin,

$$\begin{aligned}[\times 2]_* F(u,\tau) &= F(\frac{u}{2},\tau)F(\frac{u+1}{2},\tau)F(\frac{u+\tau}{2},\tau)F(\frac{u+1+\tau}{2},\tau)\\ &\sim (-1)^{(D-1)/2} \times (\text{positive real}) \times u^{D^2-1}\end{aligned}$$

and so $[\times 2]_* F(u,\tau) = (-1)^{(D-1)/2} F(u,\tau)$. □

Remark 1.2.3. As we saw in the proof, ϑ_D corresponds to one of the two nowhere-vanishing modular forms of weight $(D^2-1)/12$. Using (v) it is easy to determine which. The form arises by restriction to the zero-section of the composite isomorphism

$$[\times D]^* \mathcal{O}_E(e) \xleftarrow{\sim} \mathcal{O}_E(\ker[\times D]) \xrightarrow[\times \vartheta_D^{(E/S)}]{\sim} \mathcal{O}_E(D^2 e)$$

since $e^*[\times D]^* = e^*$. Therefore the q-expansion is D times the leading coefficient in the expansion of $\vartheta_D^{(E_\tau/\mathbb{C})}$ in powers of t, which from (v) is easily seen to be

$$(-1)^{\frac{D-1}{2}} q^{\frac{D^2-1}{12}} \prod_{n>0} (1-q^n)^{2D^2-2} = (-1)^{\frac{D-1}{2}} \Delta(\tau)^{\frac{D^2-1}{12}}$$

Remark. Suppose that E/S is an elliptic curve over an integral base S, and that $P \subset E(S)$ is a finite group of sections. Let

$$\mathfrak{D} = \sum_{x\in P} m_x(x) \in \mathbb{Z}[P]$$

be a divisor with $\sum m_x = 0$ and $\sum m_x x = e$. In the case when S is the spectrum of a field, $\mathfrak{D}$ is principal, but in general this will not be the case. For example, suppose that $P = \{e, x\}$ for a section x of order 2, disjoint from e. Then $\mathfrak{D} = 2(x) - 2(e)$ is principal if and only if $\boldsymbol{\omega}_{E/S}^{\otimes 2} = e^*\mathcal{O}(\mathfrak{D})$ is trivial.

Consider a Dedekind domain R containing $1/2$, and an ideal $A \subset R$ which has order 4 in $\operatorname{Pic} R$. Let $A^4 = (a)$ and let E/R be the elliptic curve given by the affine equation

$$y^2 = x(x^2 - a)$$

over the field of fractions of R. Take an open $U \subset \operatorname{Spec} R$ over which A becomes principal, locally generated by α, say. Then $a = \alpha^4\varepsilon$ for some unit $\varepsilon \in \mathcal{O}(U)^*$, and an equation for E over U is

$$(y/\alpha^3)^2 = (x/\alpha^2)((x/\alpha^2)^2 - \varepsilon).$$

Therefore $\boldsymbol{\omega}_{E/R}$ is locally generated over U by

$$\frac{d(x/\alpha^2)}{y/\alpha^3} = \alpha\frac{dx}{y},$$

i.e. $\boldsymbol{\omega}_{E/R} \simeq A$. So the divisor $2(0,0) - 2(e)$ is not principal on E/R.

1.3 Units and Eisenstein series

Let E be an elliptic curve over an integral base S, let $D > 1$ be an integer prime to 6, and $x \in E(S)$ a section. If x is disjoint from $\ker[\times D]$, then one obtains a unit $\vartheta_D(x) = x^*\vartheta_D \in \mathcal{O}^*(S)$ on the base. In particular, suppose that x is a torsion section of order $N > 1$, with $(N, D) = 1$. Since S is integral, x has order N at the generic point. Under either of the following conditions it is automatic that $x \cap \ker[\times D] = \emptyset$:

- N is invertible on S (then x has order N in every fibre); or
- N is divisible by at least two primes.

In the classical setting one takes S to be a modular curve (over $\mathbb{C}$) and the functions $\vartheta_D(x)$ are the *Siegel units*, studied extensively (see for example [19]).

There are at least two ways to form a logarithmic derivative from the pair (ϑ_D, x). The simplest is to form

$$\operatorname{dlog}\big(\vartheta_D(x)\big) \in \Gamma(S, \Omega^1_S)$$

which in the classical setting gives weight 2 Eisenstein series. The other way, which leads to weight 1 Eisenstein series, is to first form the "vertical" logarithmic derivative

$$\operatorname{dlog}_v \vartheta_D \in \Gamma(E - \ker[\times D], \Omega^1_{E/S}).$$

Since $\boldsymbol{\omega}_{E/S} = x^*\Omega^1_{E/S}$ (see §1.1) we obtain

$$ {}_D\mathrm{Eis}(x) = {}_D\mathrm{Eis}(E/S, x) := x^* \operatorname{dlog}_v \vartheta_D \in \Gamma(S, x^*\Omega^1_{E/S}) = \Gamma(S, \boldsymbol{\omega}_{E/S}), $$

a modular form of weight one. Notice that in this construction one can start with *any* function whose divisor is $D^2(e) - \ker[\times D]$, since it will be of the form $g\vartheta_D$ for some $g \in \mathcal{O}^*(S)$, and $\operatorname{dlog}_v g = 0$.

From property 1.2.1(iv) we have

$$ (\vartheta_D)^{D'^2} \cdot \vartheta_{D'} \circ [\times D] = (\vartheta_{D'})^{D^2} \cdot \vartheta_D \circ [\times D'] $$

and therefore

$$ D'^2 \operatorname{dlog}_v \vartheta_D - [\times D']^* \operatorname{dlog}_v \vartheta_D = D^2 \operatorname{dlog}_v \vartheta_{D'} - [\times D]^* \operatorname{dlog}_v \vartheta_{D'}. \tag{1.3.1} $$

Now $[\times D]^*$ is multiplication by D on global sections of $\Omega^1_{E/S}$. Hence (1.3.1) gives

$$ \begin{aligned} D'^2 \cdot {}_D\mathrm{Eis}(E/S, x) - D' \cdot {}_D\mathrm{Eis}(E/S, D'x) \\ = D^2 \cdot {}_{D'}\mathrm{Eis}(E/S, x) - D \cdot {}_{D'}\mathrm{Eis}(E/S, Dx). \end{aligned} $$

It follows that for any $D \equiv 1 \pmod N$, the section

$$ \mathrm{Eis}(E/S, x) := \frac{1}{D^2 - D} \cdot {}_D\mathrm{Eis}(E/S, x) \in \Gamma(S \otimes \mathbb{Z}[\frac{1}{D(D-1)}], \boldsymbol{\omega}) \tag{1.3.2} $$

is independent of D. Now if $p \nmid 2N$, there exists $D > 1$, $D \equiv 1 \pmod N$ with $(D, 6) = 1$ and $p \nmid D(D-1)$, so one can glue the various $\mathrm{Eis}(E/S, x)$ for the different D to get a section $\mathrm{Eis}(E/S, x) \in \Gamma(S \otimes \mathbb{Z}[1/2N], \boldsymbol{\omega})$. For *any* D one then has

$$ {}_D\mathrm{Eis}(E/S, x) = D^2 \,\mathrm{Eis}(E/S, x) - D \,\mathrm{Eis}(E/S, Dx). $$

Suppose $E = \mathbb{C}/\Lambda$ is an elliptic curve over $\mathbb{C}$, with $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$. Let u be the variable in the complex plane. Using the function $\sigma(z, \Lambda)^{D^2}/\sigma(Dz, \Lambda)$ (Weierstrass σ-function) in place of ϑ_D gives

$$ \operatorname{dlog}_v \vartheta_D = \big(D^2\zeta(u, \Lambda) - D\zeta(Du, \Lambda)\big)\, dz $$

and if $x \in E(\mathbb{C}) - \{e\}$ is the torsion point $(a_1\omega_1 + a_2\omega_2)/N \in N^{-1}\Lambda/\Lambda$, with $(N, D) = 1$, then

$$ \mathrm{Eis}(E/\mathbb{C}, x) = \sum_{m_i \in \frac{a_i}{N} + \mathbb{Z}} \frac{1}{(m_1\omega_1 + m_2\omega_2)\, |m_1\omega_1 + m_2\omega_2|^s} \Bigg|_{s=0} du. \tag{1.3.3} $$

On the Tate curve $\mathrm{Tate}(q)$ over $\Lambda_N = \mathbb{Z}[\boldsymbol{\mu}_N]((q^{1/N}))$ there is the canonical differential dt/t, and the level N structure

$$(\mathbb{Z}/N\mathbb{Z})^2 \to \ker[\times N], \quad (a_1, a_2) \bmod N \mapsto \zeta_N^{a_1} q^{a_2/N}.$$

If x is the point $\zeta_N^{a_1} q^{a_2/N}$ then by explicit differentiation of the infinite product in Theorem 1.2.1(v)

$$\mathrm{Eis}(\mathrm{Tate}(q)/\Lambda_N, x) = \left[B_1\left(\frac{a_2}{N}\right) - \sum_{n>0}\Bigg(\sum_{\substack{d\in\mathbb{Z},\ d|n \\ \frac{n}{d}\equiv a_2 \bmod N}} \mathrm{sgn}(d)\zeta_N^{a_1 d}\Bigg) q^{n/N}\right] dt/t$$

if $0 \le a_1 < n$, $0 < a_2 < n$. Here $B_1(X) = X - 1/2$ is the Bernoulli polynomial. (In the case $a_2 = 0 \neq a_1$ the constant term is somewhat different.) In particular, $\mathrm{Eis}(\mathrm{Tate}(q)/\Lambda_N, x)$ is holomorphic at infinity.

One can also compute the logarithmic derivative of the unit $\vartheta_D(x) \in \Lambda_N^*$. The result is most interesting if one works with absolute differentials, that is in the module of (q-adically separated) differentials

$$\hat\Omega_{\Lambda_N/\mathbb{Z}} = \Lambda_N \cdot d(q^{1/N}) \oplus \Lambda_N/I_N \cdot d\zeta_N$$

where $I_N \subset \Lambda_N$ is the annihilator of $d\zeta_N$ (and equals the ideal generated by the different of $\mathbb{Q}(\boldsymbol{\mu}_N)$). The key point is that the logarithmic derivative of a typical term in the infinite product for $\vartheta_D(x)$ is

$$\begin{aligned}&\mathrm{dlog}\left(1 - \zeta_N^{\pm a_1} q^{(m\pm a_2/N)}\right)\\ &\qquad\qquad = \pm\frac{-\zeta_N^{\pm a_1} q^{(m\pm a_2/N)}}{1-\zeta_N^{\pm a_1} q^{(m\pm a_2/N)}}(a_1 \,\mathrm{dlog}\,\zeta_N + (a_2 \pm mN)\,\mathrm{dlog}\, q^{1/N})\end{aligned}$$

whereas the corresponding term for the vertical logarithmic derivative is

$$\mathrm{dlog}(1 - t^{\pm 1} q^m)\Big|_{t=\zeta_N^{a_1} q^{a_2/N}} = \pm\frac{-\zeta_N^{\pm a_1} q^{(m\pm a_2/N)}}{1-\zeta_N^{\pm a_1} q^{(m\pm a_2/N)}}\,\mathrm{dlog}\, t.$$

Comparing gives the following striking congruence:

Proposition 1.3.4. *If $x = \zeta_N^{a_1} q^{a_2/N} \in \mathrm{Tate}(q)(\Lambda_N)$, then*

$$\mathrm{dlog}\,\vartheta_D(x) \equiv \frac{{}_D\mathrm{Eis}(\mathrm{Tate}(q)/\Lambda_N, x)}{\mathrm{dlog}\, t}(a_1\,\mathrm{dlog}\,\zeta_N + a_2\,\mathrm{dlog}\, q^{1/N}) \mod N.$$

2 Norm relations

2.1 Some elements of K-theory

For a regular, separated and noetherian scheme X, the Quillen K-groups K_iX, $i \geq 0$, together with the cup-product

$$\cup\colon K_iX \times K_jX \to K_{i+j}X$$

define a graded ring K_*X, which is a contravariant functor in X— for any morphism $f\colon X' \to X$ of regular schemes there is a graded ring homomorphism $f^*\colon K_*X \to K_*X'$. If f is *proper*, then there are also pushforward maps $f_*\colon K_iX' \to K_iX$ (group homomorphisms) which satisfy the projection formula

$$f_*(f^*a \cup b) = a \cup f_*b. \tag{2.1.1}$$

For $i = 1$ there is a canonical monomorphism

$$\mathcal{O}^*(X) \to K_1X. \tag{2.1.2}$$

For arbitrary f, the restriction of the pullback map f^* to the image of (2.1.2) is pullback on functions; if f is finite and flat, then the pushforward map f_* restricts to the norm map on functions.

In this section we are concerned with K_2. The cup-product in this case is the *universal symbol map*

$$\begin{array}{ccc} \mathcal{O}^*(X) \otimes \mathcal{O}^*(X) & \to & K_2X \\ u \otimes v & \mapsto & \{u, v\} \end{array}$$

which is alternating and satisfies the Steinberg relation: $\{u, 1-u\} = 0$ if u, $1-u \in \mathcal{O}^*(X)$. If $X = \operatorname{Spec} F$ for a field F, then the symbol map induces an *isomorphism*

$$K_2X = K_2F \xrightarrow{\sim} \Lambda^2 F^*/(\text{Steinberg relation})$$

by Matsumoto's theorem.

Returning for a moment to the general situation, let Y be a smooth (not necessarily proper) variety over a number field F. Write $\overline{Y} = Y \otimes_F \overline{\mathbb{Q}}$, $G_F = \mathrm{Gal}(\overline{\mathbb{Q}}/F)$, and let p be prime. Then if $H^{j+1}(\overline{Y}, \mathbb{Q}_p(n))$ has no G_F-invariants, there is an *Abel-Jacobi homomorphism*

$$K_{2n-j-1}Y \to H^1(G_F, H^j(\overline{Y}, \mathbb{Q}_p)(n)).$$

The condition that $H^0(G_F, H^{j+1}(\overline{Y}, \mathbb{Q}_p(n))) = 0$ can often be checked just by considering weights; if for example Y is proper, then by considering the action of an unramified Frobenius and using Deligne's theorem (Weil conjectures) one sees that it holds if $j+1 \neq 2n$.

In the case of interest here, Y is a curve and $j=1$, and $n=2$. Then the Abel-Jacobi map is even defined integrally:

$$AJ_2\colon K_2Y \to H^1(G_F, H^1(\overline{Y}, \mathbb{Z}_p)(2)). \tag{2.1.3}$$

It is constructed as follows. There is a theory of Chern classes from higher K-theory to étale cohomology: these are functorial homomorphisms, for each $q \geq 0$ and $n \in \mathbb{Z}$:

$$c_{q,n}\colon K_qY \to H^{2n-q}(Y, \mathbb{Z}_p(n)).$$

Here the cohomology on the right-hand side is continuous étale cohomology. These maps are not multiplicative, but can be made into a multiplicative map by the Chern character construction.. All we need to know here is that if α, $\alpha' \in K_1Y$ then

$$c_{1,1}(\alpha) \cup c_{1,1}(\alpha') = -c_{2,2}(\alpha \cup \alpha') \tag{2.1.4}$$

(see for example [33, p.28]). One writes $\mathrm{ch} = -c_{2,2}$.

The étale cohomology of Y is related to that of $\overline{Y}$ by the Hochschild-Serre spectral sequence:

$$E_2^{i,j} = H^i(G_F, H^j(\overline{Y}, \mathbb{Z}_p)(n)) \Rightarrow H^{i+j}(Y, \mathbb{Z}_p(n)).$$

Let $Y \hookrightarrow X$ be the smooth compactification of Y, so that $Y = X - Z$ for a finite $Z \subset X$. The $\mathbb{Z}_p$-module $H^0(\overline{Y}, \mathbb{Z}_p) = H^0(\overline{X}, \mathbb{Z}_p)$ is free of rank equal to the number of components of $\overline{X}$, and $H^2(\overline{X}, \mathbb{Z}_p) = H^0(\overline{X}, \mathbb{Z}_p)(-1)$. The module $H^1(\overline{X}, \mathbb{Z}_p)$ is the Tate module of the Jacobian of $\overline{X}$, hence is free. There is an exact sequence

$$\begin{aligned} 0 \to H^1(\overline{X}, \mathbb{Z}_p) \to H^1(\overline{Y}, \mathbb{Z}_p) \to H^0(\overline{Z}, \mathbb{Z}_p)(-1)& \\ \xrightarrow{\gamma} H^2(\overline{X}, \mathbb{Z}_p) \to H^2(\overline{Y}, \mathbb{Z}_p) \to 0.& \end{aligned}$$

The map γ is the Gysin homomorphism, mapping the class of a point $z \in \overline{Z}$ to the class of the component of $\overline{X}$ to which it belongs. Therefore all the modules $H^j(\overline{Y}, \mathbb{Z}_p)$ are free[2]. Moreover if α is an eigenvalue of a geometric Frobenius acting on $H^j(\overline{Y}, \mathbb{Q}_p)$ at a prime $v \nmid p$ of good reduction, then α is an algebraic integer satisfying

$$|\alpha| = \begin{cases} 1 & \text{if } j=0, \\ N(v)^{1/2} \text{ or } N(v) & \text{if } j=1, \text{ and} \\ N(v) & \text{if } j=2. \end{cases}$$

[2]In the case to be considered later, Y is actually *affine*, in which case one even has $H^2(\overline{Y}, \mathbb{Z}_p) = 0$.

Therefore when $n = 2$ the first column $\{E_2^{0,j}\}$ of the spectral sequence vanishes. The exact sequence of lowest degree terms then becomes:

$$\begin{aligned}0 \to H^2(G_F, H^0(\overline{Y}, \mathbb{Z}_p)(2)) &\to H^2(Y, \mathbb{Z}_p(2)) \\ &\xrightarrow{e_2} H^1(G_F, H^1(\overline{Y}, \mathbb{Z}_p)(2)) \to H^3(G_F, H^0(\overline{Y}, \mathbb{Z}_p)(2)).\end{aligned}$$

Composing the "edge homomorphism" e_2 with $\mathrm{ch} = -c_{2,2}$ defines the Abel-Jacobi homomorphism (2.1.3) — the minus sign is chosen because of (2.1.4). Notice also that the last group $H^3(G_F, H^0(\overline{Y}, \mathbb{Z}_p)(2))$ is zero if p is odd, and killed by 2 in general (see for example [24]).

We also need the Chern character into de Rham cohomology. For a Noetherian affine scheme $X = \operatorname{Spec} R$ there are homomorphisms for each $q \geq 0$

$$\mathrm{dlog} = \mathrm{dlog}_R \colon K_q R \to \Omega^q_{R/\mathbb{Z}}$$

satisfying:

(i) $\mathrm{dlog}(a \cup b) = \mathrm{dlog}\, a \wedge \mathrm{dlog}\, b$;

(ii) If $b \in R^* \subset K_1 R$ then $\mathrm{dlog}\, b = b^{-1} db \in \Omega^1_{R/\mathbb{Z}}$;

(iii) On $K_0 R$, dlog is the degree map.

(iv) If R'/R is a finite flat extension of regular rings, then $\mathrm{tr}^{\Omega}_{R'/R} \circ \mathrm{dlog}_{R'} = \mathrm{dlog}_R \circ \mathrm{tr}^K_{R'/R}$.

In (iv), $\mathrm{tr}^K_{R'/R} \colon K_q R' \to K_q R$ is the proper push-forward for $\operatorname{Spec} R' \to \operatorname{Spec} R$ (also called the transfer), and $\mathrm{tr}^{\Omega}_{R'/R} \colon \Omega^q_{R'/\mathbb{Z}} \to \Omega^q_{R/\mathbb{Z}}$ is the trace map for differentials. Since this compatibility does not seem to be documented in the literature we make some remarks about it. What follows was suggested in conversation with Gillet and Soulé.

To check the compatibility we can work locally on $\operatorname{Spec} R$, and thus assume that R is local. Therefore R' is a free R-module of rank d say. Choosing a basis gives a matrix representation $\mu \colon R' \hookrightarrow M_d(R)$. We get for every $n \geq 1$ corresponding inclusions $GL_n(R') \hookrightarrow GL_{nd}(R)$, which in the limit give an inclusion $GL(R') \hookrightarrow GL(R)$. This induces by functoriality the transfer on $K_q(-) = \pi_q(BGL(-)^+)$.

One way to define the map dlog is to use Hochschild homology (see for example [22, 1.3.11ff.]). There is a simplicial R-module $C_\bullet(R)$ with $C_q(R) = R^{\otimes q+1}$ (tensor product over $\mathbb{Z}$), whose homology is Hochschild homology $HH_*(R)$. There is also a pair of R-linear maps $\Omega^q_{R/\mathbb{Z}} \xrightarrow{\varepsilon_q} HH_q(R) \xrightarrow{\pi_q} \Omega^q_{R/\mathbb{Z}}$, whose composite is multiplication by $q!$. The map π_q is given by $r_0 \otimes r_1 \otimes \cdots \otimes r_q \mapsto r_0 dr_1 \wedge \cdots \wedge r_q$.

There is a map Dtr: $H_*(GL(R),\mathbb{Z}) \to HH_*(R)$, the *Dennis trace* (see [22, 8.4.3], which maps $r \in R^* \subset H_1(GL(R),\mathbb{Z})$ to the homology class of $r^{-1}\otimes r \in C_1(R)$. Assume that $q!$ is invertible in R. Then composing on one side with the Hurewicz map $K_q(R) \to H_q(GL(R),\mathbb{Z})$, and on the other with $(q!)^{-1}\pi_q$, defines the map dlog: $K_q(R) \to \Omega^q_{R/\mathbb{Z}}$, for any $q > 0$. It is not too hard to check directly (an exercise from [22, Ch.8]) that if a, $b \in R^*$ then $\mathrm{dlog}\{a,b\} = (ab)^{-1}da \wedge db$, which is the only part of (i) needed in what follows.

There is a trace map tr^{HH} on Hochschild homology: the representation $R' \hookrightarrow M_d(R)$ induces by functoriality a map $HH_*(R') \to HH_*(M_d(R))$, and by Morita invariance [22, 1.2.4] we have an isomorphism $HH_*(M_d(R) \xrightarrow{\sim} HH_*(R)$.

Still under the hypothesis that $q!$ is invertible, the maps ε_q and $(q!)^{-1}\pi_q$ make $\Omega^q_{R/\mathbb{Z}}$ a direct factor of $HH_q(R)$. One can then *define* the trace map $\mathrm{tr}^\Omega_{R'/R}$ as the composite $(q!)^{-1}\pi_q \circ \mathrm{tr}^{HH}_{R'/R} \circ \varepsilon_q$. (This approach to trace maps is due to Lipman [21] — see also Hübl's thesis [13].) It now is a simple exercise to check the compatibility (iv), the essential point being the transitivity [22, E1.2.2] of the generalised trace.

We need all of this only for $q \leq 2$. This means that in the reciprocity law 3.2.3 and all its consequences we need to assume that p is odd.

2.2 Level structures

Let E/S be an elliptic curve. Then for every positive integer N which is invertible on S, there exists[3] a moduli scheme $S(N)$, which is finite and étale over S, and which represents the functor on S-schemes T

$$S(N)(T) = \left\{ \begin{matrix} \text{level } N \text{ structures on } E \times_S T \\ \alpha\colon (\mathbb{Z}/N)^2_{/T} \xrightarrow{\sim} \ker[\times N]_{/T} \end{matrix} \right\}$$

More generally, for pairs (M,N) of positive integers invertible on S there is a scheme $S(M,N)$ which represents the functor

$$S(M,N)(T) = \left\{ \begin{matrix} \text{monomorphisms of } S\text{-group schemes} \\ \alpha\colon (\mathbb{Z}/M \times \mathbb{Z}/N)_{/T} \hookrightarrow E_{/T} \end{matrix} \right\}$$

The group $GL_2(\mathbb{Z}/N)$ acts freely on $S(N)$ on the right, with quotient S. One has $S(N,N) = S(N)$; in general $S(M,N)$ is a quotient of $S(N')$ where $N' = \mathrm{lcm}(M,N)$. One usually writes $S_1(N)$ for $S(N,1)$, and we will also write $S_1'(N)$ for $S(1,N)$. Of course $S_1(N)$ and $S_1'(N)$ are isomorphic, but they are different as quotients of $S(N)$. We have a lattice of subgroups of $GL_2(\mathbb{Z}/N\mathbb{Z})$, and a corresponding diagram of quotients of $S(N)$:

[3] To avoid overloading the notation we do not include the dependence on E in the notation.

$$\begin{tikzcd}[column sep=small]
& \begin{pmatrix}1&0\\0&1\end{pmatrix} \arrow[dl, dash] \arrow[ddr, dash] & \\
\begin{pmatrix}1&*\\0&*\end{pmatrix} \arrow[d, dash] & & \\
\begin{pmatrix}*&*\\0&*\end{pmatrix} \arrow[dr, dash] & & \begin{pmatrix}*&0\\ *&1\end{pmatrix} \arrow[dl, dash] \\
& \begin{pmatrix}*&*\\ *&*\end{pmatrix} &
\end{tikzcd}
\qquad
\begin{tikzcd}
& S(N) \arrow[dl, "\pi_{S(N)/S_1(N)}"'] \arrow[ddr, "\pi_{S(N)/S'_1(N)}"] & \\
S_1(N) \arrow[d, "\pi_{S_1(N)/S_0(N)}"] & & \\
S_0(N) \arrow[dr, "\pi_{S_0(N)/S}"'] & & S'_1(N) \arrow[dl, "\pi_{S'_1(N)/S}"] \\
& S &
\end{tikzcd}$$

Over $S(N)$ there is a canonical level N structure $\alpha_N\colon (\mathbb{Z}/N)^2 \xrightarrow{\sim} \ker[\times N] \subset E$, and we let y_N, $y'_N \in E(S(N))$ be the images of the generators $(1,0)$, $(0,1)$. Then y_N already belongs to $E(S_1(N))$, and y'_N to $E(S'_1(N))$.

$S_1(N)$ is canonically isomorphic to the open subscheme of $\ker[\times N]$ consisting of points in the kernel whose order is exactly N; and

$$\ker[\times N] = \coprod_{M|N} S_1(M). \tag{2.2.1}$$

The scheme $S_0(N)$ parameterises $\Gamma_0(N)$-structures on E/S; in other words, $S_0(N)(T)$ is functorially the set of cyclic subgroup schemes of rank N of $E \times_S T$. In the case $T = S_1(N)$, the morphism $\pi_{S_1(N)/S_0(N)}\colon S_1(N) \to S_0(N)$ classifies the cyclic subgroup generated by y_N.

If $M|M'$ and $N|N'$ then there is a canonical level-changing map

$$\pi_{S(M',N')/S(M,N)}\colon S(M',N') \to S(M,N)$$

induced by the inclusion $\mathbb{Z}/M \times \mathbb{Z}/N \hookrightarrow \mathbb{Z}/M' \times \mathbb{Z}/N'$. One has $y_M = (M'/M)y_{M'}$ and likewise for y'.

We also recall that all the above moduli schemes can be defined for integers M, N which are not invertible on S, using Drinfeld level structures, see [18, *passim*]. They are finite and flat over S.

Recall finally that for a positive integer N which is the product of two coprime integers ≥ 3, there is a universal elliptic curve with level N structure over the modular curve $Y(N)/\mathbb{Z}$. We shall use the standard notations $Y_\Gamma(N)$, $Y(M,N)$ without comment.

2.3 Norm relations for $\Gamma(\ell)$-structure

Now fix an integer $D > 1$ which is prime to 6. On each basechange $E \times_S T$ (where T is one of the above moduli schemes) there is the canonical function

$\vartheta_D^{(E_T/T)}$, which by Theorem 1.2.1(ii) is simply the pullback of $\vartheta_D^{(E/S)}$. Since E and D will be fixed in the discussion that follows we shall write all these functions simply as ϑ.

Consider the case of prime level ℓ. Write $y = y_\ell$, $y' = y'_\ell$, and abbreviate $S_\Gamma = S_\Gamma(\ell)$ ($\Gamma = 1$ or 0). Fix $x \in E(S)$ such that $D\ell x$ does not meet the zero section of E. Let $\lambda\colon E \times_S S_0 \to \widetilde{E}$ be the quotient by the canonical subgroup scheme of rank ℓ, generated by y. Let $\tilde{x} \in \widetilde{E}(S_0)$ be the composite

$$S_0 \xrightarrow{x} E \times_S S_0 \xrightarrow{\lambda} \widetilde{E}.$$

Write $\tilde{\vartheta} = \vartheta_D^{(\widetilde{E}/S_0)} \in \Gamma(\widetilde{E} - \ker[\times D], \mathcal{O}^*)$.

Lemma 2.3.1.

$$N_{S_1/S}(\vartheta(x+y)) = \vartheta(\ell x)\vartheta(x)^{-1}. \tag{N1}$$

$$N_{S(\ell)/S_1}(\vartheta(x+y')) = \vartheta(\ell x) \prod_{a\in\mathbb{Z}/\ell} \vartheta(x+ay)^{-1}. \tag{N2}$$

$$N_{S_1/S_0}(\vartheta(x+y)) = \tilde{\vartheta}(\tilde{x})\vartheta(x)^{-1} \tag{N3}$$

$$N_{S(\ell)/S_1}(\vartheta(x+y')) = \vartheta(\ell x)\tilde{\vartheta}(\tilde{x})^{-1}. \tag{N4}$$

$$N_{S_0/S}(\tilde{\vartheta}(\tilde{x})) = \vartheta(x)^\ell\vartheta(\ell x). \tag{N5}$$

Proof. (N1) By (2.2.1) there is a Cartesian square

$$\begin{tikzcd}
S_1 \amalg S \arrow[r, "{(x+y,x)}"] \arrow[d, squiggly] & E \arrow[d, squiggly, "{[\times\ell]}"] \\
S \arrow[r, "\ell x"] & E
\end{tikzcd}$$

Hence

$$\begin{aligned}
N_{S_1/S}(\vartheta(x+y))\vartheta(x) &= N_{S_1\amalg S/S}((x+y,x)^*\vartheta) \\
&= (\ell x)^*[\times\ell]_*\vartheta \qquad &&\text{since the square is Cartesian} \\
&= (\ell x)^*\vartheta &&\text{by 1.2.1(iii)} \\
&= \vartheta(\ell x).
\end{aligned}$$

(N2) The same argument, applied to the Cartesian square

$$\begin{tikzcd}
S(\ell) \amalg \coprod_{a\in\mathbb{Z}/\ell} S_1 \arrow[r, "{(x+y',x+ay)}"] \arrow[d, squiggly] & E \times_S S_1 \arrow[d, squiggly, "{[\times\ell]}"] \\
S_1 \arrow[r, "\ell x"] & E \times_S S_1
\end{tikzcd}$$

(N3) This comes from the Cartesian square:

$$\begin{tikzcd} S_1 \amalg S_0 \arrow[r, "(x+y,x)"] \arrow[d] & E \times_S S_0 \arrow[d, "\lambda"] \\ S_0 \arrow[r, "\tilde{x}"] & \widetilde{E} \end{tikzcd}$$

The remaining relations (N4) and (N5) are obtained by combining (N1)–(N3) and using

$$N_{S_1/S_0}(\vartheta(x+y)) = \prod_{a \in (\mathbb{Z}/\ell)^*} \vartheta(x+ay). \qquad \square$$

Lemma 2.3.2. *The norm relations* (N1)–(N5) *hold without the hypothesis that ℓ is invertible on S.*

Proof. Choose an auxiliary integer $r > 2$ prime to ℓ. Then after replacing S by an étale basechange there exists a level r structure $\beta_r\colon (\mathbb{Z}/r)^2 \to \ker[\times r]$ on E, with r invertible on S. Let $\mathcal{E}^{\mathrm{univ}} \to Y(r)$ be the universal elliptic curve with level r structure over $\mathbb{Z}[1/r]$. Then there is a unique morphism $\xi\colon S \to \mathcal{E}^{\mathrm{univ}} - \ker[\times \ell D]$ which classifies the triple $(E/S, \beta_r, x)$: there is a Cartesian square

$$\begin{tikzcd} E \arrow[r] \arrow[d] & \mathcal{E}^{\mathrm{univ}} \times_{Y(r)} (\mathcal{E}^{\mathrm{univ}} - \ker[\times \ell D]) \arrow[d, "pr_2"] \\ S \arrow[r, "\xi"] & \mathcal{E}^{\mathrm{univ}} - \ker[\times \ell D] \end{tikzcd}$$

such that β_r is the inverse image of the canonical level r structure on $\mathcal{E}^{\mathrm{univ}}$, and x is the pullback of the diagonal section $\mathcal{E}^{\mathrm{univ}} \to \mathcal{E}^{\mathrm{univ}} \times \mathcal{E}^{\mathrm{univ}}$.

By the basechange compatibility of ϑ_D, it is enough to verify the norm relations in this universal setting; but the inclusion $Y(r) \otimes \mathbb{Z}[1/\ell] \hookrightarrow Y(r)$ induces an injection on $\mathcal{O}^*$. Thus we reduce to the case in which ℓ is invertible on S. $\square$

Now consider two auxiliary integers D, D' with $(6\ell, DD') = 1$, and write $\vartheta = \vartheta_D^{(E/S)}$ and $\vartheta' = \vartheta_{D'}^{(E/S)}$. Following tradition we write $N_{\Gamma/\Gamma}$ for the push-forward maps $\pi_{\Gamma/\Gamma *}$ on K_2, but the group operation in K_2 will be written additively (for consistency with the higher K-theory case to be considered below).

Proposition 2.3.3. *In $K_2 S$ the following identity holds:*

$$\begin{aligned} &N_{S(\ell)/S}\{\vartheta(x+y), \vartheta'(x'+y')\} \\ &\qquad = \{\vartheta(\ell x), \vartheta'(\ell x')\} + \ell\{\vartheta(x), \vartheta'(x)\} - N_{S_0/S}\{\tilde{\vartheta}(\tilde{x}), \tilde{\vartheta}'(\tilde{x}')\} \end{aligned}$$

Proof. Compute using the projection formula (2.1.1) and the norm relations 2.3.1:

$$\begin{aligned}
&N_{S(\ell)/S}\{\vartheta(x+y), \vartheta'(x'+y')\}\\
&\quad = N_{S_1/S}\{\vartheta(x+y), N_{S(\ell)/S_1}\vartheta'(x'+y')\}\\
&\quad = N_{S_1/S}\{\vartheta(x+y), \vartheta'(\ell x')\tilde{\vartheta}'(\tilde{x}')^{-1}\} && \text{by (N4)}\\
&\quad = N_{S_0/S}\{\tilde{\vartheta}(\tilde{x})\vartheta(x)^{-1}, \vartheta'(\ell x')\tilde{\vartheta}'(\tilde{x}')^{-1}\} && \text{by (N3)}\\
&\quad = -N_{S_0/S}\{\tilde{\vartheta}(\tilde{x}), \tilde{\vartheta}'(\tilde{x}')\} - (\ell+1)\{\vartheta(x), \vartheta'(\ell x')\}\\
&\qquad\qquad + \{\vartheta(x), N_{S_0/S}\tilde{\vartheta}'(\tilde{x}')\} + \{N_{S_0/S}\tilde{\vartheta}(\tilde{x}), \vartheta'(\ell x')\}\\
&\quad = -N_{S_0/S}\{\tilde{\vartheta}(\tilde{x}), \tilde{\vartheta}'(\tilde{x}')\} - (\ell+1)\{\vartheta(x), \vartheta'(\ell x')\}\\
&\qquad\qquad + \{\vartheta(x), \vartheta'(x')^{\ell}\vartheta'(\ell x')\} + \{\vartheta(x)^{\ell}\vartheta(\ell x), \vartheta'(\ell x')\} && \text{by (N5)}\\
&\quad = -N_{S_0/S}\{\tilde{\vartheta}(\tilde{x}), \tilde{\vartheta}'(\tilde{x}')\} + \ell\{\vartheta(x), \vartheta'(x')\} + \{\vartheta(\ell x), \vartheta'(\ell x')\} && \square
\end{aligned}$$

Now suppose that S is a modular curve of level prime to ℓ, and E is the universal elliptic curve. Therefore $S = Y_H := Y(N)/H$ for some subgroup $H \subset GL_2(\mathbb{Z}/N)$, $E = \mathcal{E}^{\mathrm{univ}} \xrightarrow{f} Y_H$, and $S_0(\ell) = Y_{H,\ell} := Y_0(\ell, N)/H$. It is then possible to rewrite the above norm relation using the Hecke and the diamond operators, whose definitions we briefly recall.

The centre $(\mathbb{Z}/N)^* \subset GL_2(\mathbb{Z}/n)$ acts on Y_H and $\mathcal{E}^{\mathrm{univ}}$ on the right, defining the diamond operators $\langle a\rangle \in \operatorname{Aut} Y_H$, $\langle a\rangle_{\mathcal{E}} \in \operatorname{Aut}\mathcal{E}$ for $a \in (\mathbb{Z}/N)^*$. In modular language, the B-valued points of Y_H are pairs $(X/B, [\alpha_n]_H)$, where X/B is an elliptic curve and $[\alpha_N]_H$ is an H-equivalence class of level N structures $(\mathbb{Z}/N)^2 \to X$. Then $\langle a\rangle\colon (X/B, [\alpha_N]_H) \mapsto (X/B, [a\alpha_N]_H)$ is an automorphism of Y_H. The B-valued points of $\mathcal{E}^{\mathrm{univ}}$ are triples $(X/B, [\alpha_N]_H, z)$ with $z \in X(B)$, and the automorphism $\langle a\rangle_{\mathcal{E}}$ of $\mathcal{E}^{\mathrm{univ}}$ is given by $(X/B, [\alpha_N]_H, z) \mapsto (X/B, [a\alpha_N]_H, z)$.

Recall also [18, (9.4.1)] that the e_N pairing defines a morphism

$$e_N\colon Y_H \to \operatorname{Spec}\mathbb{Z}[\boldsymbol{\mu}_N]^{\det H} \tag{2.3.4}$$

$$(E/S, \alpha_N) \longmapsto e_N(\alpha_N\begin{pmatrix}1/N\\0\end{pmatrix}, \alpha_N\begin{pmatrix}0\\1/N\end{pmatrix}) \tag{2.3.5}$$

and the restriction of $\langle a\rangle^*$ to $\mathbb{Z}[\boldsymbol{\mu}_N]^{\det H}$ is then the map $\zeta \mapsto \zeta^{a^2}$ (since $\det\langle a\rangle = a^2$).

The is a commutative diagram [5, (3.17)]

$$\begin{tikzcd}
\mathcal{E}^{\mathrm{univ}} \arrow[r, "{[\times\ell]}"] \arrow[dr, "f"'] & \mathcal{E}^{\mathrm{univ}} \arrow[r, "\langle\ell\rangle_{\mathcal{E}}", "\sim"'] \arrow[d, "f"'] & \mathcal{E}^{\mathrm{univ}} \arrow[d, "f"'] \\
& Y_H \arrow[r, "\sim", "\langle\ell\rangle"'] & Y_H
\end{tikzcd}$$

and if $x\colon Y_H \to \mathcal{E}^{\mathrm{univ}}$ is any N-torsion section, $\langle \ell \rangle_{\mathcal{E}} \circ \ell x = x \circ \langle \ell \rangle$. Therefore by the basechange property 1.2.1(ii),

$$\vartheta_D(\ell x) = (\ell x)^* \vartheta_D = (\ell x)^* \langle \ell \rangle_E^* \vartheta_D = \langle \ell \rangle^* \vartheta_D(x) = \langle \ell^{-1} \rangle_* \vartheta_D(x).$$

The scheme S_0 is the quotient $Y_{H,\ell} := Y_0(N\ell, N)/H$, and $Y_0(N\ell, N)$ classifies triples (X, α_N, C) with $C \subset X$ a subgroup of rank ℓ. One then has the standard diagram [5, (3.16)]

$$\begin{tikzcd}
\mathcal{E}^{\mathrm{univ}} \arrow[d] & \mathcal{E}^{\mathrm{univ}} \times_{Y_H} Y_{H,\ell} \arrow[l, "c_{\mathcal{E}} := pr_1"'] \arrow[r, "\lambda"] \arrow[d] \arrow[rr, bend left, "w_{\mathcal{E}}"] & \widetilde{\mathcal{E}^{\mathrm{univ}}} \arrow[r, "v", "\sim"'] \arrow[d] & \mathcal{E}^{\mathrm{univ}} \times_{Y_H} Y_{H,\ell} \arrow[r, "c_{\mathcal{E}}"] \arrow[d] & \mathcal{E}^{\mathrm{univ}} \arrow[d] \\
Y_H & Y_{H,\ell} \arrow[l, "c"'] \arrow[r, equal] & Y_{H,\ell} \arrow[r, "\sim", "w"'] \arrow[d] & Y_{H,\ell} \arrow[r, "c"'] \arrow[d] & Y_H \\
 & & \operatorname{Spec} \mathbb{Z}[\boldsymbol{\mu}_N]^H \arrow[r, "\zeta \mapsto \zeta^\ell"] & \operatorname{Spec} \mathbb{Z}[\boldsymbol{\mu}_N]^H &
\end{tikzcd}$$

in which the first, third and fourth squares in the top row are Cartesian. The Hecke operator T_ℓ is by definition the correspondence $c_*(cw)^*$ on Y_H, and the correspondence $c_{\mathcal{E}*}(c_{\mathcal{E}} w_{\mathcal{E}})^*$ on $\mathcal{E}^{\mathrm{univ}}$. All the horizontal arrows are compatible with the level N structure on $\mathcal{E}^{\mathrm{univ}}$ and the quotient level N structure on $\widetilde{\mathcal{E}^{\mathrm{univ}}}$, hence $c_{\mathcal{E}} \circ w_{\mathcal{E}} \circ x = x \circ c \circ w$, and therefore

$$\tilde{\vartheta}(\tilde{x}) = (\lambda \circ x)^* \tilde{\vartheta} = (\lambda \circ x)^* (c_{\mathcal{E}} \circ v)^* \vartheta = (c \circ w)^* \vartheta(x)$$

using as always the basechange property 1.2.1(ii). Therefore

$$T_\ell \{\vartheta(x), \vartheta'(x')\} = c_*(c \circ w)^* \{\vartheta(x), \vartheta'(x')\} = N_{S_0(\ell)/S} \{\tilde{\vartheta}(\tilde{x}), \tilde{\vartheta}'(\tilde{x}')\}.$$

Finally write $z = x + y_\ell$, $z' = x' + y'_\ell$, so that $\ell x = \ell z$ and $x = (\ell z) \circ \langle \ell^{-1} \rangle$. Observe that the norm relation is invariant under the action of $GL_2(\mathbb{Z}/\ell)$, so that y_ℓ, y'_ℓ can be replaced by any basis for the ℓ-torsion of E. This yields the following reformulation of 2.3.3:

Proposition 2.3.6. *If $S = Y_H$ is a modular curve of level prime to ℓ and z, z' are torsion sections of $E/S(\ell)$ whose projections onto* $\ker[\times \ell]$ *are linearly independent, then*

$$N_{S(\ell)/S} \{\vartheta(z), \vartheta'(z')\} = (1 - T_\ell \circ \langle \ell \rangle_* + \ell \langle \ell \rangle_*) \{\vartheta(\ell z), \vartheta'(\ell z')\}.$$

2.4 Norm relations for $\Gamma(\ell^n)$-structure

We now consider norm relations in the tower $\{S(\ell^m, \ell^n)\}$.

Lemma 2.4.1. *If $m > 1$, $n \geq 0$ and $x \in E(S(\ell^m, \ell^n))$ is any section, then*

$$N_{S(\ell^m,\ell^n)/S(\ell^{m-1},\ell^n)} \colon \vartheta(x + y_{\ell^m}) \mapsto \vartheta(\ell x + y_{\ell^{m-1}}). \tag{N6}$$

Proof. If ℓ is invertible on S this follows from the Cartesian square

$$\begin{array}{ccc} S(\ell^m, \ell^n) & \xrightarrow{x+y_{\ell^m}} & E \\ \big\downarrow & & \big\downarrow [\times \ell] \\ S(\ell^{m-1}, \ell^n) & \xrightarrow{\ell x + y_{\ell^{m-1}}} & E \end{array}$$

In the general case one reduces to the universal situation exactly as in 2.3.2. □

Proposition 2.4.2. *If $(6\ell, DD') = 1$ and m, $n > 1$ then for all x, $x' \in E(S(\ell^{m-1}, \ell^{n-1}))$,*

$$N_{S(\ell^m,\ell^n)/S(\ell^{m-1},\ell^{n-1})} \colon \{\vartheta(x + y_{\ell^m}), \vartheta'(x' + y'_{\ell^n})\} \\ \mapsto \{\vartheta(\ell x + y_{\ell^{m-1}}), \vartheta'(\ell x' + y'_{\ell^{n-1}})\}.$$

Proof. This follows from the lemma since

$$\begin{aligned} &N_{S(\ell^m,\ell^n)/S(\ell^{m-1},\ell^{n-1})}\{\vartheta(x + y_{\ell^m}), \vartheta'(x' + y'_{\ell^n})\} \\ &\quad = N_{S(\ell^m,\ell^{n-1})/S}\{\vartheta(x + y_{\ell^m}), N_{S(\ell^m,\ell^n)/S(\ell^m,\ell^{n-1})}\vartheta'(x' + y'_{\ell^n})\} \\ &\quad = N_{S(\ell^m,\ell^{n-1})/S}\{\vartheta(x + y_{\ell^m}), \vartheta'(\ell x' + y'_{\ell^{n-1}})\} \\ &\quad = \{\vartheta(\ell x + y_{\ell^{m-1}}), \vartheta'(\ell x' + y'_{\ell^{n-1}})\} \end{aligned}$$ □

If E/S is a modular family over $S = Y_H$ of level prime to ℓ, then $E(Y_H)$ is finite of order prime to ℓ. Therefore

$$E(Y_H(\ell^m, \ell^n))_{\text{torsion}} = (\mathbb{Z}/\ell^m \times \mathbb{Z}/\ell^n) \times (\text{prime to } \ell),$$

so there is a well-defined projection onto $(\mathbb{Z}/\ell)^2 = \ker[\times \ell]$. Computing as in 2.3.6 we get:

Proposition 2.4.3. *Suppose that $S = Y_H$ is a modular curve of level prime to ℓ and that z, z' are torsion sections of E over $Y_H(\ell^m, \ell^n)$, with m, $n > 1$. If the projections of $\{z, z'\}$ into* $\ker[\times \ell]$ *are linearly independent, then*

$$N_{Y_H(\ell^m,\ell^n)/Y_H(\ell^{m-1},\ell^{n-1})}\{\vartheta(z), \vartheta'(z')\} = \{\vartheta(\ell z), \vartheta'(\ell z')\}.$$ □

2.5 Norm relations for products of Eisenstein series

We shall repeat the construction of the last paragraph for products of the form

$$ {}_D\mathrm{Eis}(E/S, x) \cdot {}_{D'}\mathrm{Eis}(E/S, x') \in \Gamma(S, \boldsymbol{\omega}^2_{E/S}). $$

If $g\colon S' \to S$ is a finite and flat morphism of smooth T-schemes and $E' = E \times_S S'$ then there are trace maps

$$ \mathrm{tr}_g = \mathrm{tr}_{S'/S}\colon g_*\mathcal{O}_{S'} \to \mathcal{O}_S, \quad g_*\Omega^1_{S'/T} \to \Omega^1_{S/T} $$

as well as a trace map on modular forms, defined to be the composite

$$ \begin{aligned} \mathrm{tr}_g = \mathrm{tr}_{S'/S}\colon \Gamma(S', \omega^{\otimes k}_{E'/S'}) &= \Gamma(S', g^*\omega^{\otimes k}_{E/S}) \\ &= \Gamma(S, g_*\mathcal{O}_{S'} \otimes_{\mathcal{O}_S} \omega^{\otimes k}_{E/S}) \xrightarrow{\mathrm{tr}_g} \Gamma(S, \omega^{\otimes k}_{E/S}). \end{aligned} $$

The Kodaira-Spencer map (§1.1) and the trace are not compatible.

Proposition 2.5.1. *The diagrams below commute:*

$$ \begin{tikzcd} \boldsymbol{\omega}^{\otimes 2}_{E'/S'} \arrow[r, "KS_{E'/S'}"] & \Omega^1_{S'/T} \\ g^*\boldsymbol{\omega}^{\otimes 2}_{E/S} \arrow[u, "\wr"] \arrow[r, "g^*KS_{E/S}"] & g^*\Omega^1_{S/T} \arrow[u, "g^*"'] \end{tikzcd} \qquad \begin{tikzcd} \boldsymbol{\omega}^{\otimes 2}_{E'/S'} \arrow[r, "KS_{E'/S'}"] & \Omega^1_{S'/T} \arrow[d, "\mathrm{tr}_g"] \\ g^*\boldsymbol{\omega}^{\otimes 2}_{E/S} \arrow[u, "\wr"] \arrow[r, "\deg(g)\cdot g^*KS_{E/S}"] & g^*\Omega^1_{S/T} \end{tikzcd} $$

Proof. The first commutes because of the functoriality of the Kodaira-Spencer map. Then applying $\mathrm{tr}_g \circ g^* = \deg g$ gives the second. $\square$

Lemma 2.5.2. *The notation as in Lemma 2.4.1,*

$$ \mathrm{tr}_{S(\ell^m,\ell^n)/S(\ell^{m-1},\ell^n)}\colon {}_D\mathrm{Eis}(x + y_{\ell^m}) \mapsto \ell^{-1}{}_D\mathrm{Eis}(\ell x + y_{\ell^{m-1}}). $$

Proof. Since $[\times\ell]_*\vartheta_D = \vartheta_D$, we have $\mathrm{tr}_{[\times\ell]}\colon \operatorname{dlog}\vartheta_D \mapsto \operatorname{dlog}\vartheta_D$. But on global sections of $\Omega^1_{E/S}$, $\mathrm{tr}_{[\times\ell]}$ is multiplication by ℓ. Therefore the diagram

$$ \begin{tikzcd} \Gamma(E - \ker[\times D], \Omega^1) \arrow[r, "(x+y_{\ell^n})^*"] \arrow[d, "\mathrm{tr}_{[\times\ell]}"'] & \Gamma(S(\ell^n, \ell^{n-1}), \boldsymbol{\omega}) \arrow[d, "\ell\,\mathrm{tr}_{S(\ell^n,\ell^{n-1})/S(\ell^{n-1})}"] \\ \Gamma(E - \ker[\times D], \Omega^1) \arrow[r, "(\ell x+y_{\ell^{n-1}})^*"] & \Gamma(S(\ell^{n-1}), \boldsymbol{\omega}) \end{tikzcd} $$

commutes, which gives the result. $\square$

Corollary 2.5.3. *The notations being as in 2.4.3, let g be the projection $g\colon Y_H(\ell^m, \ell^n) \to Y_H(\ell^{m-1}, \ell^{n-1})$. Then*

$$ \mathrm{tr}_g\colon {}_D\mathrm{Eis}(z) \cdot {}_{D'}\mathrm{Eis}(z') \mapsto \ell^{-2}{}_D\mathrm{Eis}(\ell z) \cdot {}_{D'}\mathrm{Eis}(\ell z') \tag{2.5.4} $$

$$ \mathrm{tr}_g\colon KS\big({}_D\mathrm{Eis}(z) \cdot {}_{D'}\mathrm{Eis}(z')\big) \mapsto \ell^2 KS\big({}_D\mathrm{Eis}(\ell z) \cdot {}_{D'}\mathrm{Eis}(\ell z')\big) \tag{2.5.5} $$

Proof. Follows from the preceding two lemmas, since $\deg(g) = \ell^4$. $\square$

A Appendix: Higher K-theory of modular varieties

A.1 Eisenstein symbols

Let $f\colon E \to S$ be an elliptic curve, and assume from now on that S is a regular scheme. For any integer $k > 0$, write E^k for the fibre product $E \times_S \cdots \times_S E$ — it is an abelian scheme of dimension k over S.

In [2], Beilinson discovered a family of canonical elements of $K_{k+1}(E^k)$. More precisely, he defined a canonical map

$$\mathbb{Q}[E_{\mathrm{tors}}]^{\mathrm{degree}=0} \to K_{k+1}(E^k) \otimes \mathbb{Q}$$

which he called the Eisenstein symbol. Here we make a modified construction which gives a norm-compatible system.

Let Γ_k be the semidirect product of the symmetric group $\mathfrak{S}_k$ and $\boldsymbol{\mu}_2^k$, which acts on E^k as follows:

- $\mathfrak{S}_k$ acts by permuting the copies of E;
- the i^{th} copy of $\boldsymbol{\mu}_2$ acts as multiplication by ± 1 in the i^{th} factor of the product.

There is a character $\varepsilon_k\colon \Gamma_k \to \boldsymbol{\mu}_2$ which is the identity on each factor $\boldsymbol{\mu}_2$ and the sign character on the symmetric group.

Γ_k has a natural realisation in $GL_k(\mathbb{Z})$ as the set of all permutation matrices with entries ± 1. Geometrically it is the group of orthogonal symmetries of a cube in n-space. In terms of this representation, ε_k is just determinant. For any $\mathbb{Z}[\Gamma_k]$-module M, write $M(\varepsilon_k)$ for the ε_k-isotypical component of $M \otimes \mathbb{Z}[1/2 \cdot k!]$.

For $x \in E(S)$ we shall consider the inclusion

$$\begin{aligned} i_x\colon E^k &\to E^{k+1} \\ (u_1, \ldots, u_k) &\mapsto (x - u_1, u_1 - u_2, \ldots, u_{k-1} - u_k, u_k). \end{aligned}$$

whose image is the subscheme

$$\left\{ (v_1, \ldots, v_{k+1}) \;\middle|\; \sum_1^{k+1} v_i = x \right\} \subset E^{k+1}.$$

For any integer $D \neq 0$ such that x is disjoint from $\ker[\times D]$, we define the following open subschemes of E^k:

$$\begin{aligned} U_{D,x}^{k\prime} &= i_x^{-1}(E - \ker[\times D])^{k+1}) \\ U_{D,x}^{k} &= \bigcap_{\gamma \in \Gamma_k} \gamma(U_{D,x}^{k\prime}). \end{aligned}$$

Observe that $U^{k\prime}_{D,x}$ and $U^k_{D,x}$ are stable under translation by $\ker[\times D]^k$, and there is an étale covering

$$[\times D]\colon U^k_{D,x} \to U^k_{1,Dx}.$$

We prove below the following lemma.

Lemma A.1.1. *If $z \in E(S)$ is any section disjoint from e, the inclusion $U^k_{1,z} \hookrightarrow (E - \{\pm z\})^k$ induces an isomorphism*

$$K_*(E - \{\pm z\})^k(\varepsilon_k) \xrightarrow{\sim} K_* U^k_{1,z}(\varepsilon_k).$$

Using this lemma we define K-theory elements, whenever $(6, D) = 1$:

$$\begin{aligned}
\vartheta^{[k]}_D &= pr_1^*(\vartheta_D) \cup \cdots \cup pr_{k+1}^*(\vartheta_D) && \in K_{k+1}(E - \ker[\times D])^{k+1} \\
{}^{(1)}\vartheta^{[k]}_D(x) &= i_x^*(\vartheta^{[k]}_D) && \in K_{k+1} U^{k\prime}_{D,x} \\
{}^{(2)}\vartheta^{[k]}_D(x) &= \frac{1}{\#\Gamma_k} \sum_{\gamma \in \Gamma_k} \varepsilon_k(\gamma)\gamma^*({}^{(1)}\vartheta^{[k]}_D(x)) && \in K_{k+1}(U^k_{D,x})(\varepsilon_k) \\
{}^{(3)}\vartheta^{[k]}_D(x) &= [\times D]_* {}^{(2)}\vartheta^{[k]}_D(x) && \in K_{k+1}(E - \{\pm Dx\})^k(\varepsilon_k)
\end{aligned}$$

We call ${}^{(i)}\vartheta^{[k]}_D(x)$ *Eisenstein symbols.* For $k = 0$ we simply define i_x to be the section $x \in E(S)$, and the Eisenstein symbol then becomes a Siegel unit:

$${}^{(i)}\vartheta^{[0]}_D(x) = \vartheta_D(x) \in \mathcal{O}^*(S).$$

If $\alpha_1, \dots, \alpha_{k+1}\colon \widetilde{E} \to E$ are isogenies of degree prime to D, then by repeated application of 1.2.1(iii) one get the norm-compatibility

$$(\alpha_1, \dots, \alpha_{k+1})_*(\vartheta^{[k]}_D) = \vartheta^{[k]}_D \tag{A.1.2}$$

Actually this is only of interest when all the α_i are equal.

Proof of Lemma A.1.1. For any T/S

$$U^k_{1,z}(T) = \left\{ (u_1, \dots, u_k) \in E(T)^k \;\middle|\; \begin{array}{l} \text{for all } i,\ u_i \neq e, \pm z; \\ \text{for all } i \neq j,\ u_i \pm u_j \neq 0. \end{array} \right\}$$

The complementary divisor $E^k - U^k_{1,z}$ is the union of the two divisors

$$V^k = \{(u_i) \mid \text{for some } i,\ u_i = e\} \cup \big\{(u_i) \mid \text{for some } i \neq j,\ u_i \pm u_j = e\big\}$$

and

$$W^k_z = \{(u_i) \mid \text{for some } i,\ u_i = \pm z\}$$

As S is regular, the K-groups in the lemma can be computed in K'-theory. From the localisation sequence, it is then enough to show that $K'_*(V^k - W^k_z)(\varepsilon_k)$ vanishes. This is a special case of the following:

Lemma A.1.3. *Let $V' \subset V^k$ be any Γ_k-invariant open subscheme. Then $K'_*(V')(\varepsilon_k)$ is trivial.*

Proof. Define a sequence of reduced closed subschemes

$$V^k = V^k_{[1]} \supset V^k_{[2]} \supset \cdots \supset V^k_{[k+1]} = \emptyset$$

inductively, by writing $V^k_{[r+1]}$ for the smallest closed subset of $V^k_{[r]}$ such that $V^k_{[r]} - V^k_{[r+1]}$ is smooth over S. Write $V'_{[r]} = V^k_{[r]} \cap V'$. Then from the definition of V^k it is easy to see that:

(i) $V^k_{[r]}$ is a union of closed subsets each given by the vanishing of a certain collection of expressions u_i, $u_i \pm u_j$, which are permuted by Γ_k;

(ii) This gives a decomposition of $V'_{[r]} - V'_{[r+1]}$ as a disjoint union $\coprod V'_{[r]\mu}$, of open and closed pieces, permuted by Γ_k, in such a way that for each μ there is some $\gamma_\mu \in \Gamma_k$ which acts trivially on $V'_{[r]\mu}$ and for which $\varepsilon_k(\gamma_\mu) = -1$.

This forces $K'_*(V'_{[r]} - V'_{[r+1]})(\varepsilon_k) = 0$ for each $r \geq 1$. In fact, if

$$\begin{aligned} c = \sum c_\mu &\in K'_*(V'_{[r]} - V'_{[r+1]}) \otimes \mathbb{Z}[1/2 \cdot k!] \\ &= \bigoplus_\mu K'_*(V'_{[r]\mu}) \otimes \mathbb{Z}[1/2 \cdot k!] \end{aligned}$$

then $\gamma^*_\mu(c) = \varepsilon_k(\gamma_\mu)c = -c$, whereas the μ-component of $\gamma^*_\mu(c)$ is evidently $+c_\mu$ by (ii). Now using the long exact sequences

$$K'_*(V'_{[r+1]}) \to K'_*(V'_{[r]}) \to K'_*(V'_{[r]} - V'_{[r+1]}) \to \dots$$

inductively (beginning with $r = k-1$) we deduce that $K'_*(V')(\varepsilon_k) = 0$. $\square$

A.2 Norm relations in higher K-groups

Here we find norm relations for the Eisenstein symbols and for cup-products, analogous to those in sections 2.3 and 2.4.

For the $\Gamma(\ell)$-structure norm relations, we use the same notation as in 2.3. In addition, write $\hat{\lambda}\colon \widetilde{E} \to E \times_S S_0$ for the isogeny dual to λ, and λ^k, $\hat{\lambda}^k$ for the isogenies on E^k, $\widetilde{E}^k$. Consider the push-forward for the morphisms:

$$\begin{aligned} [\times\ell] \times \pi_{S_1/S}&\colon E^k \times_S S_1 \to E^k \\ \lambda^k \times \pi_{S_1/S_0}&\colon E^k \times_S S_1 \to \widetilde{E}^k \\ \hat{\lambda}^k \times \pi_{S_0/S}&\colon \widetilde{E}^k \to E^k \end{aligned}$$

Fix $i \in \{1,2,3\}$ and abbreviate $\vartheta^{[k]}_D(x) = {}^{(i)}\vartheta^{[k]}_D(x)$. The symbol $\tilde{\vartheta}^{[k]}_D$ will denote the analogue on $\widetilde{E}^k$ of $\vartheta^{[k]}_D$.

Lemma A.2.1. *The following relations hold in K_{k+1}:*

$$([\times \ell] \times \pi_{S_1/S})_* \vartheta_D^{[k]}(x+y) = \vartheta_D^{[k]}(\ell x) - [\times \ell]_* \vartheta_D^{[k]}(x) \tag{EN1}$$

$$(\lambda^k \times \pi_{S_1/S_0})_* \vartheta_D^{[k]}(x+y) = \tilde{\vartheta}_D^{[k]}(\tilde{x}) - \lambda_*^k \vartheta_D^{[k]}(x) \tag{EN3}$$

$$(\hat{\lambda}^k \times \pi_{S_0/S})_* \tilde{\vartheta}_D^{[k]}(\tilde{x}) = \ell [\times \ell]_* \vartheta_D^{[k]}(x) + \vartheta_D^{[k]}(\ell x). \tag{EN5}$$

Proof. Here (EN1) and (EN3) are to be understood on $U_{D,x}^k$, and (EN5) on $(\tilde{\lambda}^k)^{-1}(U_{D,x}^k)$. The relations (EN1) and (EN3) are proved just as (N1) and (N3), by considering the Cartesian diagrams:

$$\begin{tikzcd}
E^k \times_S S_1 \amalg E^k \arrow[r, "(i_{x+y}, i_x)"] \arrow[d, "([\times\ell]\times\pi_{S_1/S}, [\times\ell])"'] & E^{k+1} \arrow[d, "{[\times\ell]}"] \\
E^k \arrow[r, "i_{\ell x}"] & E^{k+1}
\end{tikzcd}$$

and

$$\begin{tikzcd}
E^k \times_S S_1 \amalg E^k \times_S S_0 \arrow[r, "(i_{x+y}, i_x)"] \arrow[d, "(\lambda^k\times\pi_{S_1/S_0}, \lambda^k)"'] & E^{k+1} \times_S S_0 \arrow[d, "\lambda^{k+1}"] \\
\widetilde{E}^k \arrow[r, "i_{\tilde{x}}"] & \widetilde{E}^{k+1}
\end{tikzcd}$$

and using the norm-compatibility (2.5.4). Applying $\hat{\lambda}^k \times \pi_{S_0/S}$ to (EN3) gives (EN5). $\square$

We now consider cup-products of the form $\vartheta_D^{[k]} \cup \vartheta_{D'}$ in K_{k+2}. Consider the factorisation of multiplication by ℓ:

$$\begin{tikzcd}
E^k \times_S S(\ell) \arrow[r, "id\times\pi_{S(\ell)/S_1}"] \arrow[rrr, bend right, "{[\times\ell]\times\pi_{S(\ell)/S}}"'] & E^k \times_S S_1 \arrow[r, "\lambda^k\times\pi_{S_1/S_0}"] & \widetilde{E}^k \arrow[r, "\hat{\lambda}^k\times\pi_{S_0/S}"] & E^k
\end{tikzcd}$$

We compute:

$$\begin{aligned}
&(\lambda^k \times \pi_{S(\ell)/S_0})_*[\vartheta_D^{[k]}(x+y) \cup \vartheta_{D'}(x'+y')] \\
&\quad = (\lambda^k \times \pi_{S_1/S_0})_*[\vartheta_D^{[k]}(x+y) \cup (\vartheta_{D'}(\ell x') - \tilde{\vartheta}_{D'}(\tilde{x}'))] && \text{by (N4)} \\
&\quad = (\tilde{\vartheta}_D^{[k]}(\tilde{x}) - \lambda_*^k \vartheta_D^{[k]}(x)) \cup (\vartheta_{D'}(\ell x') - \tilde{\vartheta}_{D'}(\tilde{x}')) && \text{by (EN3)}
\end{aligned}$$

We need to compute the image of this cup-product under $(\hat{\lambda}^k \times \pi_{S_0/S})_*$. Taking the terms in turn:

$$(\hat\lambda^k \times \pi_{S_0/S})_*:$$

$$\begin{aligned}
\tilde\vartheta_D^{[k]}(\tilde x) \cup \vartheta_{D'}(\ell x') &\mapsto ([\times\ell]_* \vartheta_D^{[k]}(x)^\ell + \vartheta_D^{[k]}(\ell x)) \cup \vartheta_{D'}(\ell x') \quad \text{by (EN5)}\\
\lambda_*^k \vartheta_D^{[k]}(x) \cup \tilde\vartheta_{D'}(\tilde x') &\mapsto [\times\ell]_* \vartheta_D^{[k]}(x) \cup (\ell \vartheta_{D'}(x') + \vartheta_{D'}(\ell x')) \quad \text{by (N5)}\\
\lambda_*^k \vartheta_D^{[k]}(x) \cup \vartheta_{D'}(\ell x') &\mapsto (\ell+1)([\times\ell]_* \vartheta_D^{[k]}(x) \cup \vartheta_{D'}(\ell x')) \qquad \text{as } \deg \pi_{S_0/S} = \ell+1
\end{aligned}$$

Combining these gives the required generalisation of 2.3.3:

Proposition A.2.2.

$$\begin{aligned}
([\times\ell] \times \pi_{S(\ell)/S})_*(\vartheta_D^{[k]}(x+y) \cup \vartheta_{D'}(x'+y')) &= \vartheta_D^{[k]}(\ell x) \cup \vartheta_{D'}(\ell x')\\
&- (\hat\lambda^k \times \pi_{S_0/S})_*(\tilde\vartheta_D^{[k]}(\tilde x) \cup \tilde\vartheta_{D'}(\tilde x')) + \ell([\times\ell]_* \vartheta_D^{[k]}(x) \cup \vartheta_{D'}(x')). \qquad \square
\end{aligned}$$

Having got this far the analogue of 2.4.2 presents no further difficulty:

Proposition A.2.3. *If $n > 1$ and $x,\ x' \in E(S(\ell^{n-1}))$ then*

$$\begin{aligned}
([\times\ell] \times \pi_{S(\ell^n)/S(\ell^{n-1})})_*(\vartheta_D^{[k]}(x + y_{\ell^n}) \cup \vartheta_{D'}(x' + y'_{\ell^n}))&\\
= \vartheta_D^{[k]}(\ell x + y_{\ell^{n-1}}) \cup \vartheta_{D'}(\ell x' + y'_{\ell^{n-1}}).& \qquad \square
\end{aligned}$$

3 The dual exponential map

3.1 Notations

In this section K will denote a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathfrak{o}$. We fix an algebraic closure $\overline{K}$ of K. Write $\bar{\mathfrak{o}}$ for the integral closure of $\mathfrak{o}$ in $\overline{K}$, and G_K for the Galois group of $\overline{K}$ over K. We normalise all p-adic valuations such that $v(p) = 1$. Let $\widehat{\overline{K}}$ be the completion of $\overline{K}$, and write $\widehat{\bar{\mathfrak{o}}}$ for its valuation ring. Fix a uniformiser π_K of $\mathfrak{o}$.

We fix for each $n > 0$ a primitive p^n-th root of unity ζ_{p^n} in $\overline{K}$ such that $\zeta_{p^{n+1}}^p = \zeta_{p^n}$. Write $K_n = K(\zeta_{p^n})$ and denote by $\mathfrak{o}_n$ the valuation ring of K_n. Put $\mathfrak{d}_n$ = the relative different of K_n/K.

For a topological G_K-module M write $H^i(K, M)$ for the *continuous* Galois cohomology groups [38].

The cyclotomic character $\chi_{\mathrm{cycl}} \colon G_K \to \mathbb{Z}_p^*$ is defined by $g(\zeta_{p^n}) = \zeta_{p^n}^{\chi_{\mathrm{cycl}}(g)}$, for every $g \in G_K$ and $n > 0$. Its logarithm is a homomorphism from G_K to $\mathbb{Z}_p$, often viewed as an element of $H^1(K, \mathbb{Z}_p)$.

We normalise the reciprocity law of local class field theory in such a way that if L/K is unramified, then the norm residue symbol $(\pi_K, L/K)$ equals the geometric Frobenius (inverse of the Frobenius substitution $x \mapsto x^q$). This implies that for any $u \in \mathfrak{o}^*$ we have $\chi_{\mathrm{cycl}}(u, K^{\mathrm{ab}}/K) = N_{K/\mathbb{Q}_p}(u)$.

3.2 The dual exponential map for H^1 and an explicit reciprocity law

Let V be a continuous finite-dimensional representation of G_K over $\mathbb{Q}_p$. Suppose that V is *de Rham* (for generalities about p-adic representations, see for example [12]). Let $DR(V) = (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_K}$ be the associated filtered K-vector space, with the decreasing filtration $DR^i(V)$ (induced from the filtration on B_{dR}). Then Kato has defined a *dual exponential map* [16, §II.1.2]

$$\exp^*\colon H^1(K,V) \to DR^0(V)$$

which is the composite:

$$H^1(K,V) \to H^1(K, B^0_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V) = H^1(K, \mathrm{Fil}^0(B_{\mathrm{dR}} \otimes_K DR(V))) \simeq DR^0(V).$$

The last isomorphism comes from Tate's computation [37] of the groups $H^i(K, \widehat{\overline{K}}(j))$:

$$\begin{aligned} H^i(K, \widehat{\overline{K}}(j)) = 0 \quad &\text{unless } j=0 \text{ and } i=0 \text{ or } 1\text{; and} \\ K = H^0(K, \widehat{\overline{K}}) &\xrightarrow[\sim]{\cup \log \chi_{\mathrm{cycl}}} H^1(K, \widehat{\overline{K}}) \end{aligned} \tag{3.2.1}$$

together with the isomorphisms $B^j_{\mathrm{dR}}/B^{j+1}_{\mathrm{dR}} \simeq \widehat{\overline{K}}(j)$.

The group $H^1(K,V)$ classifies extensions $0 \to V \to V' \to \mathbb{Q}_p(0) \to 0$ of p-adic Galois representations, and the extension V' is de Rham if and only if its class lies in $\ker(\exp^*)$. (This follows from [3], remark before 3.8 and Lemma 3.8.1.) In particular, the kernel of $\exp^*$ is the Bloch-Kato subgroup $H^1_g(K,V) \subset H^1(K,V)$.

In some cases one can define and study the dual exponential map without reference to B_{dR}. For example, if $V = H^1(A, \mathbb{Q}_p(1))$ for an abelian variety A/K, it can be defined just using the exponential map for the analytic group $A(K)$. More generally, if the filtration on $DR(V)$ satisfies $DR^1(V) = 0$, then one only needs to use the Hodge-Tate decomposition

$$\widehat{\overline{K}} \otimes_{\mathbb{Q}_p} V \xrightarrow{\sim} \bigoplus_{i \in \mathbb{Z}} \widehat{\overline{K}}(-i) \otimes_K \mathrm{gr}^i DR(V) \tag{3.2.2}$$

since then by (3.2.1) $\exp^*$ is the natural map from $H^1(K,V)$ to

$$\begin{aligned} H^1(K, \widehat{\overline{K}} \otimes_{\mathbb{Q}_p} V) &\xrightarrow[(3.2.2)]{\sim} \bigoplus_i H^1(K, \widehat{\overline{K}}(-i) \otimes_K \mathrm{gr}^i DR(V)) \\ &= H^1(K, \widehat{\overline{K}}) \otimes_K DR^0(V) \xleftarrow[(3.2.1)]{\sim} DR^0(V) \end{aligned}$$

In what follows we shall be concerned with the case $V = H^1(Y_{\overline{K}}, \mathbb{Q}_p)(1)$ for a smooth $\mathfrak{o}$-scheme Y, which is the complement in a smooth proper $\mathfrak{o}$-scheme X of a divisor Z with relatively normal crossings. Write

$$H^i_{\mathrm{dR}}(Y/\mathfrak{o}) = H^i(X, \Omega^\bullet_{X/\mathfrak{o}}(Z))$$

(the hypercohomology of the de Rham complex of differentials with logarithmic singularities along Z). Then $DR(V)$ is just de Rham cohomology with a shift of filtration:

$$\begin{aligned} DR^{-1}(V) &= DR(V) = H^1_{\mathrm{dR}}(Y/\mathfrak{o}) \otimes_{\mathfrak{o}} K \\ DR^0(V) &= H^0(X, \Omega^1_{X/\mathfrak{o}}(\log Z)) \otimes_{\mathfrak{o}} K = \mathrm{Fil}^1 H^1_{\mathrm{dR}}(Y/\mathfrak{o}) \otimes_{\mathfrak{o}} K \\ DR^1(V) &= 0 \end{aligned}$$

Moreover the Hodge-Tate decomposition has an explicit description, essentially thanks to the work of Fontaine [11] and Coleman [4]. To compute $\exp^*$ one just needs to know the projection

$$\begin{aligned} \pi_1 \colon \widehat{\overline{K}} \otimes_{\mathbb{Q}_p} V \xrightarrow[(3.2.2)]{\sim} \widehat{\overline{K}} \otimes_{\mathfrak{o}} H^0(X, \Omega^1_{X/\mathfrak{o}}(\log Z)) \oplus \widehat{\overline{K}}(1) \otimes_{\mathfrak{o}} H^1(X, \mathcal{O}_X) \\ \longrightarrow \widehat{\overline{K}} \otimes_{\mathfrak{o}} H^0(X, \Omega^1_{X/\mathfrak{o}}(\log Z)). \end{aligned}$$

It is the limit of the maps given by the diagram

$$\begin{tikzcd} H^1(Y_{\overline{K}}, \boldsymbol{\mu}_{p^n}) \arrow[r, "\mathbf{(1)}", "\sim"'] \arrow[drr, "\pi_1 \ (\mathrm{mod}\ p^n)"'] & H^0_{\mathrm{Zar}}(Y_{\overline{K}}, \mathcal{O}^*/p^n) & H^0_{\mathrm{Zar}}(Y_{\bar{\mathfrak{o}}}, \mathcal{O}^*/p^n) \arrow[l, "\mathbf{(2)}"', "\sim"] \arrow[d, hook, "\mathrm{dlog}"] \\ & & H^0(X, \Omega^1_{X/\mathfrak{o}}(\log Z)) \otimes \bar{\mathfrak{o}}/p^n \end{tikzcd}$$

Remarks. (i) The isomorphism labelled **(1)** comes about as follows. Generally, let S be a scheme on which m is invertible, with $\boldsymbol{\mu}_m \subset \mathcal{O}_S$. An element of $H^1(S, \boldsymbol{\mu}_m)$ is an isomorphism class of finite étale coverings $S' \to S$, Galois with group $\boldsymbol{\mu}_m$. Given such an S'/S, there is an open (Zariski) covering $\{U_i\}$ of S and units $f_i \in \mathcal{O}^*(U_i)$ such that $S' \times U_i = U_i[\sqrt[m]{f_i}]$. It is easy to see that $\{f_i\}$ is a well-defined element of $H^0(Y_{\overline{K}}, \mathcal{O}_S^*/m)$, and moreover that the map thus obtained fits into an exact sequence

$$H^1_{\mathrm{Zar}}(S, \boldsymbol{\mu}_m) \to H^1_{\text{ét}}(S, \boldsymbol{\mu}_m) \xrightarrow{\mathbf{(1)}} H^0_{\mathrm{Zar}}(S, \mathcal{O}_S^*/m) \to H^2_{\mathrm{Zar}}(S, \boldsymbol{\mu}_m).$$

If S is irreducible (as is the case here) then every non-empty Zariski open subset is connected, so $\boldsymbol{\mu}_m$ is flasque for the Zariski topology, and the map **(1)** is an isomorphism.

(ii) The inclusion $j\colon Y_{\overline{K}} \hookrightarrow Y_{\overline{\mathfrak{o}}}$ induces an isomorphism $H^0(Y_{\overline{\mathfrak{o}}}, \mathcal{O}^*/p^n) \xrightarrow{\sim} H^0(Y_{\overline{K}}, \mathcal{O}^*/p^n)$ denoted **(2)** in the diagram. To see this, consider the effect of multiplication by p^n on the exact sequence

$$0 \to \mathcal{O}^*_{Y_{\overline{\mathfrak{o}}}} \to j_*\mathcal{O}^*_{Y_{\overline{K}}} \xrightarrow{v} \mathbb{Q}_{Y_{\overline{k}}} \to 0$$

(the last map is the p-adic valuation along the special fibre, taking values in the constant sheaf $\mathbb{Q}$). This shows that $\mathcal{O}^*_{Y_{\overline{\mathfrak{o}}}}/p^n \xrightarrow{\sim} (j_*\mathcal{O}^*_{Y_{\overline{K}}})/p^n$. It is therefore enough to show that $(j_*\mathcal{O}^*_{Y_{\overline{K}}})/p^n \xrightarrow{\sim} j_*(\mathcal{O}^*_{Y_{\overline{K}}}/p^n)$, because then

$$H^0(Y_{\overline{\mathfrak{o}}}, \mathcal{O}^*/p^n) \xrightarrow{\sim} H^0(Y_{\overline{\mathfrak{o}}}, j_*(\mathcal{O}^*_{Y_{\overline{K}}}/p^n)) = H^0(Y_{\overline{K}}, \mathcal{O}^*/p^n).$$

By passing to the direct limit, we can replace $\overline{K}$ by a finite extension of K. Now consider more generally an open immersion $U \hookrightarrow S$, where S is a separated noetherian scheme which is integral and regular in codimension 1. Suppose that m is invertible on U, and that $\boldsymbol{\mu}_m \subset \mathcal{O}_U$. Then $R^i j_*\boldsymbol{\mu}_m = 0$ for $i > 0$ as $\boldsymbol{\mu}_m$ is Zariski flasque. The exact sequences

$$0 \to \boldsymbol{\mu}_m \to \mathcal{O}^* \to (\mathcal{O}^*)^m \to 0$$
$$0 \to (\mathcal{O}^*)^m \to \mathcal{O}^* \to \mathcal{O}^*/m \to 0.$$

give a short exact sequence

$$0 \to (j_*\mathcal{O}^*_U)/m \to j_*(\mathcal{O}^*_U/m) \to {}_mR^1 j_*\mathcal{O}^*_U \to 0.$$

But because S is regular in codimension one, the divisor sequence

$$1 \to \mathcal{O}^*_S \to \mathcal{K}^*_S \xrightarrow{\mathrm{div}} \coprod_{\mathrm{codim}(x)=1} i_{x*}\mathbb{Z} \to 0$$

is exact, and therefore $R^1 j_*\mathcal{O}^*_U = 0$.

(iii) The simplest case (which is, however, not enough for our purposes) is when $Y = X$ is proper, when this recipe reduces to that given by Coleman: an element of $H^1(X_{\overline{K}}, \boldsymbol{\mu}_{p^n}) = \operatorname{Pic} X_{\overline{K}}[p^n]$ is the class $[D]$ of a divisor D on $X_{\overline{K}}$ such that $p^n D = \operatorname{div}(g)$ is principal. One can assume that the divisor of g on $X_{\overline{\mathfrak{o}}}$ is precisely the closure D^c of D. Put

$$\omega = \operatorname{dlog} g \in H^0(X_{\overline{\mathfrak{o}}}, \Omega^1_{X_{\overline{\mathfrak{o}}}/\overline{\mathfrak{o}}}(\operatorname{supp} D^c)).$$

Then because the residues of ω at $\operatorname{supp} D$ are $\equiv 0 \pmod{p^n}$, one has

$$\omega \pmod{p^n} \in H^0(X, \Omega^1_{X/\mathfrak{o}}) \otimes \overline{\mathfrak{o}}/p^n$$

and this defines $\pi_1([D]) \pmod{p^n} = \omega \pmod{p^n}$. (Coleman even defines such a map in the case of bad reduction.) Unfortunately we know of no reference for this description of the Hodge-Tate decomposition in the non-proper case.

Now assume that X is a smooth and proper curve over $\mathfrak{o}$, and that Y is affine. Then Z is a finite étale $\mathfrak{o}$-scheme. Recall (see also the following section) that the different $\mathfrak{d}_n = \mathfrak{d}_{K_n/K}$ is the annihilator in $\mathfrak{o}_n$ of $\Omega_{\mathfrak{o}_n/\mathfrak{o}}$. If $K/\mathbb{Q}_p$ is unramified, then $\mathfrak{o}_n = \mathfrak{o}[\zeta_{p^n}]$ and therefore $\Omega_{\mathfrak{o}_n/\mathfrak{o}}$ is generated by $\operatorname{dlog}\zeta_{p^n}$, and moreover $\mathfrak{d}_n = p^n(\zeta_p - 1)^{-1}\mathfrak{o}_n$.

Theorem 3.2.3. *Suppose that $K/\mathbb{Q}_p$ is unramified (and that $p > 2$). There exists an integer c such that for every $n > 0$ the following diagram commutes up to p^c-torsion:*

$$\begin{tikzcd}
K_2(Y \otimes \mathfrak{o}_n) \otimes \boldsymbol{\mu}_{p^n}^{\otimes -1} \arrow[r, "ch"] \arrow[d, "\operatorname{dlog}"'] & H^2(Y \otimes K_n, \boldsymbol{\mu}_{p^n}) \arrow[d, "\text{Hochschild-Serre}"] \\
H^0(X \otimes \mathfrak{o}_n, \Omega^2_{X\otimes\mathfrak{o}_n/\mathfrak{o}}(\log Z))(-1) \arrow[d, equal] & H^1(K_n, H^1(Y \otimes \overline{K}, \boldsymbol{\mu}_{p^n})) \arrow[d, "\pi_1 \ (\mathrm{mod}\ p^{n-1})"] \\
\Omega^1_{\mathfrak{o}_n/\mathfrak{o}}(-1) \otimes \operatorname{Fil}^1 H^1_{\mathrm{dR}}(Y/\mathfrak{o}) & H^1(K_n, \bar{\mathfrak{o}}/p^{n-1}) \otimes_{\mathfrak{o}} \operatorname{Fil}^1 H^1_{\mathrm{dR}}(Y/\mathfrak{o}) \\
\mathfrak{o}_n/\mathfrak{d}_n \otimes_{\mathfrak{o}} \operatorname{Fil}^1 H^1_{\mathrm{dR}}(Y/\mathfrak{o}) \arrow[r] \arrow[u, "\operatorname{dlog}\zeta_{p^n}\otimes[\zeta_{p^n}]^{-1}\mapsto 1", "\wr"'] & \mathfrak{o}_n/p^{n-1} \otimes_{\mathfrak{o}} \operatorname{Fil}^1 H^1_{\mathrm{dR}}(Y/\mathfrak{o}) \arrow[u, "\cup(1/p^n)\log\chi_{\mathrm{cycl}}"']
\end{tikzcd}$$

Corollary 3.2.4. *(The explicit reciprocity law) The following diagram commutes:*

$$\begin{tikzcd}
\varprojlim_n \left(K_2(Y \otimes \mathfrak{o}_n) \otimes \boldsymbol{\mu}_{p^n}^{\otimes -1}\right) \arrow[r, "\mathrm{HS}\circ ch"] \arrow[d, "\operatorname{dlog}"'] & \varprojlim_n H^1(K_n, H^1(Y \otimes \overline{K}, \boldsymbol{\mu}_{p^n})) \arrow[d] \\
\varprojlim_n H^0(X \otimes \mathfrak{o}_n, \Omega^2_{X\otimes\mathfrak{o}_n/\mathfrak{o}}(\log Z))(-1) \arrow[d, equal] & \varprojlim_{n\ge m} H^1(K_m, H^1(Y \otimes \overline{K}, \boldsymbol{\mu}_{p^n})) \arrow[d, "\wr"] \\
\varprojlim_n \Omega^1_{\mathfrak{o}_n/\mathfrak{o}}(-1) \otimes \operatorname{Fil}^1 H^1_{\mathrm{dR}}(Y/\mathfrak{o}) \arrow[d] & H^1(K_m, H^1(Y \otimes \overline{K}, \mathbb{Z}_p)(1)) \arrow[d, "\exp^*"] \\
\varprojlim_n \mathfrak{o}_n/\mathfrak{d}_n \otimes_{\mathfrak{o}} \operatorname{Fil}^1 H^1_{\mathrm{dR}}(Y/\mathfrak{o}) \arrow[r, "(\frac{1}{p^n}\operatorname{tr}_{K_n/K_m})_{n\ge m}"] & K_m \otimes_{\mathfrak{o}} \operatorname{Fil}^1 H^1_{\mathrm{dR}}(Y/\mathfrak{o})
\end{tikzcd}$$

Remarks. (i) The assumption that $K/\mathbb{Q}_p$ is unramified is not essential for the proof, and is only included to simplify the statement. In general the situation is completely analogous to 3.3.15 below. The case $p = 2$ is excluded only because we do not know a reference for the compatibility of the trace maps in this case, cf. §2.1.

(ii) The maps "Hochschild-Serre" comes from the Hochschild-Serre spectral sequence with finite coefficients (cf. §2.1); since Y is affine, $H^2(Y \otimes \overline{K}, \boldsymbol{\mu}_{p^n}) = 0$.

(iv) For a discussion of the map dlog, see §2.1. *A priori* its target is the group $H^0(Y \otimes \mathfrak{o}_n, \Omega^2_{X\otimes\mathfrak{o}_n/\mathfrak{o}})(-1)$. We just explain why its image is contained in the submodule of differentials with logarithmic singularities along Z. By making an unramified basechange, one is reduced to the case when Z is a union of sections. Let A be the local ring of $X \otimes \mathfrak{o}_n$ at a closed point of Z, and t a local equation for Z. Then by the localisation sequence, one sees that $K_2A[t^{-1}]$ is generated by K_2A and symbols $\{u,t\}$ with $u \in \mathfrak{o}_n^*$, and $\operatorname{dlog}\{u,t\} = u^{-1}du \wedge \operatorname{dlog} t$.

Proof. First we explain precisely what are the transition maps in the various inverse systems in the diagram. In the Galois cohomology groups they are given by corestriction and reduction mod p^n. The finite flat morphisms $Y \otimes \mathfrak{o}_{n+1} \to Y \otimes \mathfrak{o}_n$ induce compatible trace maps (cf. §2.1)

$$K_2(Y \otimes \mathfrak{o}_{n+1}) \to K_2(Y \otimes \mathfrak{o}_n) \quad \text{and} \quad \Omega^2_{Y\otimes\mathfrak{o}_{n+1}/\mathfrak{o}} \to \Omega^2_{Y\otimes\mathfrak{o}_n/\mathfrak{o}}$$

which are the maps in the first and second inverse systems in the left-hand side of the diagram. In the system $(\Omega^1_{\mathfrak{o}_n/\mathfrak{o}})_n$ the transition maps are trace, and in the remaining system $(\mathfrak{o}_n/\mathfrak{d}_n)_n$ the maps are $\frac{1}{p}\operatorname{tr}_{K_{n+1}/K_n}$. (For the compatibility of these various maps, see 3.3.12 below.)

From the discussion above, the diagram below commutes:

$$\begin{tikzcd}
H^1(K_m, H^1(Y \otimes \overline{K}, \mathbb{Q}_p)(1)) \arrow[r, "\pi_1"] \arrow[d, "\exp^*"'] & H^1(K_m, \widehat{\overline{K}} \otimes_{\mathfrak{o}} \operatorname{Fil}^1 H^1_{\mathrm{dR}}(Y/\mathfrak{o})) \arrow[d, equal] \\
K_m \otimes_{\mathfrak{o}} \operatorname{Fil}^1 H^1_{\mathrm{dR}}(Y/\mathfrak{o}) \arrow[r, "\sim", "\cup \log\chi_{\mathrm{cycl}}"'] & H^1(K_m, \widehat{\overline{K}}) \otimes_{\mathfrak{o}} \operatorname{Fil}^1 H^1_{\mathrm{dR}}(Y/\mathfrak{o})
\end{tikzcd}$$

To deduce the corollary from the theorem it is thus only necessary to take inverse limits and use the commutativity (cf. Proposition 3.3.10 below) of the following diagram

$$\begin{tikzcd}
K_n \arrow[r, "\frac{1}{p^n}\log\chi_{\mathrm{cycl}}"] \arrow[d, "\frac{1}{p^{n-m}}\operatorname{tr}_{K_n/K_m}"'] & H^1(K_n, \widehat{\overline{K}}) \arrow[d, "\mathrm{cor}"] \\
K_m \arrow[r, "\frac{1}{p^m}\log\chi_{\mathrm{cycl}}"] & H^1(K_m, \widehat{\overline{K}}).
\end{tikzcd} \tag{3.2.5}$$

□

Remarks. (i) Consider the special case $Y = \mathbb{A}^1 - \{0\} = \operatorname{Spec}\mathfrak{o}[t,t^{-1}]$. Let $(u_n) \in \varprojlim \mathfrak{o}_n^*$ be a universal norm. By applying the corollary to the norm-compatible symbols $\{u_n, t\} \in K_2(Y \otimes \mathfrak{o}_n)$ one recovers a form of Iwasawa's cyclotomic explicit reciprocity law, which will be proved more directly in 3.3.15 below.

(ii) Theorem 3.2.3 is proved in section 3.4 below. It is much easier than the general cases considered by Kato in [17], first because one is not working with coefficients in a general formal group, and secondly because the assumption that $X/\mathfrak{o}$ is smooth makes for considerable simplifications. In the non-smooth case there is an analogous statement which is needed to compute the image of Kato's Euler system when p divides the conductor.

3.3 Fontaine's theory

We shall review here some of the theory of differentials for local fields developed by Fontaine [11], and as a warm-up for the next section, show how it gives a version of Iwasawa's explicit reciprocity law.

Recall (see for example [34, §III.6–7]) that if K'/K is a finite extension then its valuations ring $\mathfrak{o}'$ equals $\mathfrak{o}[x]$ for some $x \in \mathfrak{o}'$. This implies that the module of Kähler differentials $\Omega_{\mathfrak{o}'/\mathfrak{o}}$ is a cyclic $\mathfrak{o}'$-module, generated by dx, and that its annihilator is the relative different $\mathfrak{d}_{K'/K}$.

The module $\Omega_{\overline{\mathfrak{o}}/\mathfrak{o}}$ equals the direct limit of $\Omega_{\mathfrak{o}'/\mathfrak{o}}$ taken over all finite extensions of K in $\overline{K}$. In particular, it is torsion.

Theorem 3.3.1. [11] *There is a short exact sequence of $\overline{\mathfrak{o}}$-modules*

$$0 \to \mathfrak{a}(1) \to \overline{K}(1) \xrightarrow{\alpha} \Omega_{\overline{\mathfrak{o}}/\mathfrak{o}} \to 0$$

where $\mathfrak{a} = \mathfrak{a}_{\overline{\mathfrak{o}}/\mathfrak{o}}$ is the fractional ideal

$$\mathfrak{a}_{\overline{\mathfrak{o}}/\mathfrak{o}} = (\zeta_p - 1)^{-1}\mathfrak{d}_{K/\mathbb{Q}_p}^{-1}\overline{\mathfrak{o}} \subset \overline{K}$$

and where $\alpha\colon \overline{K}(1) := \mathbb{Z}_p(1) \otimes \overline{K} \to \Omega_{\overline{\mathfrak{o}}/\mathfrak{o}}$ is the unique $\overline{\mathfrak{o}}$-linear map satisfying

$$\alpha([\zeta_{p^m}]_m \otimes p^{-n}) = \operatorname{dlog} \zeta_{p^n} = \frac{d\zeta_{p^n}}{\zeta_{p^n}}$$

for any $n \geq 0$. □

Remark 3.3.2. In particular, for any $n \geq 0$ the annihilator of $\operatorname{dlog} \zeta_{p^n} \in \Omega_{\overline{\mathfrak{o}}/\mathfrak{o}}$ is $p^n \mathfrak{a} \cap \overline{\mathfrak{o}}$.

From 3.3.1 we get the fundamental canonical isomorphism

$$\widehat{\mathfrak{a}}_{\overline{\mathfrak{o}}/\mathfrak{o}}(1) \xrightarrow{\sim} T_p \Omega_{\overline{\mathfrak{o}}/\mathfrak{o}} \qquad (3.3.3)$$

which is $\overline{\mathfrak{o}}$-linear, and maps $(\zeta_{p^n})_n \in \mathbb{Z}_p(1) \subset \widehat{\mathfrak{a}}(1)$ to $(\operatorname{dlog} \zeta_{p^n})_n$.

Suppose that $K''/K'/K$ are finite extensions. Then there is an exact sequence of differentials

$$\Omega_{\mathfrak{o}'/\mathfrak{o}} \otimes_{\mathfrak{o}'} \mathfrak{o}'' \to \Omega_{\mathfrak{o}''/\mathfrak{o}} \to \Omega_{\mathfrak{o}''/\mathfrak{o}'} \to 0$$

(the "first exact sequence", [23, 26.H]), which is exact on the left as well by the multiplicativity of the different (or alternatively by the argument in the footnote on page 420). Passing to the direct limit over K'' gives a short exact sequence

$$0 \to \Omega_{\mathfrak{o}'/\mathfrak{o}} \otimes_{\mathfrak{o}'} \overline{\mathfrak{o}} \to \Omega_{\overline{\mathfrak{o}}/\mathfrak{o}} \to \Omega_{\overline{\mathfrak{o}}/\mathfrak{o}'} \to 0 \tag{3.3.4}$$

At this point, recall that for any short exact sequence $0 \to X \to Y \to Z \to 0$ of abelian groups, there is an inverse system of long exact sequences

$$0 \to {}_{p^n}X \to {}_{p^n}Y \to {}_{p^n}Z \to X/p^n \to Y/p^n \to Z/p^n \to 0. \tag{3.3.5}$$

If the inverse systems ${}_{p^n}M$ (for $M = X,\ Y,\ Z$) satisfy the Mittag-Leffler condition (ML) then the inverse limit sequence

$$0 \to T_pX \to T_pY \to T_pZ \to \varprojlim X/p^n \to \varprojlim Y/p^n \to \varprojlim Z/p^n \to 0$$

is also exact (a special case of EGA 0, 13.2.3). Note that $({}_{p^n}M)$ satisfies (ML) in two particular cases:

- the torsion subgroup of M is p-divisible (then ${}_{p^n}M \to {}_{p^{n-1}}M$ is surjective);
- the p-primary torsion subgroup of M has finite exponent (then $({}_{p^n}M)$ is ML-zero).

Applying these considerations to (3.3.4), since $\Omega_{\overline{\mathfrak{o}}/\mathfrak{o}}$ and $\Omega_{\overline{\mathfrak{o}}/\mathfrak{o}'}$ are divisible and $\Omega_{\mathfrak{o}'/\mathfrak{o}}$ is killed by a power of p, we get an exact sequence

$$0 \to T_p\Omega_{\overline{\mathfrak{o}}/\mathfrak{o}} \to T_p\Omega_{\overline{\mathfrak{o}}/\mathfrak{o}'} \to \Omega_{\mathfrak{o}'/\mathfrak{o}} \otimes_{\mathfrak{o}'} \overline{\mathfrak{o}} \to 0. \tag{3.3.6}$$

Now pass to continuous Galois cohomology. This gives a long exact sequence since the surjection in (3.3.6) has a continuous set-theoretic section (this is obvious here as $\Omega_{\mathfrak{o}'/\mathfrak{o}} \otimes_{\mathfrak{o}'} \overline{\mathfrak{o}}$ is discrete). We are only interested in the connecting map, and define δ to be the composite homomorphism:

$$\delta = \delta_{K'} \colon \Omega_{\mathfrak{o}'/\mathfrak{o}} \hookrightarrow H^0(K', \Omega_{\mathfrak{o}'/\mathfrak{o}} \otimes_{\mathfrak{o}'} \overline{\mathfrak{o}}) \xrightarrow{\text{connecting}} H^1(K', T_p\Omega_{\overline{\mathfrak{o}}/\mathfrak{o}}).$$

The map "reduction mod p^n" : $T_p\Omega_{\overline{\mathfrak{o}}/\mathfrak{o}} \to {}_{p^n}\Omega_{\overline{\mathfrak{o}}/\mathfrak{o}}$ induces a map on cohomology, which when composed with $\delta_{K'}$ gives

$$\delta_{K'} \pmod{p^n} : \Omega_{\mathfrak{o}'/\mathfrak{o}} \to H^1(K', {}_{p^n}\Omega_{\overline{\mathfrak{o}}/\mathfrak{o}}).$$

Lemma 3.3.7. *(i) The following diagram commutes:*

$$\begin{tikzcd}
\mathfrak{o}'^{*} \arrow[r, "\text{Kummer}"] \arrow[d, "\mathrm{dlog}"'] & H^1(K', \boldsymbol{\mu}_{p^n}) \arrow[d, "\mathrm{dlog}"] \\
\Omega_{\mathfrak{o}'/\mathfrak{o}} \arrow[r, "\delta_{K'} \bmod p^n"] & H^1(K', {}_{p^n}\Omega_{\bar{\mathfrak{o}}/\mathfrak{o}})
\end{tikzcd}$$

(ii) For any nonzero $x \in \mathfrak{o}'$

$$\delta_{K'}(dx) \ (\mathrm{mod}\ p^n) = x \,\mathrm{dlog}(\mathrm{Kummer}(x))$$

Proof. (i) Simply compute: if $u \in \mathfrak{o}'^{*}$ then fix a sequence (u_m) in $\bar{\mathfrak{o}}^{*}$ with $u_0 = u$, $u_{m+1}^p = u_m$. The composite dlog ∘ "Kummer" maps u to the class of the cocycle

$$g \mapsto \mathrm{dlog}(u_n^{g-1}) \in {}_{p^n}\Omega_{\bar{\mathfrak{o}}/\mathfrak{o}}.$$

Now compute the effect of $\delta_{K'}$ on $\mathrm{dlog}\, u$: first lift $\mathrm{dlog}\, u$ in the exact sequence (3.3.6) to the element $(\mathrm{dlog}\, u_m)_m \in T_p(\Omega_{\bar{\mathfrak{o}}/\mathfrak{o}'})$, then act by $g-1$ to get the desired cocycle. So the commutativity is trivial.

(ii) If x is a unit this is equivalent to (i). For the general case one simply calculates as in (i). □

Lemma 3.3.8. *Let $n \geq 1$ and assume that $\boldsymbol{\mu}_{p^n} \subset K'$. If $p \neq 2$, then the diagram*

$$\begin{tikzcd}
\mathfrak{o}'/p^n(1) \arrow[r, "\cup \frac{1}{p^n} \log \chi_{\mathrm{cycl}}"] \arrow[d, "1\otimes[\zeta_{p^n}] \mapsto \mathrm{dlog}\,\zeta_{p^n}"'] & H^1(K', \mathfrak{a}_{\bar{\mathfrak{o}}/\mathfrak{o}}/p^n)(1) \arrow[d, "\wr\ (3.3.3)"] \\
\Omega^1_{\mathfrak{o}'/\mathfrak{o}} \arrow[r, "\delta_{K'} \bmod p^n"] & H^1(K', {}_{p^n}\Omega^1_{\bar{\mathfrak{o}}/\mathfrak{o}})
\end{tikzcd}$$

commutes. For $p = 2$ it commutes mod 2^{n-1}.

Proof. All the maps are $\mathfrak{o}'$-linear, so it is enough to compute the image of $1 \otimes [\zeta_{p^n}]$. We have $\chi_{\mathrm{cycl}}(g) \equiv 1 \ (\mathrm{mod}\ p^n)$ for all $g \in G_K$, hence $\log \chi_{\mathrm{cycl}}(g) \equiv 0 \ (\mathrm{mod}\ p^n)$ and so if $p \neq 2$ then

$$\frac{1}{p^n} \log \chi_{\mathrm{cycl}}(g) \equiv \frac{1}{p^n}(\chi_{\mathrm{cycl}}(g) - 1) \quad \mathrm{mod}\ p^n.$$

In the proof of 3.3.7 one can take $u_m = \zeta_{p^{m+n}}$ for all $m \geq 0$, and then $\delta_{K'}(\mathrm{dlog}\,\zeta_{p^n}) \in H^1(K', T_p(\Omega_{\bar{\mathfrak{o}}/\mathfrak{o}}))$ is represented by the cocycle

$$g \mapsto (\mathrm{dlog}\,\zeta_{p^{m+n}}^{g-1})_m \in T_p(\Omega_{\bar{\mathfrak{o}}/\mathfrak{o}})$$

and $\zeta_{p^{m+n}}^{g-1} = \zeta_{p^{m+n}}^{\chi_{\mathrm{cycl}}(g)-1} = \zeta_{p^m}^{(\chi_{\mathrm{cycl}}(g)-1)/p^n}$. Applying the inverse of (3.3.3) maps this to the class of the cocycle

$$\begin{aligned} g \mapsto \frac{1}{p^n}(\chi_{\mathrm{cycl}}(g)-1) \otimes (\zeta_{p^m})_m \qquad &\in \mathfrak{a}_{\bar{\mathfrak{o}}/\mathfrak{o}}(1) \\ \equiv \frac{1}{p^n} \log \chi_{\mathrm{cycl}}(g) \quad \bmod p^n. & \end{aligned}$$

The reader will make the necessary modifications when $p=2$. □

We now need some elementary facts about cyclotomic extensions of local fields. Our chosen normalisation of the reciprocity law of local class field theory identifies the homomorphisms

$$\log \chi_{\mathrm{cycl}} \in H^1(K, \mathbb{Z}_p) = \mathrm{Hom}_{\mathrm{cts}}(\mathrm{Gal}(\bar{K}/K), \mathbb{Z}_p)$$

and

$$\log \circ N_{K/\mathbb{Q}_p} : K^* \longrightarrow \mathbb{Z}_p.$$

As observed in the proof of the previous lemma, if $\boldsymbol{\mu}_{p^m} \subset K$ then $\log \chi_{\mathrm{cycl}} \equiv 0 \pmod{p^m}$.

Lemma 3.3.9. *Suppose that $\boldsymbol{\mu}_{p^m} \subset K$. Then for any finite extension K'/K the diagram*

$$\begin{array}{ccc} \mathfrak{o}' & \xrightarrow{\cup \frac{1}{p^m} \log \chi_{\mathrm{cycl}}} & H^1(K', \widehat{\bar{\mathfrak{o}}}) \\ {\scriptstyle \mathrm{tr}_{K'/K}} \big\downarrow & & \big\downarrow {\scriptstyle \mathrm{cor}} \\ \mathfrak{o} & \xrightarrow[\cup \frac{1}{p^m} \log \chi_{\mathrm{cycl}}]{} & H^1(K, \widehat{\bar{\mathfrak{o}}}) \end{array}$$

commutes.

Proof. The statement follows from the projection formula for cup-product in group cohomology, since on H^0 the corestriction

$$\mathrm{cor} \colon H^0(K', \widehat{\bar{\mathfrak{o}}}) = \mathfrak{o}' \longrightarrow H^0(K, \widehat{\bar{\mathfrak{o}}}) = \mathfrak{o}$$

equals $\mathrm{tr}_{K'/K}$. □

Recall that $K_n := K(\zeta_{p^n})$. Let ℓ be the largest integer such that $\boldsymbol{\mu}_{p^\ell} \subset K$. Then if $n > m \geq \ell$, direct calculation gives

$$\mathrm{tr}_{K_n/K_m}\left(\mathfrak{o}[\zeta_{p^n}]\right) = \begin{cases} p^{n-m}\mathfrak{o}[\zeta_{p^m}] & \text{if } m > 0 \\ p^{n-1}\mathfrak{o} & \text{if } m = 0. \end{cases}$$

Define, for any $n > m \geq 0$

$$t_{n,m} := \frac{1}{p^{n-m}} \mathrm{tr}_{K_n/K_m} : K_n \to K_m.$$

Proposition 3.3.10. *If $n > m \geq \max(\ell, 1)$ the diagram*

$$\begin{tikzcd}
\mathfrak{o}[\zeta_{p^n}] \arrow[r, "\cup\frac{1}{p^{n-1}}\log\chi_{\mathrm{cycl}}"] \arrow[d, "t_{n,m}"'] & H^1(K_n, \widehat{\bar{\mathfrak{o}}}) \arrow[d, "\mathrm{cor}"] \\
\mathfrak{o}[\zeta_{p^m}] \arrow[r, "\cup\frac{1}{p^{m-1}}\log\chi_{\mathrm{cycl}}"] & H^1(K_m, \widehat{\bar{\mathfrak{o}}})
\end{tikzcd}$$

is commutative. If $n > \ell = 0$, the diagram

$$\begin{tikzcd}
\mathfrak{o}[\zeta_{p^n}] \arrow[r, "\cup\frac{1}{p^{n-1}}\log\chi_{\mathrm{cycl}}"] \arrow[d, "pt_{n,0}"'] & H^1(K_n, \widehat{\bar{\mathfrak{o}}}) \arrow[d, "\mathrm{cor}"] \\
\mathfrak{o} \arrow[r, "\cup\log\chi_{\mathrm{cycl}}"] & H^1(K, \widehat{\bar{\mathfrak{o}}})
\end{tikzcd}$$

is commutative.

Proof. For $n = 1$, the second diagram commutes by 3.3.9 with $K' = K_1$. By transitivity of trace and corestriction, the lemma will be proved if we verify the commutativity of the first diagram for $n = m + 1 > 1$. Take the diagram of 3.3.9 for K_{m+1}/K_m and factorise:

$$\begin{tikzcd}
\mathfrak{o}[\zeta_{p^{m+1}}] \arrow[rr, "\cup p^{-m}\log\chi_{\mathrm{cycl}}"] \arrow[d, "\mathrm{tr}_{K_{m+1}/K_m}"] \arrow[dd, bend right=60, "\frac{1}{p}\mathrm{tr}_{K_{m+1}/K_m}"'] & & H^1(K_{m+1}, \widehat{\bar{\mathfrak{o}}}) \arrow[d, "\mathrm{cor}"] \\
p\mathfrak{o}[\zeta_{p^m}] \arrow[r, "\cup p^{-m}\log\chi_{\mathrm{cycl}}"] & H^1(K_m, \mathfrak{o}_m) \arrow[r] & H^1(K_m, \widehat{\bar{\mathfrak{o}}}) \\
\mathfrak{o}[\zeta_{p^m}] \arrow[u, "\times p"] \arrow[ur, "\cup p^{1-m}\log\chi_{\mathrm{cycl}}"'] & &
\end{tikzcd}$$

The bottom triangle commutes since $H^1(K_m, \mathfrak{o}_m) = \mathrm{Hom}_{\mathrm{cts}}(\mathrm{Gal}(\bar{K}/K_m), \mathfrak{o}_m)$ is torsion-free. Hence the entire diagram is commutative, and going round the outside gives what we need. □

Now consider $\mathfrak{d}_n = \mathfrak{d}_{K_n/K}$. From the definition of $\mathfrak{d}_n^{-1}$ as the largest fractional ideal of K_n whose trace is contained in $\mathfrak{o}$, it is an easy exercise to check

$$\mathfrak{d}_n^{-1} \subset p^{-n}\mathfrak{o}[\zeta_{p^n}].$$

By [37, Propn. 5] the difference $v_p(\mathfrak{d}_n) - n$ is bounded, so for some c independent of n,

$$p^c\mathfrak{o}_n \subset \mathfrak{o}[\zeta_{p^n}] \subset \mathfrak{o}_n.$$

Since $\Omega_{\mathfrak{o}_n/\mathfrak{o}}$ is cyclic with annihilator $\mathfrak{d}_n$, the homomorphism

$$\mathfrak{o}[\zeta_{p^n}]/p^n \xrightarrow{x \mapsto x \operatorname{dlog} \zeta_{p^n}} \Omega_{\mathfrak{o}_n/\mathfrak{o}} \tag{3.3.11}$$

is well-defined, and its kernel and cokernel are killed by a bounded power of p, by remark 3.3.2.

Proposition 3.3.12. *Let $n > m \geq \max(\ell, 1)$. Then the diagram*

$$\begin{tikzcd}
\mathfrak{o}[\zeta_{p^n}] \arrow[r, "\times \operatorname{dlog} \zeta_{p^n}"] \arrow[d, "t_{n,m}"'] & \Omega_{\mathfrak{o}_n/\mathfrak{o}} \arrow[d, "\mathrm{tr}"] \\
\mathfrak{o}[\zeta_{p^m}] \arrow[r, "\times \operatorname{dlog} \zeta_{p^m}"] & \Omega_{\mathfrak{o}_m/\mathfrak{o}}
\end{tikzcd}$$

commutes.

Proof. It is enough to compute what happens when $m = n - 1$. Taking $1, \zeta_{p^n}, \dots, \zeta_{p^n}^{p-1}$ as basis for $\mathfrak{o}[\zeta_{p^n}]$ over $\mathfrak{o}[\zeta_{p^{n-1}}]$, for $1 \leq j < p$

$$\mathrm{tr}(\zeta_{p^n}^j \operatorname{dlog} \zeta_{p^n}) = \mathrm{tr}(j^{-1} d\zeta_{p^n}^j) = j^{-1} d(\mathrm{tr}\, \zeta_{p^n}^j) = 0$$

and for $j = 0$

$$\mathrm{tr}(\operatorname{dlog} \zeta_{p^n}) = \operatorname{dlog}(N_{K_n/K_{n-1}} \zeta_{p^n}) = \operatorname{dlog} \zeta_{p^{n-1}}. \qquad \square$$

Therefore passing to the inverse limit gives a homomorphism

$$\varprojlim_{t_{-,-}} \mathfrak{o}[\zeta_{p^n}] = \varprojlim_{t_{-,-}} \mathfrak{o}[\zeta_{p^n}]/p^n \longrightarrow \varprojlim_{\mathrm{tr}} \Omega_{\mathfrak{o}_n/\mathfrak{o}} \tag{3.3.13}$$

which becomes an isomorphism when tensored with $\mathbb{Q}$. (If $K/\mathbb{Q}_p$ is unramified, then (3.3.13) is itself an isomorphism.) By [38, Proposition 2.2], the canonical map $H^1(K_m, \mathbb{Z}_p) \to \varprojlim_n H^1(K_m, \mathbb{Z}/p^n)$ is an isomorphism. Inverting both of these arrows yields a diagram

$$\begin{tikzcd}
\mathbb{Q} \otimes \varprojlim_{\mathrm{norm}} \mathfrak{o}_n^* \arrow[r, "\mathrm{Kummer}"] \arrow[d, "\operatorname{dlog}"'] & \mathbb{Q} \otimes \varprojlim_n H^1(K_n, \boldsymbol{\mu}_n) \arrow[d, "\zeta_{p^n} \mapsto 1"] \\
\mathbb{Q} \otimes \varprojlim_{\mathrm{trace}} \Omega_{\mathfrak{o}_n/\mathfrak{o}} \arrow[d, "\operatorname{dlog} \zeta_{p^n} \mapsto 1"', "\wr"] & \mathbb{Q} \otimes \varprojlim_n H^1(K_n, \mathbb{Z}/p^n) \arrow[d, "\mathrm{cor}"] \\
\mathbb{Q} \otimes \varprojlim_{t_{-,-}} \mathfrak{o}[\zeta_{p^n}]/p^n \arrow[d, "(t_{n,m})_n"'] & \mathbb{Q} \otimes \varprojlim_n H^1(K_m, \mathbb{Z}/p^n) \arrow[d] \\
K_m \arrow[r, "\sim", "\cup \frac{1}{p^m} \log \chi_{\mathrm{cycl}}"'] & H^1(K_m, \widehat{\overline{K}})
\end{tikzcd} \tag{3.3.14}$$

where down the right-hand side all the inverse limits are with respect to the corestriction maps and reduction mod p^n. We then have the following version of the classical explicit reciprocity law of Artin-Hasse and Iwasawa. Without loss of generality we can assume m chosen so that $\boldsymbol{\mu}_{p^{m+1}} \not\subset K$.

Theorem 3.3.15. *The diagram* (3.3.14) *is commutative.*

Proof. At finite level, replacing $\mathbb{Z}/p^n$ with $\bar{\mathfrak{o}}/p^n$, one has the diagram:

$$\begin{tikzcd}
\mathfrak{o}_n^* \arrow[r, "\text{Kummer}"] \arrow[d, "\mathrm{dlog}"'] & H^1(K_n, \boldsymbol{\mu}_{p^n}) \arrow[d, "\mathrm{dlog}"'] \arrow[dd, bend left, "\zeta_{p^n} \mapsto 1"] \\
\Omega^1_{\mathfrak{o}_n/\mathfrak{o}} \arrow[r, "\delta_{K_n} \bmod p^n"] & H^1(K_n, {}_{p^n}\Omega_{\bar{\mathfrak{o}}/\mathfrak{o}}) \\
\mathfrak{o}[\zeta_{p^n}]/p^n \arrow[r, "\cup \frac{1}{p^n} \log \chi_{\mathrm{cycl}}"] \arrow[u, "1 \mapsto \mathrm{dlog}\, \zeta_{p^n}"] \arrow[d, "t_{n,m}"'] & H^1(K_n, \bar{\mathfrak{o}}/p^n) \arrow[u, "1 \mapsto \mathrm{dlog}\, \zeta_{p^n}"] \arrow[d, "\mathrm{cor}"] \\
\mathfrak{o}[\zeta_{p^m}]/p^n \arrow[r, "\cup \frac{1}{p^m} \log \chi_{\mathrm{cycl}}"] & H^1(K_m, \bar{\mathfrak{o}}/p^n).
\end{tikzcd}$$

This is for $m > 0$; for $m = 0$ the bottom arrow should read $p^{-1}\mathfrak{o}/p^n \to H^1(K, p^{-1}\bar{\mathfrak{o}}/p^n)$. The top two squares commute by 3.3.7, 3.3.8 respectively. The bottom square commutes up to p-torsion by (3.2.5). All maps are compatible with passing to the inverse limit. As remarked after equation (3.3.11), the left-hand map labelled "$1 \mapsto \mathrm{dlog}\, \zeta_{p^n}$" has cokernel and kernel killed by a bounded power of p, and by (3.3.3) the same is true for the one on the right. Therefore passing to the limit and tensoring with $\mathbb{Q}$ one obtains the theorem. ☐

Remark. One can use 3.3.7(ii) to describe the image of an arbitrary element of K_n^* under the Kummer map in a similar way.

Here is the relation with the usual form of the explicit reciprocity law. Let $u = (u_n)_n \in \varprojlim \mathfrak{o}_n^*$ be a universal norm. Its image down the left hand side of the diagram (3.3.14) equals (with an obvious abuse of notation)

$$\Phi(u) := \lim_{n \to \infty} \frac{1}{p^{n-m}} \operatorname{tr}_{K_n/K_m} \left(\frac{\mathrm{dlog}\, u_n}{\mathrm{dlog}\, \zeta_{p^n}} \right) \in K_m.$$

Going round the other way, use the expression of the Kummer map in terms of the Hilbert symbol, which we write as a bilinear map $[-,-]_n \colon K_n^* \times K_n^* \to \mathbb{Z}/p^n$ given by

$$(\sqrt[p^n]{x})^{(a, K_n^{\mathrm{ab}}/K_n) - 1} = \zeta_{p^n}^{[x,a]_n}$$

Thus u_n is mapped to the cocycle in $H^1(K_n, \mathbb{Z}/p^n)$ which takes the norm residue symbol $(a, K_n^{\mathrm{ab}}/K_n)$ to $[u_n, a]_n$. By the compatibility of the norm residue symbol with norm and corestriction, one gets that the image of the family u in $H^1(K_m, \mathbb{Z}_p)$ is represented by the cocycle (i.e. homomorphism)

$$(a, K_m^{\mathrm{ab}}/K_m) \longmapsto \lim_{n\to\infty} [u_n, a]_n \in \mathbb{Z}_p$$

Therefore the reciprocity law says that this homomorphism, and the homomorphism

$$g \longmapsto p^{-m}\Phi(u) \log \chi_{\mathrm{cycl}}(g)$$

represent the same cohomology class in $H^1(K_m, \widehat{\overline{K}})$.

Proposition 3.3.16. [35, III.A7 ex. 2] *Let* $c_K\colon \mathrm{Hom}_{\mathbb{Q}_p}(K, K) \to K$ *be the unique map such that for all* $T \in \mathrm{Hom}_{\mathbb{Q}_p}(K, K)$ *and all* $x \in K$*,*

$$\mathrm{tr}([\times x] \circ T) = \mathrm{tr}_{K/\mathbb{Q}_p}(x\, c_K(T)).$$

Then the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbb{Q}_p}(K, K) \arrow[r, "c_K"] \arrow[d, "\circ \log"', "\wr"] & K \arrow[dd, "\wr"', "\log \chi_{\mathrm{cycl}}"] \\
\mathrm{Hom}_{\mathrm{cts}}(\mathfrak{o}^*, K) & \\
\mathrm{Hom}_{\mathrm{cts}}(G_K^{\mathrm{ab}}, K) \arrow[u, "\text{local CFT}", "\wr"'] \arrow[r] & H^1(G_K, \widehat{\overline{K}})
\end{tikzcd}$$

is commutative. □

Remark. Because of the normalisation of the reciprocity law of local class field theory used here (see §3.1), this differs from the statement in [35] by a sign.

Now the composite

$$\mathrm{Hom}_{\mathbb{Q}_p}(K, \mathbb{Q}_p) \hookrightarrow \mathrm{Hom}_{\mathbb{Q}_p}(K, K) \xrightarrow{\ c_K\ } K$$

is the inverse of the isomorphism $K \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Q}_p}(K, \mathbb{Q}_p)$ given by the trace form. Therefore, for every $a \in \mathfrak{o}_m^*$,

$$\lim_{n\to\infty} [u_n, a]_n = p^{-m}\, \mathrm{tr}_{K_m/\mathbb{Q}_p}(\Phi(u) \log a)$$

which is the "limit form" of the classical explicit reciprocity law [20, Ch. 9, Thm. 1.2].

3.4 Big local fields

This section reviews the generalisation by Hyodo [14, esp. §4] and Faltings [10, §2] of Fontaine's theory to local fields with imperfect residue field. We consider fields $L \supset \mathbb{Q}_p$ such that:

$$\begin{array}{l} L \text{ is complete with respect to a discrete valuation, and} \\ \text{its residue field } \ell \text{ satisfies } [\ell : \ell^p] = p^r < \infty. \end{array} \tag{3.4.1}$$

Fix such a field L, and write A for its ring of integers. If $R \subset A$ is any subring, define

$$\widehat{\Omega}_{A/R} := \varprojlim \Omega_{A/R}/p^n \Omega_{A/R}.$$

Fix also an algebraic closure $\bar{L}$ of L, and let $\bar{A}$ be the integral closure of A in $\bar{L}$. For any B with $A \subset B \subset \bar{A}$ and any subring $R \subset B$ set

$$\widehat{\Omega}_{B/R} = \varinjlim_{A'} \widehat{\Omega}_{A'/A' \cap R}$$

the limit running over all finite extensions A'/A contained in B.

Let $K \subset L$ be a finite extension of $\mathbb{Q}_p$, with ring of integers $\mathfrak{o}$ and uniformiser π_K. Then π_K is prime in A if and only if $A/\mathfrak{o}$ is formally smooth (by [23], (28.G) and Theorems 62, 82).

Let L'/L be a finite extension with valuation ring A'. Then A' is finite over A (being the normalisation of a complete DVR in a finite extension), and is a relative complete intersection (by EGA IV 19.3.2). Therefore the first exact sequence of differentials is exact on the left as well[4]

$$0 \to A' \otimes_A \Omega_{A/\mathfrak{o}} \to \Omega_{A'/\mathfrak{o}} \to \Omega_{A'/A} \to 0.$$

[4]More generally, if A'/A is a relative complete intersection of integral domains which is generically smooth, then for any $R \subset A$ the first exact sequence is exact on the left. For an elementary proof, write A' as the quotient B/I of a polynomial algebra B over A by an ideal I generated by a regular sequence. Then one has a split exact sequence

$$0 \to \Omega_{A/R} \otimes B \to \Omega_{B/R} \to \Omega_{B/A} \to 0 \tag{1}$$

as well as exact sequences, for $? = A$ or R,

$$I/I^2 \to \Omega_{B/?} \otimes_B A' \to \Omega_{A'/?} \to 0. \tag{2}$$

Applying the tensor product $\otimes_B A'$ to (1), and using (2) and the snake lemma, gives the exact sequence

$$N_{A'/A} \to A' \otimes_A \Omega_{A/R} \to \Omega_{A'/R} \to \Omega_{A'/A} \to 0.$$

where $N_{A'/A} = \ker(I/I^2 \to \Omega_{B/A} \otimes_B A')$. Since A'/A is generically smooth the map $I/I^2 \to \Omega_{B/A}$ is generically an injection, hence $N_{A'/A}$ is torsion. Now I/I^2 is projective since I is a regular ideal; therefore $N_{A'/A} = 0$.

As in (3.3.5), we get an exact sequence of inverse systems

$$ {}_{p^n}\Omega_{A'/A} \to A' \otimes_A \Omega_{A/\mathfrak{o}}/p^n \to \Omega_{A'/\mathfrak{o}}/p^n \to \Omega_{A'/A}/p^n \to 0. $$

Since $\Omega_{A'/A}$ is a finite A'-module, the inverse system $({}_{p^n}\Omega_{A'/A})$ is ML-zero, and so passing to the inverse limit gives an exact sequence:

$$ 0 \to A' \otimes_A \widehat{\Omega}_{A/\mathfrak{o}} \to \widehat{\Omega}_{A'/\mathfrak{o}} \to \Omega_{A'/A} \to 0. \tag{3.4.2} $$

Proposition 3.4.3. *(i) $\widehat{\Omega}_{A/\mathfrak{o}}$ is a finite A-module, generated by elements of the form dy, $y \in A^*$.*

(ii) If $T_1, \dots T_r \in A$ are elements whose whose images in ℓ form a p-basis, then $\{\operatorname{dlog} T_i\}$ is a basis for the vector space $\widehat{\Omega}_{A/\mathfrak{o}} \otimes_A L$.

(iii) If π_K is prime in A, then $\widehat{\Omega}_{A/\mathfrak{o}}$ is free over A.

Proof. By [23] pp. 211–212, A is a finite extension of a complete DVR B in which p is prime. Then $A_0 = B\mathfrak{o}$ is a complete DVR with uniformiser π_K, and A/A_0 is finite and totally ramified. Let $k = \mathfrak{o}/\pi_K\mathfrak{o}$. One knows (*loc. cit.*, Thm. 86) that the image of $\{dT_i\}$ is an ℓ-basis for $\Omega_{\ell/k}$, and therefore (by Nakayama's lemma) $\widehat{\Omega}_{A_0/\mathfrak{o}} = \bigoplus A_0 \cdot dT_i = \bigoplus A_0 \cdot \operatorname{dlog} T_i$, proving (iii). To deduce (i) and (ii), it is enough to apply the exact sequence (3.4.2) to $A/A_0/\mathfrak{o}$. □

Taking the direct limit of (3.4.2) over all finite extensions L'/L, one gets an exact sequence

$$ 0 \to \bar{A} \otimes_A \widehat{\Omega}_{A/\mathfrak{o}} \to \widehat{\Omega}_{\bar{A}/\mathfrak{o}} \to \Omega_{\bar{A}/A} \to 0 $$

of $\bar{A}$-modules. Now apply (3.3.5) again. Since $x\,dy = pz^{p-1}x\,dz$ if $y = z^p$, one sees (using 3.4.3(i)) that $\widehat{\Omega}_{\bar{A}/\mathfrak{o}}$ and $\Omega_{\bar{A}/A}$ are divisible. Therefore, since $\widehat{\Omega}_{A/\mathfrak{o}}$ is finitely generated, one can pass to the limit to get an exact sequence of $\widehat{\bar{A}}$-modules

$$ 0 \to T_p(\widehat{\Omega}_{\bar{A}/\mathfrak{o}}) \to T_p(\Omega_{\bar{A}/A}) \xrightarrow{\pi} \widehat{\bar{A}} \otimes_A \widehat{\Omega}_{A/\mathfrak{o}} \to 0. \tag{3.4.4} $$

Because $\widehat{\Omega}_{A/\mathfrak{o}}$ is a finite A-module, the map π has a continuous set-theoretic section (write $\widehat{\Omega}_{A/\mathfrak{o}} = P \oplus N$ with P free and N torsion; over $\widehat{\bar{A}} \otimes P$ one has a continuous linear section of π by freeness, and $\widehat{\bar{A}} \otimes N$ is discrete, so over it one can take any section).

One then has Hyodo's generalisation [14, (4-2-2)] of 3.3.1 (see also [10, §2b)]):

Proposition 3.4.5. *Let $\mathfrak{a}_{\bar{\mathfrak{o}}/\mathfrak{o}}$ be as in 3.3.1 above, and put $\mathfrak{a}_{\bar{A}/\mathfrak{o}} = \mathfrak{a}_{\bar{\mathfrak{o}}/\mathfrak{o}}\bar{A} \subset \bar{L}$. Then there is an exact sequence of $\bar{A}$-modules and Galois-equivariant maps*

$$0 \to \mathfrak{a}_{\bar{A}/\mathfrak{o}}(1) \xrightarrow{\subseteq} \bar{L}(1) \xrightarrow{\alpha} \widehat{\Omega}_{\bar{A}/\mathfrak{o}} \xrightarrow{\beta} \bar{L}^r \to 0$$

where α is given by the same formula as in 3.3.1, and where the map β is a split surjection, with right inverse

$$\begin{aligned} \bar{L}^r &\to \widehat{\Omega}_{\bar{A}/\mathfrak{o}} \\ (a_1/p^n, \ldots, a_r/p^n) &\mapsto \sum a_i \operatorname{dlog}(T_i^{p^{-n}}) \quad (a_i \in \bar{A}). \end{aligned} \qquad \square$$

Remark. Hyodo states this only in the case $K = \mathbb{Q}_p$, but his proof works in general. The key point (which underlies Faltings' approach to p-adic Hodge theory) is that the extension $\bar{L}/L(\boldsymbol{\mu}_{p^\infty}, T_i^{p^{-\infty}})$ is almost unramified (cf. the proof of Proposition 3.4.12 below), which shows that $\widehat{\Omega}_{\bar{A}/\mathfrak{o}}$ is generated as an $\widehat{\bar{A}}$-module by the forms $\operatorname{dlog}\zeta_{p^n}$, $\operatorname{dlog} T_i^{p^{-n}}$.

Corollary 3.4.6. *There is a unique isomorphism*

$$\widehat{\mathfrak{a}}_{\bar{A}/\mathfrak{o}}(1) \xrightarrow{\sim} T_p(\widehat{\Omega}_{\bar{A}/\mathfrak{o}}) \tag{3.4.7}$$

which maps $(\zeta_{p^n})_n \in \mathbb{Z}_p(1)$ to $(\operatorname{dlog}\zeta_{p^n})_n$. $\square$

Remark. Comparing (3.3.3) and (3.4.7) we have in particular

$${}_{p^n}\widehat{\Omega}_{\bar{A}/\mathfrak{o}} = {}_{p^n}\Omega_{\bar{\mathfrak{o}}/\mathfrak{o}} \otimes_{\bar{\mathfrak{o}}} \bar{A}. \tag{3.4.8}$$

Now consider as before the connecting homomorphism attached to the Galois cohomology of (3.4.4), for a (not necessarily finite) extension L'/L contained in $\bar{L}$:

$$\delta_{L'/L} \colon \widehat{\Omega}_{A/\mathfrak{o}} \otimes_A \widehat{A}' \to H^1(L', T_p\widehat{\Omega}_{\bar{A}/\mathfrak{o}}) \tag{3.4.9}$$

For $L' = L$ we write δ_L for $\delta_{L'/L}$. If L'/L is finite the maps $\delta_{L'}$, $\delta_{L'/L}$ are related by a commutative diagram

$$\begin{tikzcd} \widehat{\Omega}_{A/\mathfrak{o}} \otimes_A A' \arrow[d, "\text{canonical}"'] \arrow[dr, "\delta_{L'/L}"] & \\ \widehat{\Omega}_{A'/\mathfrak{o}} \arrow[r, "\delta_{L'}"] & H^1(L', T_p\widehat{\Omega}_{\bar{A}/\mathfrak{o}}) \end{tikzcd} \tag{3.4.10}$$

(because the exact sequence (3.4.4) is functorial in A). If L'/L is infinite, we define $\delta_{L'} \colon \widehat{\Omega}_{A'/\mathfrak{o}} \to H^1(L', T_p\widehat{\Omega}_{\bar{A}/\mathfrak{o}})$ as the direct limit of the maps $\delta_{L''}$, for finite subextensions $L \subset L'' \subset L'$; the analogue of (3.4.10) still holds.

The following lemma is proved just the same way as 3.3.7.

Lemma 3.4.11. *For any algebraic extension L'/L, the following diagram commutes:*

$$\begin{tikzcd} A'^{*} \arrow[r, "\text{Kummer}"] \arrow[d, "\mathrm{dlog}"'] & H^1(L', \boldsymbol{\mu}_{p^n}) \arrow[d, "\mathrm{dlog}"] \\ \widehat{\Omega}_{A'/\mathfrak{o}} \arrow[r, "\delta_{L'} \bmod p^n"] & H^1(L', {}_{p^n}\widehat{\Omega}_{\bar{A}/\mathfrak{o}}) \end{tikzcd}$$

□

Proposition 3.4.12. *Let L_∞/L be an algebraic extension which contains all p-th power roots of unity, with valuation ring A_∞, and whose residue field extension is separable. Suppose that $r = 1$, so that $[l\colon l^p] = p$. Then for $j \geq 2$, $H^j(L_\infty, T_p\widehat{\Omega}_{\bar{A}/\mathfrak{o}})$ is killed by the maximal ideal $\mathfrak{m}_\infty \subset A_\infty$, and the kernel and cokernel of*

$$\delta_{L_\infty/L}\colon \widehat{\Omega}_{A/\mathfrak{o}} \otimes_A \widehat{A_\infty} \to H^1(L_\infty, T_p\widehat{\Omega}_{\bar{A}/\mathfrak{o}})$$

are killed by a power of p.

Proof. Initially there is no need to make any assumption on r. Choose units $T_1, \ldots, T_r \in A^*$ whose images in l form a p-basis. Consider the extensions $M = L(T_i^{p^{-\infty}}, \ldots T_i^{p^{-\infty}})$ and $M_\infty = ML_\infty$. Let B, B_∞ be the valuation rings of M, M_∞. Then the residue field of M is perfect, so Tate's theory [37] applies; in particular, the groups $H^i(M_\infty, \widehat{\bar{A}})$ are $\mathfrak{m}_\infty$-torsion for $i > 0$. Therefore, using the Hochschild-Serre spectral sequence and the fact that $\mathfrak{m}_\infty^2 = \mathfrak{m}_\infty$, the inflation map

$$H^j(M_\infty/L_\infty, \widehat{B_\infty}) = H^j(M_\infty/L_\infty, H^0(M_\infty, \widehat{\bar{A}})) \to H^j(L_\infty, \widehat{\bar{A}}) \tag{3.4.13}$$

is an isomorphism up to $\mathfrak{m}_\infty$-torsion. Now by Kummer theory and the hypothesis on the residue fields, $\mathrm{Gal}(M_\infty/L_\infty) \simeq \mathbb{Z}_p(1)^r$ (the isomorphism being determined by the choice of $\{T_i\}$). Therefore if $r = 1$

$$H^j(M_\infty/L_\infty, \widehat{B_\infty}(1)) = 0 \quad \text{for all } j > 1\text{, and} \tag{3.4.14}$$

$$H^1(M_\infty/L_\infty, \widehat{B_\infty}(1)) \simeq \big(\widehat{B_\infty}\big)_{\mathbb{Z}_p(1)}. \tag{3.4.15}$$

Now by 3.4.6 there exists a (non-canonical!) isomorphism of $\mathrm{Gal}(\bar{L}/L_\infty)$-modules $T_p\widehat{\Omega}_{\bar{A}/\mathfrak{o}} \simeq \widehat{\bar{A}}$. Combining this and equations (3.4.14) and (3.4.13), one sees that that $H^j(L_\infty, T_p\widehat{\Omega}_{\bar{A}/\mathfrak{o}})$ is killed by $\mathfrak{m}_\infty$ for all $j > 1$.

For the second part, we compute the coinvariants in (3.4.15). First observe that the ring $A' = A[T^{p^{-n}}]$ is finite over A, and that $\pi_A A'$ is a maximal ideal in it. Therefore A' is a discrete valuation ring, hence is the valuation ring of $L(T^{p^{-n}})$. It follows that any element of $\widehat{B_\infty}$ has the form

$$b = \sum_{a\in\mathbb{Q}_p/\mathbb{Z}_p} b_a T^a$$

where $T = T_1$ and $b_a \in \widehat{A_\infty}$, with $b_a \to 0$ as $|a|_p \to \infty$. Let $\gamma \in \mathrm{Gal}(M_\infty/L_\infty)$ be the topological generator for which $\gamma(T^{1/p^r}) = \zeta_{p^r} T^{1/p^r}$, for each $r \geq 1$. If b is divisible by $(1-\zeta_p)$, then $b = b_0 + (1-\gamma)b'$, where

$$b' = \sum_{0\neq x/p^r\in\mathbb{Q}_p/\mathbb{Z}_p} (1-\zeta_{p^r}^x)^{-1} b_{x/p^r} T^{x/p^r} \in \widehat{B_\infty}.$$

From this one sees that the inclusion $\widehat{A_\infty} \subset \widehat{B_\infty}$ induces an injection

$$\widehat{A_\infty} \hookrightarrow H^1(M_\infty/L_\infty, \widehat{B_\infty}(1)) \tag{3.4.16}$$

whose cokernel is killed by $(1-\zeta_p)$. Now there is a diagram

$$\begin{tikzcd}
\widehat{A_\infty} \arrow[r, "(3.4.16)"] \arrow[dd, hook, "1\mapsto \mathrm{dlog}\, T"'] & H^1(M_\infty/L_\infty, \widehat{B_\infty}(1)) \arrow[d, hook, "\mathrm{infl}"] \\
 & H^1(L_\infty, \widehat{\bar{A}}(1)) \arrow[d, hook, "(3.4.7)"] \\
\widehat{\Omega}_{A/\mathfrak{o}} \otimes \widehat{A_\infty} \arrow[r, "\delta_{L_\infty/L}"] & H^1(L_\infty, T_p\widehat{\Omega}_{\bar{A}/\mathfrak{o}})
\end{tikzcd}$$

in which the vertical arrows have kernel and cokernel killed by a power of p (by 3.4.3, 3.4.6 and (3.4.13)). It remains to check that it is commutative, which having got this far is an easy exercise. $\square$

A similar computation can be carried out for all $r > 1$, using the isomorphism $\mathrm{Gal}(M_\infty/L_\infty) \simeq \mathbb{Z}_p(1)^r$ and the Koszul complex. In this way Hyodo computed the cohomology of $\widehat{\bar{L}}(j)$ over L, generalising Tate's result. His final result (not needed here) is:

Theorem 3.4.17. [14, Theorem 1] *There are canonical isomorphisms*

$$H^q(L, \widehat{\bar{L}}(j)) \xrightarrow{\sim} \begin{cases} \widehat{\Omega}^q_{A/\mathfrak{o}} \otimes \mathbb{Q} & \text{if } j = q \\ \widehat{\Omega}^{q-1}_{A/\mathfrak{o}} \otimes \mathbb{Q} & \text{if } j = q-1 \\ 0 & \text{otherwise} \end{cases}$$

compatible with cup-product. For $j = q-1 = 0$ it is given by cup-product with $\log\chi_{\mathrm{cycl}}$ and for $q = j = 1$ by (3.4.7) *and* (3.4.9). $\square$

3.5 Proof of Theorem 3.2.3

Theorem 3.2.3 is proved by reducing to the setting of the previous section. Recall that X is a smooth and proper curve over the ring of integers $\mathfrak{o}$ of a finite unramified extension $K/\mathbb{Q}_p$. Assume that X is connected and that $\Gamma(X, \mathcal{O}_X) = \mathfrak{o}$ (otherwise first replace K by an unramified extension). Let $\eta \in X$ be the generic point of the special fibre. Write also:

$$A = \widehat{\mathcal{O}_{X.\eta}};\ L = \text{field of fractions of } A;$$

$$L_n = L(\boldsymbol{\mu}_{p^n});\ A_n = \text{integral closure of } A \text{ in } L_n;$$

The fields L, L_n satisfy the hypothesis (3.4.1), with $r = 1$. There is an obvious localisation map $\phi\colon\ \operatorname{Spec} A \to Y$. Note that since $A/\mathfrak{o}$ is formally smooth we actually have $A_n = A \otimes \mathfrak{o}_n$; and by 3.4.3, $\widehat{\Omega}_{A/\mathfrak{o}}$ is a free A-module of rank 1.

Now use the fact that the map

$$\phi^*\colon\ \mathrm{Fil}^1 H^1_{\mathrm{dR}}(Y/\mathfrak{o}) = H^0(X, \Omega^1_{X/\mathfrak{o}}(\log Z)) \to \widehat{\Omega}_{A/\mathfrak{o}}$$

is injective and its cokernel is torsion-free (this holds because the fibres of $X/\mathfrak{o}$ are connected). This means that the diagram in Theorem 3.2.3 can be localised to $\operatorname{Spec} A$ without losing information. We shall write down the localised diagram and then explain why it implies 3.2.3.

Proposition 3.5.1. *There exists an integer c such that for every $n > 0$ the following diagram commutes up to p^c-torsion:*

$$\begin{tikzcd}
(K_2(A_n) \otimes \boldsymbol{\mu}_{p^n}^{\otimes -1})^0 \arrow[r, "\mathrm{ch}"] \arrow[d, "\mathrm{dlog}"'] & H^2(L_n, \boldsymbol{\mu}_{p^n})^0 \arrow[d, "\text{Hochschild-Serre}"] \\
\Omega^2_{A_n/\mathfrak{o}}(-1) \arrow[d, equal] & H^1(K_n, H^1(L\overline{K}, \boldsymbol{\mu}_{p^n})) \arrow[d, equal] \\
\Omega^1_{\mathfrak{o}_n/\mathfrak{o}}(-1) \otimes_{\mathfrak{o}} \widehat{\Omega}_{A/\mathfrak{o}} \arrow[d, "\mathrm{dlog}\,\zeta_{p^n} \otimes [\zeta_{p^n}]^{-1} \mapsto 1"', "\wr"] & H^1(K_n, (A\overline{\mathfrak{o}})^*/p^n) \arrow[d, "\mathrm{dlog}"] \\
\mathfrak{o}_n/\mathfrak{d}_n \otimes_{\mathfrak{o}} \widehat{\Omega}_{A/\mathfrak{o}} \arrow[r, "\cup(1/p^n)\log\chi_{\mathrm{cycl}}"] & H^1(K_n, \widehat{\Omega}_{A/\mathfrak{o}} \otimes \overline{\mathfrak{o}}/p^{n-1})
\end{tikzcd}$$

Remarks. (i) Since $A/\mathfrak{o}$ is formally smooth, the valuation ring of $L\overline{K}$ is simply $A\overline{\mathfrak{o}}$.

(ii) We have written

$$H^2(L_n, \boldsymbol{\mu}_{p^n})^0 = \ker\left(\mathrm{res}\colon H^2(L_n, \boldsymbol{\mu}_{p^n}) \to H^2(L\overline{K}, \boldsymbol{\mu}_{p^n})\right)$$
$$(K_2(A_n) \otimes \boldsymbol{\mu}_{p^n}^{\otimes -1})^0 = \ker\left(\mathrm{ch}\colon K_2(A_n) \otimes \boldsymbol{\mu}_{p^n}^{\otimes -1} \to H^2(L\overline{K}, \boldsymbol{\mu}_{p^n})\right).$$

The map marked "Hochschild-Serre" is then the first edge-homomorphism from the Hochschild-Serre spectral sequence.

(iii) Concerning the bottom right-hand corner: the natural map is

$$\mathrm{dlog}\colon (A\bar{\mathfrak{o}})^*/p^n \to \widehat{\Omega}_{A\bar{\mathfrak{o}}/\mathfrak{o}}/p^n$$

but as $A/\mathfrak{o}$ is formally smooth

$$\widehat{\Omega}_{A\bar{\mathfrak{o}}/\mathfrak{o}} = (\widehat{\Omega}_{A/\mathfrak{o}} \otimes \bar{\mathfrak{o}}) \oplus (\Omega_{\bar{\mathfrak{o}}/\mathfrak{o}} \otimes_{\mathfrak{o}} A)$$

and the second summand is divisible.

(iv) To deduce Theorem 3.2.3 from the proposition, it is enough, by what has already been said, to show that there is a map from the diagram in 3.2.3 to the diagram above. Since the composite $K_2(Y \otimes \mathfrak{o}_n) \to K_2(A_n) \to H^2(L\overline{K}, \mu_{p^n}^{\otimes 2})$ factors through $H^2(Y \otimes \overline{K}, \mu_{p^n}^{\otimes 2}) = 0$, one obtains the map $K_2(Y \otimes \mathfrak{o}_n) \to K_2(A_n)^0$. The only remaining thing to check is that the diagram

$$\begin{tikzcd}
H^1(K_n, H^1(Y \otimes \overline{K}, \boldsymbol{\mu}_{p^n})) \arrow[r] \arrow[dd, "\pi_1 \pmod{p^n}"] & H^1(K_n, H^1(L\overline{K}, \boldsymbol{\mu}_{p^n})) \arrow[d, equal] \\
 & H^1(K_n, (A\bar{\mathfrak{o}})^*/p^n) \arrow[d, "\mathrm{dlog}"] \\
H^1(K_n, \bar{\mathfrak{o}}(1)/p^n) \otimes_{\mathfrak{o}} \mathrm{Fil}^1\, H^1_{\mathrm{dR}}(Y/\mathfrak{o}) \arrow[r] & H^1(K_n, \widehat{\Omega}_{A/\mathfrak{o}} \otimes \bar{\mathfrak{o}}/p^n)
\end{tikzcd}$$

commutes, but this follows from the description of π_1 given in §3.2.

Proof of 3.5.1. We reduce the diagram to the (smaller) diagrams in the following three lemmas. By (3.4.7), ${}_{p^n}\widehat{\Omega}_{\bar{A}/\mathfrak{o}}$ is free over $\bar{A}/p^n$ of rank one, and by ${}_{p^n}\widehat{\Omega}^{\otimes 2}_{\bar{A}/\mathfrak{o}}$ we mean its tensor square as $\bar{A}/p^n$-module.

Lemma 3.5.2. *For any m, n the diagram below commutes:*

$$\begin{tikzcd}
K_2(A_m) \arrow[r, "\mathrm{ch}"] \arrow[d, "\mathrm{dlog}"'] & H^2(L_m, \boldsymbol{\mu}_{p^n}^{\otimes 2}) \arrow[d] \\
\widehat{\Omega}^2_{A_m/\mathfrak{o}} \arrow[r, "\wedge^2 \delta_{L_m}"] & H^2(L_m, {}_{p^n}\widehat{\Omega}^{\otimes 2}_{\bar{A}/\mathfrak{o}})
\end{tikzcd}$$

in which the unlabelled arrow is induced by $\mathrm{dlog}\colon \boldsymbol{\mu}_{p^n} \to {}_{p^n}\widehat{\Omega}_{\bar{A}/\mathfrak{o}}$.

Proof. Since A_m is local the symbol $A_m^* \otimes A_m^* \to K_2(A_m)$ is surjective. Since the Chern character is compatible with cup-product, the compatibility follows by Lemma 3.4.11. ☐

This reduces the computation to Galois cohomology. Write

$$H^2(L_n, {}_{p^n}\widehat{\Omega}^{\otimes 2}_{\bar{A}/\mathfrak{o}})^0 = \ker\left[H^2(L_n, {}_{p^n}\widehat{\Omega}^{\otimes 2}_{\bar{A}/\mathfrak{o}}) \to H^2(L\overline{K}, {}_{p^n}\widehat{\Omega}^{\otimes 2}_{\bar{A}/\mathfrak{o}})\right].$$

Lemma 3.5.3. *(i) The composite map*

$$\widehat{\Omega}^2_{A_n/\mathfrak{o}} \xrightarrow{\wedge^2\delta_{L_n}} H^2(L_n, {}_{p^n}\widehat{\Omega}^{\otimes 2}_{\bar{A}/\mathfrak{o}}) \longrightarrow H^2(L\overline{K}, {}_{p^n}\widehat{\Omega}^{\otimes 2}_{\bar{A}/\mathfrak{o}})$$

equals zero.

(ii) The following diagram is commutative:

$$\begin{tikzcd}
\Omega_{\mathfrak{o}_n/\mathfrak{o}} \otimes \widehat{\Omega}_{A/\mathfrak{o}} = \widehat{\Omega}^2_{A_n/\mathfrak{o}} \arrow[rr, "\wedge^2\delta_{L_n}"] \arrow[d, "\delta_{K_n/K}\otimes\mathrm{id}"] & & H^2(L_n, {}_{p^n}\widehat{\Omega}^{\otimes 2}_{\bar{A}/\mathfrak{o}})^0 \arrow[dd, "\text{Hochschild-Serre}"] \\
H^1(K_n, {}_{p^n}\widehat{\Omega}_{\bar{\mathfrak{o}}/\mathfrak{o}} \otimes_{\mathfrak{o}} \widehat{\Omega}_{A/\mathfrak{o}}) \arrow[d, "H^1(\mathrm{id}\otimes\delta_{L\overline{K}/L})"] & & \\
H^1(K_n, {}_{p^n}\widehat{\Omega}_{\bar{\mathfrak{o}}/\mathfrak{o}} \otimes_{\bar{\mathfrak{o}}} H^1(L\overline{K}, {}_{p^n}\widehat{\Omega}_{\bar{A}/\mathfrak{o}})) \arrow[rr, equal] & & H^1(K_n, H^1(L\overline{K}, {}_{p^n}\widehat{\Omega}^{\otimes 2}_{\bar{A}/\mathfrak{o}}))
\end{tikzcd}$$

(iii) The map $H^1(\mathrm{id}\otimes\delta_{L\overline{K}/L})$ has kernel and cokernel killed by a bounded power of p.

Proof. (i) The cup-product $\wedge^2\delta_{L_n}$ factorises as

$$\Omega_{\mathfrak{o}_n/\mathfrak{o}} \otimes \widehat{\Omega}_{A/\mathfrak{o}} \xrightarrow{\delta_{K_n}\otimes\delta_{L_n/L}} H^1(K_n, {}_{p^n}\Omega_{\bar{\mathfrak{o}}/\mathfrak{o}}) \otimes H^1(L_n, {}_{p^n}\widehat{\Omega}_{\bar{A}/\mathfrak{o}}) \to H^2(L_n, {}_{p^n}\widehat{\Omega}^{\otimes 2}_{\bar{A}/\mathfrak{o}})$$

and so its composition with the restriction to $L\overline{K}$ is zero (it factors through $H^1(\overline{K}, {}_{p^n}\Omega_{\bar{\mathfrak{o}}/\mathfrak{o}}) \otimes H^1(L\overline{K}, {}_{p^n}\widehat{\Omega}_{\bar{A}/\mathfrak{o}}) = 0$).

(ii) The bottom equality comes from (3.4.8). The commutativity is a general fact. We have groups

$$\Gamma = \mathrm{Gal}(\overline{L}/L_n) \supset \Delta = \mathrm{Gal}(\overline{L}/L\overline{K}), \quad \Gamma/\Delta = \mathrm{Gal}(L\overline{K}/L_n) = \mathrm{Gal}(\overline{K}/K_n)$$

and two exact sequences of Γ-modules

$$\begin{aligned} 0 \to A \ \to B \ \to C \ \to 0 \\ 0 \to A' \to B' \to C' \to 0 \end{aligned}$$

given by (3.3.6) and (3.4.4) respectively. On the first Δ acts trivially. So we have the following diagram

$$\begin{tikzcd}
H^0(\Gamma/\Delta, C) \otimes H^0(\Gamma, C') \arrow[r, "\delta\otimes\delta'"] \arrow[dd, "\delta\otimes\mathrm{id}"'] & H^1(\Gamma/\Delta, A) \otimes H^1(\Gamma, A') \arrow[d, "\cup"] \\
& \ker\bigl[\mathrm{res}\colon H^2(\Gamma, A\otimes A') \to H^2(\Delta, A\otimes A')\bigr] \arrow[d, "\mathrm{HS}"] \\
H^1(\Gamma/\Delta, A) \otimes H^0(\Gamma, C') \arrow[d, "\cup"'] & H^1(\Gamma/\Delta, H^1(\Delta, A\otimes A')) \\
H^1(\Gamma/\Delta, A\otimes H^0(\Delta, C')) \arrow[r, "H^1(\mathrm{id}\otimes\delta')"] & H^1(\Gamma/\Delta, A\otimes H^1(\Delta, A')) \arrow[u]
\end{tikzcd}$$

and it is a simple, if tedious, exercise to check this commutes.

(iii) Follows from Proposition 3.4.12 applied to $L_\infty = L\overline{K}$. □

Lemma 3.5.4. *The following diagram commutes:*

$$\begin{tikzcd}
H^1(L\overline{K}, \boldsymbol{\mu}_{p^n}^{\otimes 2}) \arrow[r, "\mathrm{dlog}\otimes\mathrm{dlog}"] & H^1(L\overline{K}, {}_{p^n}\widehat{\Omega}^{\otimes 2}_{\bar{A}/\mathfrak{o}}) \arrow[d, equal] \\
(A\bar{\mathfrak{o}})^*/p^n(1) \arrow[u, "\mathrm{Kummer}", "\wr"'] \arrow[d, "\mathrm{dlog}"'] & {}_{p^n}\widehat{\Omega}_{\bar{\mathfrak{o}}/\mathfrak{o}} \otimes_{\bar{\mathfrak{o}}} H^1(L\overline{K}, {}_{p^n}\widehat{\Omega}_{\bar{A}/\mathfrak{o}}) \\
\bar{\mathfrak{o}}/p^n(1) \otimes \widehat{\Omega}_{A/\mathfrak{o}} \arrow[r, "x\otimes\zeta_{p^n}\otimes\omega\mapsto x\,\mathrm{dlog}\,\zeta_{p^n}\otimes\omega"] & {}_{p^n}\Omega_{\bar{\mathfrak{o}}/\mathfrak{o}} \otimes_{\mathfrak{o}} \widehat{\Omega}_{A/\mathfrak{o}} \arrow[u, "\mathrm{id}\otimes\delta_{L\overline{K}/L}"']
\end{tikzcd}$$

Proof. This follows from Lemma 3.4.11. □

As $K/\mathbb{Q}_p$ is unramified, we have $\mathfrak{a} = \mathfrak{a}_{\bar{\mathfrak{o}}/\mathfrak{o}} = (\zeta_p - 1)^{-1}\bar{\mathfrak{o}}$ by 3.3.1, so $\mathfrak{d}_n\bar{\mathfrak{o}} = p^n\mathfrak{a}$ and $\bar{\mathfrak{o}}/\mathfrak{d}_n\bar{\mathfrak{o}} \hookrightarrow \mathfrak{a}/p^n$.

We now can make a big diagram:

$$\begin{tikzcd}
(K_2A_n\otimes\mathbb{Z}/p^n)^0 \arrow[r] \arrow[d] & H^2(L_n, \boldsymbol{\mu}_{p^n}^{\otimes 2})^0 \arrow[r] \arrow[d] & H^1(K_n, H^1(L\overline{K}, \boldsymbol{\mu}_{p^n}^{\otimes 2})) \arrow[d] \\
\widehat{\Omega}^2_{A_n/\mathfrak{o}} \arrow[r] \arrow[d, equal] & H^2(L_n, {}_{p^n}\widehat{\Omega}^{\otimes 2}_{\bar{A}/\mathfrak{o}})^0 \arrow[r] & H^1(K_n, H^1(L\overline{K}, {}_{p^n}\widehat{\Omega}^{\otimes 2}_{\bar{A}/\mathfrak{o}})) \arrow[d, equal] \\
\Omega_{\mathfrak{o}_n/\mathfrak{o}} \otimes \widehat{\Omega}_{A/\mathfrak{o}} \arrow[r] \arrow[d, "\mathrm{dlog}\,\zeta_{p^n}\mapsto 1\otimes\zeta_{p^n}", "\wr"'] & H^1(K_n, {}_{p^n}\Omega_{\bar{\mathfrak{o}}/\mathfrak{o}} \otimes_{\mathfrak{o}} \widehat{\Omega}_{A/\mathfrak{o}}) \arrow[r, "(*)"] & H^1(K_n, {}_{p^n}\Omega_{\bar{\mathfrak{o}}/\mathfrak{o}} \otimes_{\bar{\mathfrak{o}}} H^1(L\overline{K}, {}_{p^n}\widehat{\Omega}_{\bar{A}/\mathfrak{o}})) \\
\mathfrak{o}_n/\mathfrak{d}_n(1) \otimes \widehat{\Omega}_{A/\mathfrak{o}} \arrow[r, "(\dagger)"] & H^1(K_n, \mathfrak{a}/p^n(1) \otimes_{\mathfrak{o}} \widehat{\Omega}_{A/\mathfrak{o}}) \arrow[u, "\wr"] & H^1(K_n, (A\bar{\mathfrak{o}})^*/p^n)(1) \arrow[l] \arrow[uuu, bend right, "\wr"']
\end{tikzcd}$$

To save space we have not labelled most of the arrows: they can be found in the corresponding places in the subdiagrams 3.5.2–3.5.4, apart from the arrow labelled (†), which is $\cup(1/p^n)\log\chi_{\mathrm{cycl}}$. The top left square commutes by 3.5.2, and the top right square by functoriality of the Hochschild-Serre spectral sequence. The rectangle in the middle commutes by 3.5.3, and the bottom left square by 3.3.8. The remaining part of the diagram (the right-hand hexagon) commutes by 3.5.4

Going round the outside of the diagram in both directions gives two maps

$$(K_2 A_n \otimes \mathbb{Z}/p^n)^0 \longrightarrow H^1(K_n, \mathfrak{a}/p^n(1) \otimes_{\mathfrak{o}} \widehat{\Omega}_{A/\mathfrak{o}})$$

and it is enough to show that their difference is killed by a bounded power of p. This follows from the commutativity of the diagram, since the kernel of the arrow marked $(*)$ is killed by a bounded power of p, by 3.5.3(iii). $\square$

4 The Rankin-Selberg method

In this section we calculate the projection of the product of two weight one Eisenstein series onto a cuspidal Hecke eigenspace, using the Rankin-Selberg integral. In order to separate the Euler factors more easily, we work semi-adelically, regarding modular forms as functions on $(\mathbb{C}-\mathbb{R})\times GL_2(\mathbb{A}_f)$. The passage from classical to adelic modular forms is well-known, but we review the correspondence briefly in §4.2 since there is more than one possible normalisation. The same applies to the discussion of Eisenstein series in section §4.3.

4.1 Notations

G denotes the algebraic group GL_2, with the standard subgroups

$$P = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}, \quad U = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}, \quad Z = \left\{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right\}$$

If R is a ring and H is G or any of the above subgroups, write H_R for the group of R-valued points of H. If $R \subset \mathbb{R}$ then H_R^+ denotes $\{h \in H_R \mid \det(h) > 0\}$.

The ring of finite adeles of $\mathbb{Q}$ is $\mathbb{A}_f = \hat{\mathbb{Z}}\otimes_{\mathbb{Z}}\mathbb{Q}$. If $\chi = \prod \chi_p \colon \mathbb{A}_f^*/\mathbb{Q}_{>0}^* \to \mathbb{C}^*$ is a character (continuous homomorphism) and M is a multiple of the conductor of χ, then $\chi_{\mathrm{mod}\,M} \colon (\mathbb{Z}/M\mathbb{Z})^* \to \mathbb{C}^*$ denotes the associated (not necessarily primitive) Dirichlet character: for $a \in \hat{\mathbb{Z}}^*$, $\chi(a) = \chi_{\mathrm{mod}\,M}(a \bmod M)$. Of course this means that if $(p, M) = 1$ then $\chi_{\mathrm{mod}\,M}(p \bmod M) = \chi_p(p)^{-1}$.

Write finite idelic and p-adic modulus as $|-|_f$, $|-|_p$, and archimedean absolute value as $|-|_\infty$. If there can be no confusion we drop the subscripts.

Write also H_f, H_p in place of $H_{\mathbb{A}_f}$, $H_{\mathbb{Q}_p}$, and define the standard congruence subgroups

$$G_p \supset K_p = G_{\mathbb{Z}_p} \supset K_0(p^\nu) = \left\{ h \in K_p \mid h \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod p^\nu \right\}$$
$$\supset K_1(p^\nu) = \left\{ h \in K_p \mid h \equiv \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \bmod p^\nu \right\}.$$

Haar measure on all the groups encountered is to be normalised in the usual way: on $\mathbb{Q}_p$ the additive measure dx gives $\mathbb{Z}_p$ measure 1, and on $\mathbb{Q}_p^*$ the multiplicative measure d^*x gives $\mathbb{Z}_p^*$ measure 1. On G_f, G_p the subgroups $G_{\hat{\mathbb{Z}}}$, K_p have measure 1.

Fix additive characters $\psi_p \colon \mathbb{Q}_p \to \mathbb{C}^*$, $\psi_f = \prod \psi_p \colon \mathbb{A}_f \to \mathbb{C}^*$ by requiring $\psi_p(x/p^n) = e^{2\pi i x/p^n}$ for every $x \in \mathbb{Z}$.

$\mathfrak{H}$ denotes the upper-half plane, and $\mathfrak{H}^\pm = \mathbb{C} - \mathbb{R}$. The group $G_\mathbb{R}$ acts on $\mathfrak{H}^\pm$ by linear fractional transformations. Put

$$j(\gamma, \tau) = \det\gamma \cdot (c\tau + d)^{-1} \quad \text{if } \tau \in \mathfrak{H}^\pm,\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_\mathbb{R}$$

so that $j(\gamma,\tau)(1,-\tau)\gamma^{-1} = (1, -\gamma(\tau))$.

Write $\mathcal{S}(\mathbb{A}_f^2)$ for the space of locally constant functions $\mathbb{A}_f^2 \to \mathbb{C}$ of compact support. The group G_f acts on $\mathcal{S}(\mathbb{A}_f^2)$ by the rule

$$(g\phi)(\underline{x}) = \phi(g^{-1}\underline{x}), \quad \phi \in \mathcal{S}(\mathbb{A}_f^2),\ \underline{x} \in \mathbb{A}_f^2. \tag{4.1.1}$$

If $\delta \in \mathbb{A}_f^*$ and $\phi \in \mathcal{S}(\mathbb{A}_f^2)$, write $[\delta]\phi$ for the function

$$[\delta]\phi \colon \underline{x} \mapsto \phi(\delta^{-1}\underline{x}). \tag{4.1.2}$$

So in particular, if ϕ is the characteristic function of an open compact subset $X \subset \mathbb{A}_f^2$, then $[\delta]\phi$ is the characteristic function of δX.

4.2 Adelic modular forms

In the adelic setting, a holomorphic modular form of weight k is a function

$$F \colon \mathfrak{H}^\pm \times G_f \to \mathbb{C}$$

which is holomorphic in the first variable, and satisfies:

(i) For every $\gamma \in G_\mathbb{Q}$, $F(\gamma(\tau), \gamma g) = j(\gamma,\tau)^{-k} F(\tau, g)$;

(ii) There exists an open compact subgroup $K \subset G_f$ such that $F(\tau, gh) = F(\tau, g)$ for all $h \in K$;

(iii) F is holomorphic at the cusps.

Any modular form F has a Fourier expansion

$$F(\tau, g) = \sum_{m \in \mathbb{Q}} a_m(g) e^{2\pi i m \tau}, \quad \tau \in \mathfrak{H},\ g \in G_f$$

where $a_m(g) = 0$ when $m < 0$ (this is the meaning of condition (iii)). Put $A(g) = a_1(g)$, the *Whittaker function* attached to F. Then A is a locally constant function on G_f which satisfies

$$A\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} g\right) = \psi_f(-b) A(g) \quad \text{for all } b \in \mathbb{A}_f. \tag{4.2.1}$$

One can recover the remaining Fourier coefficients (apart from the constant term) from $A(g)$ by

$$a_m(g) = m^k A\left(\begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix} g\right) \quad \text{if } 0 < m \in \mathbb{Q}.$$

It is convenient to introduce the normalised Whittaker function

$$A^*(g) = A(g) |\det g|_f^{-k/2}. \tag{4.2.2}$$

The group G_f acts on adelic modular forms by right translation, and the translates of F by G_f generate an admissible representation, call it π. From the definition π is isomorphic to the representation generated by $A(g)$; the representation generated by $A^*(g)$ is isomorphic to the twist $\pi \otimes |\det|_f^{-k/2}$. If π is irreducible it has a factorisation $\pi = \otimes' \pi_p$, and the centre of G_f acts on the space of π via a character. With the normalisation used here, there is a (finite order) character $\varepsilon\colon \mathbb{A}_f^*/\mathbb{Q}_{>0}^* \to \mathbb{C}^*$ with $\varepsilon(-1) = (-1)^k$ such that

$$\pi\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = \varepsilon(a) |a|_f^k \quad \text{for all } a \in \mathbb{A}_f^*.$$

This means that

$$A^*\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} g\right) = \varepsilon(a) A^*(g) \quad \text{for all } a \in \mathbb{A}_f^* \text{ and } g \in G_f.$$

If F comes from a newform on $\Gamma_1(N)$ then $A(g)$ is factorisable: there are functions $A_p\colon G_p \to \mathbb{C}$, satisfying $A_p(1) = 1$, $\prod A_p(g_p) = A(g)$ for all $g = (g_p) \in G_f$ and such that

$$A_p\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} g\right) = \varepsilon_p(a) |a|_p^k \psi_p(-b) A_p(g) \quad \text{for all } a \in \mathbb{Q}_p^* \text{ and } b \in \mathbb{Q}_p. \tag{4.2.3}$$

Suppose A_p is K_p-invariant. Then π_p is unramified, and its local L-function is given by

$$L(\pi_p, s) = \sum_{r\geq 0} A_p \begin{pmatrix} p^r & 0 \\ 0 & 1 \end{pmatrix} p^{r(k-s)} = \sum_{r\geq 0} A_p^* \begin{pmatrix} p^r & 0 \\ 0 & 1 \end{pmatrix} p^{r(k/2-s)}$$
$$= \frac{1}{(1-\alpha_p p^{-s})(1-\alpha'_p p^{-s})}$$

where

$$\alpha_p + \alpha'_p = p^{k/2} A_p^* \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \alpha_p \alpha'_p = \varepsilon_p(p) p^{k-1}$$

(this is the normalisation of L-functions which gives the functional equation for $s \leftrightarrow k-1-s$; it differs from that of Jacquet-Langlands by a shift).

Complex conjugation of Fourier coefficients defines an involution of the space of modular forms. In representation theoretic terms, this becomes the isomorphism

$$\bar{\pi} \simeq \pi \otimes \varepsilon^{-1}. \tag{4.2.4}$$

If $\lambda\colon \mathbb{A}_f^*/\mathbb{Q}_{>0}^* \to \mathbb{C}^*$ is any character of finite order and F is an adelic modular form of weight k, so is

$$F \otimes \lambda\colon (\tau, g) \mapsto \lambda(\det g) F(\tau, g). \tag{4.2.5}$$

To go from adelic to classical modular forms, let $K(n)$ be the standard level n subgroup of $G_{\hat{\mathbb{Z}}}$. Then

$$G_{\mathbb{Q}} \backslash \mathfrak{H}^{\pm} \times G_f / K(n) = G_{\mathbb{Z}} \backslash \mathfrak{H}^{\pm} \times G_{\hat{\mathbb{Z}}}/K(n) \simeq Y(n)(\mathbb{C}).$$

where the last isomorphism is normalised in such a way that the point $(\tau, h) \in \mathfrak{H}^{\pm} \times GL_2(\mathbb{Z}/n\mathbb{Z})$ corresponds to the elliptic curve $E_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ with level structure

$$\begin{aligned} \alpha_{\tau,h}\colon \underline{v} &\mapsto (1/n, -\tau/n) \cdot h \cdot \underline{v} \\ (\mathbb{Z}/n\mathbb{Z})^2 &\to (\frac{1}{n}\mathbb{Z} + \frac{\tau}{n}\mathbb{Z})/(\mathbb{Z} + \tau\mathbb{Z}) = \ker[\times n]_{E_\tau}. \end{aligned}$$

Write z for the coordinate on E_τ, and let F be an adelic modular form which is invariant under $K(n)$. It corresponds to the classical modular form over $\mathbb{C}$

$$(E_\tau, \alpha_{\tau,h}) \longmapsto F(\tau, \tilde{h}) du^{\otimes k} \in H^0(E_\tau, \boldsymbol{\omega}^{\otimes k})$$

where $h \in G_{\mathbb{Z}/n\mathbb{Z}}$ and $\tilde{h} \in G_{\hat{\mathbb{Z}}}$ is any lifting of h.

The map (2.3.4) $e_N\colon Y(N) \to \operatorname{Spec} \mathbb{Q}(\boldsymbol{\mu}_N)$ is then given on complex points by

$$e_N\colon (\tau, g) \mapsto \psi_f(\pm \det g/N) \quad \text{if } g \in G_{\hat{\mathbb{Z}}}$$

(the sign depends on the normalisation of the e_N-pairing).

4.3 Eisenstein series

Here we establish notations for Eisenstein series in the framework of the previous section. The results quoted can be obtained easily from those found in classical references (probably [31, Chapter VII] is closest to what is found here).

Let $\phi \in \mathcal{S}(\mathbb{A}_f^2)$, with the action (4.1.1) of G_f. The series

$$E_{k,s}(\phi)(\tau, g) = \sum_{0 \neq \underline{m} \in \mathbb{Q}^2} (g\phi)(\underline{m})(m_1 - m_2\tau)^{-k}|m_1 - m_2\tau|^{-2s}$$

is absolutely convergent for $k + 2\,\mathrm{Re}(s) > 2$, with a meromorphic continuation, and satisfies

$$E_{k,s}(\phi)(\gamma\tau, \gamma g) = j(\gamma, \tau)^{-k}|j(\gamma, \tau)|^{-2s} E_{k,s}(\phi)(\tau, g) \quad \text{for all } \gamma \in G_{\mathbb{Q}}.$$

The functions $E_k(\phi) := E_{k,0}(\phi)$ are holomorphic (and therefore modular of weight k) if $k \geq 3$ or $k = 1$; if $k = 2$ they are holomorphic provided that $\int_{\mathbb{A}_f^2} \phi = 0$.

The map $\phi \mapsto E_{k,s}(\phi)$ is G_f-equivariant. In particular, if $\delta = d \in \mathbb{Q}^*$, then in the notation of (4.1.2)

$$E_{k,s}([d]\phi) = d^{-k}|d|_\infty^{-2s} E_{k,s}(\phi). \tag{4.3.1}$$

One can rewrite the Eisenstein series as a sum over the group. If $f\colon G_f \to \mathbb{C}$ is a locally constant function satisfying

$$f\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} g\right) = a^{-k}|a|_\infty^{-2s} f(g) \quad \text{for all } a,\, d \in \mathbb{Q}^*,\, b \in \mathbb{A}_f \tag{4.3.2}$$

then define

$$E_{k,s,f}(\tau, g) = \sum_{\gamma \in P_{\mathbb{Q}}^+ \backslash G_{\mathbb{Q}}^+} f(\gamma g) j(\gamma, \tau)^k |j(\gamma, \tau)|^{2s}. \tag{4.3.3}$$

The relation between the two definitions is that $E_{k,s}(\phi) = E_{k,s,f}$ with

$$f(g) = \sum_{x \in \mathbb{Q}^*} (g\phi)\begin{pmatrix} x \\ 0 \end{pmatrix} x^{-k}|x|_\infty^{-2s}.$$

Moreover every $E_{k,s,f}$ is an $E_{k,s}(\phi)$ for some ϕ.

In the normalisation used here, the Whittaker function of $E_k(\phi)$ is

$$B(g) = \frac{(2\pi i)^k}{(k-1)!} \lim_{s \to 0} \sum_{y \in \mathbb{Q}^*} y^{-k}|y|_\infty^{1-2s} \int_{\mathbb{A}_f} \psi_f(-x/y)(g\phi)\begin{pmatrix} x \\ y \end{pmatrix} dx$$

which can be obtained without too much difficulty from the classical formulae — see e.g. [31, pp.156–7 & 164ff.].

One can decompose the Eisenstein series under the action of the centre of G_f. It is more convenient to replace f and B by the normalised functions (cf. (4.2.2) above)

$$f^*(g) = f(g)|\det g|_f^{-(k+2s)/2}, \quad B^*(g) = B(g)|\det g|_f^{-k/2}$$

and then to write

$$f^*(g) = \sum_\chi f^*_\chi(g), \quad B^*(g) = \sum_\chi B^*_\chi(g)$$

where the functions $f^*_\chi(g)$, $B^*_\chi(g)$ are zero unless $\chi(-1) = (-1)^k$, in which case

$$\begin{aligned} f^*_\chi(g) &= \int\limits_{\mathbb{A}_f^*/\mathbb{Q}^*_{>0}} \chi(a) f^*\left(\begin{pmatrix} a^{-1} & 0 \\ 0 & a^{-1}\end{pmatrix} g\right) d^*a \\ &= 2|\det g|_f^{-(k+2s)/2} \int\limits_{\mathbb{A}_f^*} \chi(x)|x|_f^{k+2s}(g\phi)\begin{pmatrix} x \\ 0\end{pmatrix} d^*x \end{aligned}$$

and

$$\begin{aligned} B^*_\chi(g) &= \int\limits_{\mathbb{A}_f^*/\mathbb{Q}^*_{>0}} \chi(a) B^*\left(\begin{pmatrix} a^{-1} & 0 \\ 0 & a^{-1}\end{pmatrix} g\right) d^*a \\ &= 2\frac{(2\pi i)^k}{(k-1)!}|\det g|_f^{-k/2} \int\limits_{\mathbb{A}_f \times \mathbb{A}_f^*} \chi(y)|y|_f^{k+2s-1}\psi_f(-x/y)(g\phi)\begin{pmatrix} x \\ y\end{pmatrix} dx\, d^*y \Bigg|_{s=0} \end{aligned}$$

If $\phi = \prod \phi_p$ is factorisable, with ϕ_p equal to the characteristic function of $\mathbb{Z}_p^2$ for almost all p, then the expressions above factorise and one has

$$f^*_\chi(g) = 2\prod_p f^*_{\chi_p}(g_p), \quad B^*_\chi(g) = 2\frac{(2\pi i)^k}{(k-1)!}\prod_p B^*_{\chi_p}(g_p) \quad \text{for } g = (g_p) \in G_f$$

where the functions $f^*_{\chi_p}$, $B^*_{\chi'_p}$ are given by local integrals

$$f^*_{\chi_p}(g_p) = |\det g_p|_p^{-(k+2s)/2} \int\limits_{\mathbb{Q}_p^*} \chi_p(x)|x|_p^{k+2s}(g_p\phi_p)\begin{pmatrix} x \\ 0\end{pmatrix} d^*x \qquad (4.3.4)$$

$$\begin{aligned} B^*_{\chi_p}(g_p) &= |\det g_p|_p^{-k/2} \int\limits_{\mathbb{Q}_p\times\mathbb{Q}_p^*} \chi_p(y)|y|_p^{k+2s-1}\psi_p(-x/y)(g_p\phi_p)\begin{pmatrix} x \\ y \end{pmatrix} dx\, d^*y\Big|_{s=0} \\ &= |\det g_p|_p^{-k/2} \int\limits_{\mathbb{Q}_p\times\mathbb{Q}_p^*} \chi_p(y)|y|_p^{k+2s}\psi_p(-x)(g_p\phi_p)\begin{pmatrix} xy \\ y \end{pmatrix} dx\, d^*y\Big|_{s=0} \end{aligned} \tag{4.3.5}$$

In fact the integral in (4.3.5) is a finite sum, because the x-integral is a finite linear combination of integrals of the form

$$\int\limits_{t+p^\nu\mathbb{Z}_p} \psi_p(-x/y)\, dx$$

which vanish if y is sufficiently close to 0. So one can omit s from the formula.

Because of (4.3.2) and (4.2.3) the functions $f^*_{\chi_p}$ and $B^*_{\chi_p}$ are determined by their restrictions to the subgroup

$$\begin{pmatrix} \mathbb{Q}_p^* & 0 \\ 0 & 1 \end{pmatrix} K_p \subset G_p$$

and these are given by

$$f^*_{\chi_p}\left(\begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix} h\right) = \chi_p(m)|m|^{(k+2s)/2} f^*_{\chi_p}(h) \tag{4.3.6}$$

$$B^*_{\chi_p}\left(\begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix} h\right) = |m|_p^{-k/2+1} \int\limits_{\mathbb{Q}_p\times\mathbb{Q}_p^*} \chi_p(y)|y|_p^{k-1}\psi_p(-mx/y)(h\phi_p)\begin{pmatrix} x \\ y \end{pmatrix} dx\, d^*y \tag{4.3.7}$$

$$= |m|_p^{-k/2+1} \int\limits_{\mathbb{Q}_p\times\mathbb{Q}_p^*} \chi_p(y)|y|_p^{k}\psi_p(-mx)(h\phi_p)\begin{pmatrix} xy \\ y \end{pmatrix} dx\, d^*y \tag{4.3.8}$$

4.4 The Rankin-Selberg integral

Let F, G be adelic modular forms of weights $k+l$, k respectively, at least one of which is a cusp form, and let $E_{l,s,f}$ be the Eisenstein series (4.3.3). The product

$$\Omega = E_{l,s,f} G\overline{F} y^{k+l+s-2} |\det g|^{-k-l-s} d\tau \wedge d\bar\tau$$

is a left $G^+_{\mathbb{Q}}$-invariant form on $\mathfrak{H}\times G_f$, and the aim of this and the following sections is to compute the inner product

$$\langle E_{l,s,f}G, F\rangle := \int\limits_{G^+_{\mathbb{Q}}\backslash\mathfrak{H}\times G_f} \Omega\, dg$$

which is a Rankin-Selberg integral.

Proposition 4.4.1. *Let $A(g)$, $B(g)$ be the Whittaker functions of F and G. Then*

$$\langle E_{l,s,f}G, F\rangle = -\frac{i\Gamma(k+l+s-1)}{(4\pi)^{k+l+s-1}}$$
$$\times \int\limits_{\mathbb{A}_f^* \times G_{\hat{\mathbb{Z}}}} f\left(\begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix} h\right) B\left(\begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix} h\right) \overline{A\left(\begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix} h\right)} |m|^{-k-l-s-1} d^*m\, dh.$$

Proof. A very similar calculation is done in [30, §5]; here we simply write the equations with little comment:

$$\begin{aligned}
\langle E_{l,s,f}G, F\rangle &= \int\limits_{G_{\mathbb{Q}}^+ \backslash \mathfrak{H} \times G_f} \sum_{\gamma \in P_{\mathbb{Q}}^+ \backslash G_{\mathbb{Q}}^+} f(\gamma g) j(\gamma,\tau)^l |j(\gamma,\tau)|^{2s} \\
&\qquad \times G(\tau,g)\overline{F(\tau,g)} y^{k+l+s-2} |\det g|_f^{-k-l-s} d\tau \wedge d\bar{\tau}\, dg \\
&= -2i \int\limits_{P_{\mathbb{Q}}^+ \backslash \mathfrak{H} \times G_f} f(g) G(\tau,g)\overline{F(\tau,g)} y^{k+l+s-2} |\det g|_f^{-k-l-s} dx\, dy\, dg \\
&= -2i \int\limits_{P_{\mathbb{Q}}^+ \backslash \mathfrak{H} \times G_f} f(g) \sum_{m \in \mathbb{Q}_{>0}^*} B\overline{A}\left(\begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix} g\right) \\
&\qquad \times e^{-4\pi m y} y^{k+l+s-2} |\det g|_f^{-k-l-s} dx\, dy\, dg \\
&= -2i \int\limits_{Z_{\mathbb{Q}} U_{\mathbb{Q}} \backslash \mathfrak{H} \times G_f} f(g) B(g)\overline{A(g)} e^{-4\pi y} y^{k+l+s-2} |\det g|_f^{-k-l-s} dx\, dy\, dg \\
&= -\frac{2i\Gamma(k+l+s-1)}{(4\pi)^{k+l+s-1}} \int\limits_{Z_{\mathbb{Q}} N_{\mathbb{Q}} \backslash \mathbb{R} \times G_f} f(g) B(g)\overline{A(g)} |\det g|_f^{-k-l-s} dx\, dg
\end{aligned}$$

To get the final result, use the parameterisation

$$\begin{aligned}
\pi\colon \mathbb{A}_f \times \mathbb{A}_f^* \times G_{\hat{\mathbb{Z}}}/\{\pm 1\} &\longrightarrow G_f/Z_{\mathbb{Q}} \\
(b,m,h) &\longmapsto \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix} h
\end{aligned}$$

in terms of which integration is given by

$$\int\limits_{G_f/Z_{\mathbb{Q}}} \Phi(g)\, dg = \int\limits_{\mathbb{A}_f \times \mathbb{A}_f^* \times G_{\hat{\mathbb{Z}}}/\{\pm 1\}} (\pi^*\Phi)(b,m,h) |y|^{-1} db\, d^*m\, dh$$

Since $f(g)B(g)\overline{A(g)}$ is invariant by $g \mapsto \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} g$, the integral in the last line above splits as a product

$$\int\limits_{\mathbb{Q}\backslash\mathbb{R}\times\mathbb{A}_f} dx\, db \int\limits_{\mathbb{A}_f^*\times G_{\hat{\mathbb{Z}}}/\{\pm 1\}} fB\overline{A}\left(\begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix} h\right) d^*m\, dh$$

and the first factor equals 1 by choice of Haar measure. □

Now suppose:

- F is a cusp form, belonging to an irreducible $\pi = \otimes' \pi_p$, with central character ε, whose Whittaker function $A(g) = \prod A_p(g_p)$ is factorisable;
- $G = E_k(\phi')$ is an Eisenstein series and $f = f_\phi$ for factorisable functions $\phi = \prod \phi_p$, $\phi' = \prod \phi'_p \in \mathcal{S}(\mathbb{A}_f^2)$.

Then the integral in the previous proposition can be decomposed under the action of the centre and then factorised, giving:

Proposition 4.4.2. *Under the above hypotheses:*

$$\langle E_{l,s}(\phi)E_k(\phi'), F\rangle = C \sum_{\chi,\chi'} \prod_p I_p(\chi_p, \chi'_p)$$

where $C = \dfrac{i^{k-1}\Gamma(k+l+s-1)}{2^{k+2l+2s-4}(k-1)!}$ *and*

$$I_p(\chi_p, \chi'_p) = \int\limits_{\mathbb{Q}_p^*\times K_p} f^*_{\chi_p} B^*_{\chi'_p} \overline{A^*_p}\left(\begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix} h\right) |m|^{-1} d^*m\, dh$$

and $f^*_{\chi_p}$, $B^*_{\chi'_p}$ *are as in* (4.3.4), (4.3.5) *above. The sum is over all pairs of characters* χ, $\chi' \colon \mathbb{A}_f^*/\mathbb{Q}^*_{>0} \to \mathbb{C}^*$ *such that* $\chi\chi' = \varepsilon$ *and* $\chi(-1) = (-1)^l$.

Remark 4.4.3. It will become clear in the computation that follows that the sum over characters is actually a finite sum.

4.5 Local integrals

Write $\mathrm{char}\left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right]$ for the characteristic function of a subset $X \times Y \subset \mathbb{A}_f^2$. The next proposition will compute the local integral $I_p(\chi_p, \chi'_p)$ for almost all primes.

Proposition 4.5.1. *Suppose that*

$$\phi_p = \phi'_p = \phi^0_p := \operatorname{char}\begin{bmatrix}\mathbb{Z}_p \\ \mathbb{Z}_p\end{bmatrix}$$

and that A_p is K_p-invariant, with $A_p(1) = 1$. Then

$$I_p(\chi_p, \chi'_p) = \begin{cases} L(\pi_p, k+l+s-1)L(\pi_p \otimes \chi_p'^{-1}, l+s) & \text{if } \chi_p \text{ is unramified} \\ 0 & \text{otherwise.} \end{cases}$$

Remark 4.5.2. Since π_p has a K_p-invariant vector, ε is unramified at p. Therefore since $\chi\chi' = \varepsilon$, either both or neither of χ_p, χ'_p are unramified.

Proof. (See [15, §15.9].) If $\phi_p = \phi'_p = \phi^0_p$ then by (4.3.6) for $h \in K_p$, $m \in \mathbb{Q}_p^*$

$$\begin{aligned} f^*_{\chi_p}\left(\begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix} h\right) &= \chi_p(m)|m|^{(l+2s)/2} \int\limits_{\mathbb{Z}_p - \{0\}} \chi_p(x)|x|^{l+2s} d^*x \\ &= \begin{cases} L(\chi_p, l+2s) & \text{if } \chi_p \text{ is unramified} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Moreover by (4.3.7)

$$B^*_{\chi'_p}\left(\begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix} h\right) = |m|^{1-k/2} \int\limits_{\mathbb{Z}_p \times \mathbb{Z}_p - \{0\}} \chi'_p(y)|y|^{k-1}\psi_p(-mx/y)dx\, dy$$

where the x-integral equals 1 is $m/y \in \mathbb{Z}_p$ and vanishes otherwise. The y-integral then vanishes if $\chi'_p\big|_{\mathbb{Z}_p^*} \neq 1$, giving

$$B^*_{\chi'_p}\left(\begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix} h\right) = \begin{cases} p^{-rk/2} \sum\limits_{0 \le j \le r} \chi'_p(p)^j p^{(r-j)(k-1)} & \\ \qquad \text{if } r = v_p(m) \ge 0 \text{ and } \chi'_p \text{ is unramified} & \\ 0 \qquad \text{otherwise.} & \end{cases} \tag{4.5.3}$$

Thus $I_p(\chi_p, \chi'_p) = 0$ unless both χ, χ' are unramified at p, which we now assume. Then $f^*_{\chi_p}$, $B^*_{\chi'_p}$ are K_p-invariant and

$$\begin{aligned} I_p(\chi_p, \chi'_p) &= \sum_{r \in \mathbb{Z}} p^r f^*_{\chi_p} B^*_{\chi'_p} \overline{A^*_p}\begin{pmatrix} p^r & 0 \\ 0 & 1 \end{pmatrix} \\ &= L(\chi_p, l+2s) \sum_{r \ge 0} \chi(p)^r p^{r(1-l/2-s)} B^*_{\chi'_p} \overline{A^*_p}\begin{pmatrix} p^r & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Now from (4.5.3)

$$\sum_{r\geq 0} B^*_{\chi'_p}\begin{pmatrix} p^r & 0\\ 0 & 1\end{pmatrix} T^r = \frac{1}{(1-\chi'_p(p)p^{-k/2}T)(1-p^{k/2-1}T)}$$

and therefore by [15, Lemma 15.9.4] one gets

$$I_p(\chi_p,\chi'_p) = L(\chi_p, l+2s)\frac{L(\bar\pi_p\otimes\chi_p\chi'_p, k+l+s-1)L(\bar\pi_p\otimes\chi_p, l+s)}{L(\varepsilon_p^{-1}\chi'_p\chi_p^2, l+2s)}.$$

Since $\varepsilon = \chi\chi'$ and $\bar\pi \simeq \pi\otimes\varepsilon^{-1}$ (4.2.4) the result follows. $\square$

Corollary 4.5.4. *Under the hypotheses of Proposition 4.4.2, let S be a finite set of primes such that, for every $p\notin S$, $\phi_p = \phi'_p = \phi^0_p$, A_p is K_p-invariant and $A_p(1) = 1$. Then*

$$\langle E_{l,s}(\phi)E_k(\phi'), F\rangle = \\ C\cdot L_S(\pi, k+l+s-1)\sum_{\chi,\chi'} L_S(\pi\otimes\chi'^{-1}, l+s)\prod_{p\in S} I_p(\chi_p,\chi'_p)$$

where the sum is over characters χ, χ' unramified outside S, with $\chi\chi'=\varepsilon$ and $\chi(-1) = (-1)^l$.

Here L_S denotes the L-function with Euler factors at all $p\in S$ removed. At other primes we use the following choice for ϕ_p:

Proposition 4.5.5. *Let $t\in\mathbb{Q}_p$ with $v_p(t) = -\nu < 0$. Suppose that*

$$\phi_p = \phi_p^{1,t} := \mathrm{char}\begin{bmatrix} t+\mathbb{Z}_p\\ \mathbb{Z}_p\end{bmatrix}.$$

Let $m\in\mathbb{Q}_p^*$, $h = \begin{pmatrix} a & b\\ c & d\end{pmatrix}\in K_p$. Then

$$f^*_{\chi_p}\left(\begin{pmatrix} m & 0\\ 0 & 1\end{pmatrix} h\right) = \begin{cases} \left(1-\dfrac{1}{p}\right)^{-1} p^{\nu(l+2s-1)}\chi_p(amt)|m|^{l/2+s} & \\ & \textit{if } \mathrm{cond}\,\chi_p\leq\nu \textit{ and } h\in K_0(p^\nu)\\ 0 & \textit{otherwise.}\end{cases}$$

Proof. Straightforward calculation from (4.3.4). $\square$

At the bad primes for F we are going to choose ϕ'_p in such a way as to make the local factor be simply a constant.

The standard way to achieve this is to use a suitable Atkin-Lehner operator to replace the coefficient of q^n in the q-expansion by zero whenever $p|n$. In representation-theoretic language, this means to use the vector in the Kirillov model which is the characteristic function of $\mathbb{Z}_p^*$. (See [6, Thm. 2.5.6], and also compare [30, 4.5.4].) For Eisenstein series it is easy to write down a parameter ϕ'_p which does the trick, although possibly this does not give the best constant in 4.6.3 below.

Proposition 4.5.6. *Suppose*

$$\phi'_p = \phi_p^{2,t'} := \operatorname{char}\begin{bmatrix} t' + \mathbb{Z}_p \\ \mathbb{Z}_p^* \end{bmatrix} - \frac{1}{p}\operatorname{char}\begin{bmatrix} t' + p^{-1}\mathbb{Z}_p \\ \mathbb{Z}_p^* \end{bmatrix}$$

where $t' \in \mathbb{Q}_p^*$, $v_p(t') = -\mu \leq 0$. *Then for all* $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0(p^{\mu+1}) \cap K_0(p^2)$

$$B^*_{\chi'_p}\left(\begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix} h\right) = \begin{cases} \chi'_p(-amt') \displaystyle\int_{p^{-\mu}\mathbb{Z}_p^*} \chi'_p(y)^{-1}\psi_p(y)\, d^*y & \text{if } \operatorname{cond}\chi'_p \leq \mu \text{ and } m \in \mathbb{Z}_p^* \\ 0 & \text{otherwise.} \end{cases}$$

Remark. In the special case $\mu = 0$, this becomes

$$\phi'_p = \phi_p^{2,1} = \operatorname{char}\begin{bmatrix} \mathbb{Z}_p \\ \mathbb{Z}_p^* \end{bmatrix} - \frac{1}{p}\operatorname{char}\begin{bmatrix} p^{-1}\mathbb{Z}_p \\ \mathbb{Z}_p^* \end{bmatrix}$$

and for all $h \in K_0(p^2)$ and all $m \in \mathbb{Q}_p^*$

$$B^*_{\chi'_p}\left(\begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix} h\right) = \begin{cases} 1 & \text{if } \chi'_p \text{ is unramified and } |m| = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. First consider what happens when $\phi'_p = \operatorname{char}\begin{bmatrix} t' + \mathbb{Z}_p \\ \mathbb{Z}_p^* \end{bmatrix}$. For every $h \in K_0(p^{\mu+1})$ one has

$$h\phi'_p = \operatorname{char}\begin{bmatrix} at' + \mathbb{Z}_p \\ \mathbb{Z}_p^* \end{bmatrix}$$

which gives

$$B^*_{\chi'_p}\left(\begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix} h\right) = |m|^{1-k/2} \int_{(at'+\mathbb{Z}_p)\times\mathbb{Z}_p^*} \chi'_p(y)\psi_p(-mx/y)\, dx\, d^*y \tag{4.5.7}$$

and the x-integral vanishes for $m \notin \mathbb{Z}_p$, and equals $\psi_p(-amt'/y)$ otherwise. Therefore for $m \in \mathbb{Z}_p$ (4.5.7) becomes

$$|m|^{1-k/2} \int\limits_{\mathbb{Z}_p^*} \chi'_p(y)\psi_p(-amt'/y)d^*y = |m|^{1-k/2}\chi'_p(-amt') \int\limits_{mt'\mathbb{Z}_p^*} \chi'_p(y)^{-1}\psi_p(y)\, d^*y$$

Now if $B\colon G_p \to \mathbb{C}$ is *any* Whittaker function (i.e. satisfies (4.2.1) and is locally constant) which is invariant under $K_1(p^\nu)$, some $\nu \geq 1$, then the function

$$\widetilde{B} = B - \frac{1}{p} \sum_{x \bmod p} \begin{pmatrix} 1 & p^{-1}x \\ 0 & 1 \end{pmatrix} B$$

satisfies, for every $h \in K_0(p^\nu) \cap K_0(p^2)$,

$$\widetilde{B}\left(\begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix} h\right) = \mathrm{char}_{\mathbb{Z}_p^*}(m) B\left(\begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix} h\right).$$

Since

$$(t' + p^{-1}\mathbb{Z}_p) \times \mathbb{Z}_p^* = \coprod_{x \bmod p} \begin{pmatrix} 1 & p^{-1}x \\ 0 & 1 \end{pmatrix} \left[(t' + \mathbb{Z}_p) \times \mathbb{Z}_p^*\right]$$

the result follows. □

From Propositions 4.5.5 and 4.5.6 one obtains:

Proposition 4.5.8. *Suppose that A_p is invariant under $K_1(p^\nu)$ and that $A_p(1) = 1$. Let $t,\ t' \in \mathbb{Q}_p^*$ with $v_p(t) = -\nu$, $v_p(t') = -\mu$ and $\nu > \mu \geq 0$, $\nu \geq 2$. Then if $\phi_p = \phi_p^{1,t}$ and $\phi'_p = \phi_p^{2,t'}$,*

$$I_p(\chi_p, \chi'_p) = \begin{cases} \left(1 - \dfrac{1}{p^2}\right)^{-1} p^{\nu(l+2s-2)}\chi_p(t)\chi'_p(-t') \displaystyle\int\limits_{p^{-\mu}\mathbb{Z}_p^*} \chi'_p(y)^{-1}\psi_p(y)\, d^*y & \textit{if } \mathrm{cond}\,\chi'_p \leq \mu \\ 0 & \textit{otherwise.} \end{cases}$$

Remark. Since $\varepsilon = \chi\chi'$ one has $\mathrm{cond}\,\chi'_p \leq \mu \implies \mathrm{cond}\,\chi_p \leq \nu$.

Proof. If $\mathrm{cond}\,\chi'_p > \mu$ then $I_p(\chi_p, \chi'_p) = 0$. Otherwise, if $h \in K_0(p^\nu)$ and A_p is $K_1(p^\nu)$-invariant, one has $A_p^*(h) = \varepsilon_p(a)A_p^*(1) = \chi_p\chi'_p(a)A_p^*(1)$, so that

$I_p(\chi_p, \chi'_p)$ equals

$$\left(1-\frac{1}{p}\right)^{-1} p^{\nu(l+2s-1)} \int\limits_{p^{-\mu}\mathbb{Z}_p^*} \chi'_p(y)^{-1}\psi_p(y)\, d^*y \int\limits_{K_0(p^\nu)} \chi_p(at)\chi'_p(-at')\overline{A_p^*(h)}\, dh$$

$$= \operatorname{vol} K_0(p^\nu) \left(1-\frac{1}{p}\right)^{-1} p^{\nu(l+2s-1)} \int\limits_{p^{-\mu}\mathbb{Z}_p^*} \chi'_p(y)^{-1}\psi_p(y)\, d^*y \chi_p(t)\chi'_p(-t')\overline{A_p^*(1)}$$

and $\operatorname{vol} K_0(p^\nu) = [K_p\colon K_0(p^\nu)]^{-1} = (1+1/p)^{-1}p^{-\nu}$. □

There is just one more case to consider, in order to compute the image of the Euler system in the cyclotomic tower.

Proposition 4.5.9. *Suppose that*

$$\phi'_p = (\phi_p^{1,t'})^{\mathrm{tr}} = \operatorname{char}\begin{bmatrix} \mathbb{Z}_p \\ t' + \mathbb{Z}_p \end{bmatrix}, \quad v_p(t') = -\nu < 0.$$

Then for all $m \in \mathbb{Q}_p^*$ *and* $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0(p^\nu)$,

$$B^*_{\chi'_p}\left(\begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix} h\right) = \begin{cases} \left(1-\dfrac{1}{p}\right)^{-1} p^{\nu(k-2)}|m|^{(1-k/2)}\chi'_p(dt')\psi_p(-mb/d) & \\ \qquad\qquad\qquad\qquad \textit{if } \operatorname{cond}\chi'_p \le \nu \textit{ and } v_p(m) \ge -\nu \\ 0 \qquad\qquad\qquad\qquad \textit{otherwise.} \end{cases}$$

Proof. One has $h\phi'_p = \operatorname{char}\begin{bmatrix} bt' + \mathbb{Z}_p \\ dt' + \mathbb{Z}_p \end{bmatrix}$ and $dt' + \mathbb{Z}_p = dt'(1+p^\nu\mathbb{Z}_p)$, therefore by (4.3.7)

$$B^*_{\chi'_p} = |m|^{-k/2+1} \int\limits_{dt'(1+p^\nu\mathbb{Z}_p)} \chi'_p(y)|y|^{k-1}\Bigg(\int\limits_{bt'+\mathbb{Z}_p} \psi_p(-mx/y)\, dx\Bigg) d^*y$$

$$= \begin{cases} 0 & \text{if } v_p(m) < -\nu \\ |m|^{-k/2+1}\chi'_p(dt')p^{\nu(k-1)} \displaystyle\int\limits_{1+p^\nu\mathbb{Z}_p} \chi'_p(y)\psi_p(-mb/dy)\, d^*y & \text{if } v_p(m) \ge -\nu. \end{cases}$$

If $v_p(m) \ge -\nu$ and $y \in 1 + p^\nu\mathbb{Z}_p$ then $\psi_p(-mb/dy) = \psi_p(-mb/d)$ and the result follows. □

Corollary 4.5.10. *Suppose there exists a character* $\lambda_p\colon \mathbb{Q}_p^* \to \mathbb{C}^*$ *such that* $\pi_p \otimes \lambda_p^{-1}$ *is unramified, and that* A_p^* *is the twist of the spherical vector; that is,* $A_p^* \otimes \lambda_p^{-1}\colon g \mapsto \lambda_p(\det g)^{-1}A_p(g)$ *is* K_p*-invariant and* $A_p(1) = 1$*. Put*

$$\phi_p = \phi_p^{1,t},\ \phi'_p = (\phi_p^{1,t'})^{\mathrm{tr}} \quad \textit{with } v_p(t) = v_p(t') = -\nu < 0.$$

Then if $\chi'_p\lambda_p^{-1}$ *is unramified and* $\operatorname{cond}\lambda_p \le \nu$*,*

$$I_p(\chi_p,\chi'_p) = \left(1-\frac{1}{p}\right)^{-1}\left(1-\frac{1}{p^2}\right)^{-1} p^{\nu(k+l+2s-4)}\chi_p(t)\chi'_p(t')L(\pi_p\otimes\chi_p'^{-1}, l+s)$$

and otherwise $I_p(\chi_p,\chi'_p)=0$*.*

Proof. The central character $\varepsilon_p = \chi_p\chi'_p$ of π_p is λ_p^2 times an unramified character, so one of $\chi_p\lambda_p^{-1}$, $\chi'_p\lambda_p^{-1}$ is unramified if and only if both are. By Propositions 4.5.5 and 4.5.9, $I_p(\chi_p,\chi'_p)=0$ whenever $\operatorname{cond}\chi_p$ or $\operatorname{cond}\chi'_p > \nu$. Otherwise

$$\begin{aligned} I_p(\chi_p,\chi'_p) &= \left(1-\frac{1}{p}\right)^{-2} p^{\nu(k+l+2s-3)}\chi_p(t)\chi'_p(t') \\ &\quad\times \int\limits_{\mathbb{Q}_p^*\times K_0(p^\nu)} \chi_p(am)\chi'_p(d)|m|^{s+(l-k)/2}\overline{A_p^*\left(\begin{pmatrix} m & 0\\ 0 & 1\end{pmatrix}h\right)}\, d^*m\, dh \\ &= \left(1-\frac{1}{p}\right)^{-2} p^{\nu(k+l+2s-3)}\chi_p(t)\chi'_p(t') \\ &\quad\times \sum_{r\ge 0}\chi_p(p)^r p^{-r(l-k+2s)/2}\lambda_p(p)^{-r}\overline{A_p^*\otimes\lambda_p^{-1}\begin{pmatrix} p^r & 0\\ 0 & 1\end{pmatrix}} \\ &\quad\times \int\limits_{K_0(p^\nu)} \chi_p(a)\chi'_p(d)\lambda(\det h)\, dh. \end{aligned}$$

The integral in the last expression vanishes unless $\chi_p\lambda_p^{-1}$ and $\chi'_p\lambda_p^{-1}$ are unramified, in which case $I_p(\chi_p,\chi'_p)$ becomes

$$\begin{aligned} &\left(1-\frac{1}{p}\right)^{-1}\left(1-\frac{1}{p^2}\right)^{-1} p^{\nu(k+l+2s-4)}\chi_p(t)\chi'_p(t') \\ &\qquad\times\sum_{r\ge 0}(\chi_p\lambda_p^{-1})(p)^r\overline{A_p^*\otimes\lambda_p^{-1}\begin{pmatrix} p^r & 0\\ 0 & 1\end{pmatrix}}p^{-r(l-k+2s)/2} \qquad (4.5.11)\\ &= \left(1-\frac{1}{p}\right)^{-1}\left(1-\frac{1}{p^2}\right)^{-1} p^{\nu(k+l+2s-4)}\chi_p(t)\chi'_p(t')L(\bar{\pi}_p\otimes\chi_p, l+s) \end{aligned}$$

giving the desired expression from (4.2.4). □

Remark 4.5.12. One can do exactly the same computation merely assuming that $A_p^*\otimes\lambda_p^{-1}$ is $K_1(p^\nu)$-invariant; $I_p(\chi_p,\chi'_p)$ vanishes unless $\chi'_p\lambda_p^{-1}$ is unramified, in which case one gets the formula (4.5.11).

4.6 Putting it all together

We change notation slightly from the previous sections. Begin with a cusp form F of weight $k+l$, generating an irreducible $\pi = \otimes' \pi_p$, and whose Whittaker function $A(g) = \prod A_p(g_p)$ factorises, with $A_p(1) = 1$ for all p. Let ε be the character of the centre of G_f on A^*. Assume that the following data is given:

(i) Disjoint finite sets of primes S and T, such that if $p \notin S$ then A_p is K_p-invariant. (In particular this means that ε is unramified outside S.)

(ii) A character $\lambda\colon \mathbb{A}_f^*/\mathbb{Q}_{>0}^* \to \mathbb{C}^*$, unramified outside T.

(iii) For each $p \in S$, elements t_p, $t'_p \in \mathbb{Q}_p^*$ with $v_p(t_p) = -\nu_p$, $v_p(t'_p) = -\mu_p$, such that $\nu_p > \mu_p \geq 0$, $\nu_p \geq 2$, and A_p is $K_1(p^{\nu_p})$-invariant.

(iv) For each $p \in T$, elements t_p, $t'_p \in \mathbb{Q}_p^*$ with $v_p(t_p) = v_p(t'_p) = -\nu_p$ where $\nu_p \geq \max(\operatorname{cond} \lambda_p, 1)$.

Put

$$N = \prod_{p \in S} p^{\nu_p}, \quad M = \prod_{p \in S} p^{\mu_p}, \quad R = \prod_{p \in T} p^{\nu_p}.$$

Denote by t, t' the finite ideles whose components at primes $p \in S \cup T$ are t_p, t'_p, and which are 1 elsewhere. Pick $y \in \mathbb{Z}$ such that

$$y \equiv -\frac{p^{\mu_p} t'_p}{N t_p} \mod p^{\mu_p} \quad \text{for all } p \in S \tag{4.6.1}$$

(note that the right-hand side belongs to $\mathbb{Z}_p^*$) — thus y is well-defined mod M. The integral to compute is

$$\langle E_{l,s}(\phi) E_k(\phi') \otimes \lambda^{-1}, F \rangle = \langle E_{l,s}(\phi) E_k(\phi'), F \otimes \lambda \rangle$$

where

$$F \otimes \lambda(\tau, g) = \lambda(\det g) F(\tau, g)$$

is the twist of F by λ, and ϕ, ϕ' are given as follows:

- For $p \in S$, $\phi_p = \phi_p^{1,t_p}$ and $\phi'_p = \phi_p^{2,t'_p}$.
- For $p \in T$, $\phi_p = \phi_p^{1,t_p}$ and $\phi'_p = (\phi_p^{1,t'_p})^{\mathrm{tr}}$
- For $p \notin S \cup T$, $\phi_p = \phi'_p = \phi_p^0$.

We can then assemble the previous calculations. Put $\chi = \varepsilon\lambda\theta$ and $\chi' = \lambda\theta^{-1}$ for a variable character θ — thus $\chi\chi' = \varepsilon\lambda^2$, the central character of $A^* \otimes \lambda$. Then only those θ satisfying the following conditions contribute to the sum:

- $\theta(-1) = (-1)^k\lambda(-1)$;
- If $p \notin S$ then θ_p is unramified;
- If $p \in S$ then $\operatorname{cond}\theta_p \leq \mu_p$.

So $\operatorname{cond}\theta | M$, $\operatorname{cond}\varepsilon | N$ and $\operatorname{cond}\lambda | R$. This gives:

$$\langle E_{l,s}(\phi)E_k(\phi'), F\otimes\lambda\rangle = C\cdot L_{S\cup T}(\pi\otimes\lambda, l+k+s-1) \times \sum_{\substack{\theta(-1)=(-1)^k\lambda(-1)\\ \operatorname{cond}\theta|M}} L_{S\cup T}(\pi\otimes\theta, l+s)\prod_{p\in S\cup T} I_p(\chi_p,\chi'_p)$$

by 4.5.4, 4.5.8 and 4.5.9, where

$$\prod_{p\in S} I_p(\chi_p,\chi'_p) = N^{l+2s-2}\prod_{p\in S}\left(1-\frac{1}{p^2}\right)^{-1}\chi_p(t_p)\chi'_p(-t'_p)\int_{p^{-\mu_p}\mathbb{Z}_p^*}\chi'_p(y)^{-1}\psi_p(y)\,d^*y$$

with

$$\begin{aligned}\prod_{p\in S}\chi_p(t_p)\chi'_p(-t'_p) &= \prod_{p\in S}\lambda_p(p)^{-\mu_p-\nu_p}\varepsilon_p(t_p)\theta_p(-t_p/t'_p)\\ &= \theta_{\mathrm{mod}\,M}(y)\prod_{p\in S}\varepsilon_p(t_p)\theta_p(p)^{\mu_p}\lambda_p(MN)^{-1},\end{aligned}$$

and

$$\prod_{p\in T} I_p(\chi_p,\chi'_p) = R^{k+l+2s+4}\prod_{p\in T}\left(1-\frac{1}{p}\right)^{-1}\left(1-\frac{1}{p^2}\right)^{-1}\chi_p(t_p)\chi'_p(t'_p)L(\pi_p\otimes\theta_p, l+s)$$

and for each $p\in T$, $\chi_p(t_p)\chi'_p(t'_p) = \varepsilon_p(t_p)\lambda_p(t_pt'_p)$. This gives

$$\begin{aligned}\langle E_{l,s}(\phi)E_k(\phi'), F\otimes\lambda\rangle = {}& C\cdot N^{l+2s-2}R^{k+l+2s-4}\prod_{p\in S\cup T}\left(1-\frac{1}{p^2}\right)^{-1}\prod_{p\in T}\left(1-\frac{1}{p}\right)^{-1}\\ &\times\varepsilon(t)\prod_{p\in T}\lambda_p(MNt_pt'_p)\cdot L_{S\cup T}(\pi\otimes\lambda, l+k+s-1)\\ &\times\sum_{\substack{\theta(-1)=(-1)^k\lambda(-1)\\ \operatorname{cond}\theta|M}}\Bigg(\prod_{p|M}\theta_p(p)^{\mu_p}\int_{p^{-\mu_p}\mathbb{Z}_p^*}\chi'_p(y)^{-1}\psi_p(y)\,d^*y\\ &\qquad\times\theta_{\mathrm{mod}\,M}(y)\cdot L_S(\pi\otimes\theta, l+s)\Bigg).\end{aligned} \tag{4.6.2}$$

In the third line of this expression, the product over $p|M$ can be rewritten in terms of a classical Gauss sum as

$$\varphi(M)^{-1} \sum_{x\in(\mathbb{Z}/M\mathbb{Z})^*} \theta_{\mathrm{mod}\,M}(x)e^{2\pi ix/M}.$$

(Here φ is Euler's totient function.) The sum over characters θ in (4.6.2) then becomes (combining odd and even characters)

$$\begin{aligned}\frac{1}{2}\varphi(M)^{-1} \sum_{\mathrm{cond}\,\theta|M} \sum_{x\in(\mathbb{Z}/M\mathbb{Z})^*} \sum_{\substack{m\geq 1\\(m,N)=1}} &\left[1+(-1)^k\theta\lambda(-1)\right] \\ &\times \theta_{\mathrm{mod}\,M}(xy^{-1}m^{-1})e^{2\pi ix/M}a_m m^{-l-s} \\ = \sum_{\substack{m\geq 1\\(m,N)=1}} \frac{1}{2}&\left[\psi_f(my/M)+(-1)^k\lambda(-1)\psi_f(-my/M)\right]a_m m^{-l-s}\end{aligned}$$

by the character orthogonality relations.

For $\alpha \in \mathbb{A}_f$ write

$$L_S(\pi, s; \alpha) = \sum_{\substack{m\geq 1\\(m,N)=1}} \psi_f(m\alpha)a_m m^{-s}$$

for the twisted Dirichlet series.

Theorem 4.6.3. *Under the above hypotheses*

$$\begin{aligned}\langle E_{l,s}(\phi)E_k(\phi')\otimes\overline{\lambda}, F\rangle &= CN^{l+2s-2}R^{k+l+2s}\#GL_2(\mathbb{Z}/R\mathbb{Z})^{-1} \\ &\times \varepsilon(t)\prod_{p\in T}\lambda_p(MNt_pt'_p)\prod_{p\in S}\left(1-\frac{1}{p^2}\right)^{-1}L_{S\cup T}(\pi\otimes\lambda, l+k+s-1) \\ &\times \big(L_S(\pi, l+s; y/M)+(-1)^k\lambda(-1)L_S(\pi, l+s; -y/M)\big)\end{aligned}$$

with C as in Proposition 4.4.2. □

5 The Euler systems

5.1 Modular curves

We can at last give Kato's construction of an Euler system in the Galois cohomology of the modular curve $Y(N)$ over a family of abelian extensions of $\mathbb{Q}$. We assume throughout that p is a prime not dividing N.

Pick auxiliary integers D, $D' > 1$ which are prime to $6Np$, and put

$$\mathcal{R}'_p = \{\text{squarefree positive integers prime to } NpDD'\}$$
$$\mathcal{R}_p = \{r = r_0 p^m \mid r_0 \in \mathcal{R}'_p,\ m \geq 1\}$$

We suppose that, for each $r \in \mathcal{R}_p$, we are given points z_r, $z'_r \in \mathcal{E}^{\text{univ}}(Y(Nr)) \simeq (\mathbb{Z}/Nr)^2$, such that:

- If r and $rs \in \mathcal{R}_p$, then $sz^{(\prime)}_{rs} = z^{(\prime)}_r$ — i.e., one has elements of the inverse limit

  $$(z_r)_r,\ (z'_r)_r \in \varprojlim_{r \in \mathcal{R}_p} \ker[\times Nr] \simeq \mathbb{Z}_p^2 \times \prod_{\ell \nmid NpDD'} \mathbb{Z}/\ell$$

  of torsion points on the universal elliptic curve.

- For every $r \in \mathcal{R}_p$, the points Nz_r and Nz'_r generate $\ker[\times r]$ (in particular, the orders of z_r, z'_r are multiples of r).

- If $r = p^m$ then the orders of z_r, z'_r are divisible by a prime other than p.

Remarks. (i) The first condition implies that there exists $e \in \mathbb{Z}_p^*$ such that for every $r = r_0 p^m \in \mathcal{R}_p$, the Weil pairing of z_r and z'_r is

$$e_{Nr}(z_r, z'_r) = \zeta_{p^m}^{er_0^{-1}} \times (\text{prime-to-}p \text{ root of } 1). \tag{5.1.1}$$

(ii) The third condition is really only added for convenience. It ensures that for every r the points $z^{(\prime)}_r$ are not of prime power order, which means that they do not meet the zero section of $\mathcal{E}^{\text{univ}}/Y(Nr)$ in any characteristic.

It follows from (ii) that the modular units $\vartheta_D(z_r)$, $\vartheta_{D'}(z'_r)$ actually belong to $\mathcal{O}^*(Y(Nr)_{/\mathbb{Z}})$, for any $r \in \mathcal{R}_p$. Define

$$\widetilde{\sigma}_r = \{\vartheta_D(z_r), \vartheta_{D'}(z'_r)\} \in K_2\big(Y(Nr)\big)$$

and also

$$\sigma_r = N_{Y(Nr)/Y(N)\otimes\mathbb{Q}(\boldsymbol{\mu}_r)} \widetilde{\sigma}_r \in K_2\big(Y(N) \otimes \mathbb{Q}(\boldsymbol{\mu}_r)\big);$$

by what was just said, these belong to the images of K_2 of the models over $\operatorname{Spec} \mathbb{Z}$.

Let $T_\ell = T_{\ell, Y(N)}$, $\langle a \rangle = \langle a \rangle_{Y(N)}$ denote the Hecke correspondence and diamond operators as in §2.3 above. If $(\ell, r) = 1$ write $\operatorname{Frob}_\ell \in \operatorname{Gal}(\mathbb{Q}(\boldsymbol{\mu}_r)/\mathbb{Q})$ for the *geometric* Frobenius automorphism, so that $\operatorname{Frob}_\ell = \varphi_\ell^{-1}$ where $\varphi_\ell\colon \zeta_r \mapsto \zeta_r^\ell$ is the arithmetic Frobenius substitution. For every finite field extension

L'/L write simply $N_{L'/L}$ for the norm map $K_2(Y(N)\otimes L') \to K_2(Y(N)\otimes L)$. Notice that if $\ell \nmid Nr$ then

$$N_{Y(Nr)/Y(N)\otimes\mathbb{Q}(\boldsymbol{\mu}_r)} \circ T_{\ell,Y(Nr)} = (T_{\ell,Y(N)} \otimes \varphi_\ell) \circ N_{Y(Nr)/Y(N)\otimes\mathbb{Q}(\boldsymbol{\mu}_r)}$$
$$N_{Y(Nr)/Y(N)\otimes\mathbb{Q}(\boldsymbol{\mu}_r)} \langle \ell \rangle_{Y(Nr)} = (\langle \ell \rangle_{Y(N)} \otimes \varphi_\ell^2) \circ N_{Y(Nr)/Y(N)\otimes\mathbb{Q}(\boldsymbol{\mu}_r)}$$

since T_ℓ acts as φ_ℓ on the constant field and $\langle \ell \rangle$ acts as φ_ℓ^2, by §2.3. Therefore from 2.3.6 and 2.4.3 one obtains:

Theorem 5.1.2. *Let $r \in \mathcal{R}_p$. Then:*

(i) $N_{\mathbb{Q}(\boldsymbol{\mu}_{rp})/\mathbb{Q}(\boldsymbol{\mu}_r)} \sigma_{rp} = \sigma_r$.

(ii) If ℓ is prime and $(\ell, NDD'r) = 1$ then

$$N_{\mathbb{Q}(\boldsymbol{\mu}_{\ell r})/\mathbb{Q}(\boldsymbol{\mu}_r)} \sigma_{\ell r} = (1 - T_\ell \langle \ell \rangle_* \otimes \mathrm{Frob}_\ell + \ell \langle \ell \rangle_* \otimes \mathrm{Frob}_\ell^2)\, \sigma_r.$$

Now write

$$\mathbb{T}_{p,N} = H^1(Y(N) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathbb{Z}_p(1))$$

and consider, for $r = r_0 p^m$, the homomorphisms

$$\begin{array}{c}
\varprojlim_{n\geq m} K_2(Y(N)\otimes \mathbb{Q}(\boldsymbol{\mu}_{r_0p^n})) \otimes \boldsymbol{\mu}_{p^n}^{\otimes -1} \\
\Big\downarrow \mathrm{AJ} \\
\varprojlim_{n\geq m} H^1(\mathbb{Q}(\boldsymbol{\mu}_{r_0p^n}), H^1(Y(N)\otimes_{\mathbb{Q}} \overline{\mathbb{Q}}, \boldsymbol{\mu}_{p^n})) \\
\Big\downarrow \mathrm{cor} \\
\varprojlim_{n\geq m} H^1(\mathbb{Q}(\boldsymbol{\mu}_{r_0p^m}), H^1(Y(N)\otimes_{\mathbb{Q}} \overline{\mathbb{Q}}, \boldsymbol{\mu}_{p^n})) \\
\Big\| \\
H^1(\mathbb{Q}(\boldsymbol{\mu}_r), \mathbb{T}_{p,N})
\end{array}$$

By 5.1.2(i), the family $\{\sigma_{r_0p^n} \otimes [\zeta_{p^n}]^{-1},\ n \geq m\}$ is an element of the first group. (This twisting of elements of K_2 was used first by Soulé.) Let

$$\xi_r = \xi_r(Y(N)) \in H^1(\mathbb{Q}(\boldsymbol{\mu}_r), \mathbb{T}_{p,N})$$

be its image. On the one hand, $\mathrm{Gal}(\mathbb{Q}(\boldsymbol{\mu}_r)/\mathbb{Q})$ acts on $H^1(\mathbb{Q}(\boldsymbol{\mu}_r), \mathbb{T}_{p,N})$, since $\mathbb{T}_{p,N}$ is a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-module; on the other, the level N Hecke operators T_ℓ, $\langle \ell \rangle$ act by functoriality. Also Frob_ℓ acts as ℓ^{-1} on $\boldsymbol{\mu}_{p^n}$. By Theorem 5.1.2 the classes ξ_r therefore satisfy Euler system-like identities:

Corollary 5.1.3. *(i) For all* $r \in \mathcal{R}_p$, $\mathrm{cor}_{\mathbb{Q}(\boldsymbol{\mu}_{rp})/\mathbb{Q}(\boldsymbol{\mu}_r)} \xi_{rp} = \xi_r$.

(ii) If ℓ *is prime and* r, $\ell r \in \mathcal{R}_p$ *then*

$$\mathrm{cor}_{\mathbb{Q}(\boldsymbol{\mu}_{\ell r})/\mathbb{Q}(\boldsymbol{\mu}_r)} \xi_{\ell r} = \left(1 - \ell^{-1} T_\ell \langle \ell \rangle_* \mathrm{Frob}_\ell + \ell^{-1} \langle \ell \rangle_* \mathrm{Frob}_\ell^2\right) \xi_r.$$

In the next section we will pass to an elliptic curve and get an Euler system in the sense of §2 of [29].

Recall from §1.3 the definition of the weight 1 Eisenstein series (for any N and any $D > 1$ which is prime to $6N$)

$$_D\mathrm{Eis}(z) = z^* \operatorname{dlog}_v \vartheta_D \in H^0(X(N), \boldsymbol{\omega}).$$

defined for any $0 \neq z \in \mathcal{E}^{\mathrm{univ}}(Y(N)) = (\mathbb{Z}/N\mathbb{Z})^2$. The form $_D\mathrm{Eis}(z)$ extends to $X(N)_{\mathbb{Z}}$ provided the order of z is divisible by at least 2 primes.

Recall from §1.1 the Kodaira-Spencer isomorphism

$$KS_N := KS_{Y(N)} \colon H^0(X(N), \boldsymbol{\omega}^{\otimes 2}) \xrightarrow{\sim} H^0(X(N), \Omega^1_{X(N)/\mathbb{Q}}(\log \text{cusps}))$$

identifying holomorphic modular forms of weight 2 and differentials with at worst simple poles at cusps. Let $Y(N)^{\mathrm{ord}}$ be the complement in $Y(N)_{/\mathbb{Z}}$ of the (finite) set of supersingular points in characteristic dividing N. The scheme $Y(N)^{\mathrm{ord}}$ is smooth over $\mathbb{Z}[\boldsymbol{\mu}_N]$ by [18, Cor. 10.9.2].

Proposition 5.1.4. *The Kodaira-Spencer map divided by* N *extends to a homomorphism of sheaves on* $Y(N)^{\mathrm{ord}}$

$$\frac{1}{N} KS_N \colon \boldsymbol{\omega}^{\otimes 2}_{\mathcal{E}^{\mathrm{univ}}/Y(N)^{\mathrm{ord}}} \to \Omega^1_{Y(N)^{\mathrm{ord}}/\mathbb{Z}[\boldsymbol{\mu}_N]}$$

with logarithmic singularities at the cusps.

Proof. The Kodaira-Spencer map takes the modular form $f(q^{1/N})\,(dt/t)^{\otimes 2}$ to the differential $f(q^{1/N})\,dq/q = Nf(q^{1/n}) \operatorname{dlog}(q^{1/N})$. So on q-expansions it is divisible by N. The result follows by the q-expansion principle. □

Remark. One knows that (always assuming that N is the product of two coprime integers, each ≥ 3) the scheme $X(N)$ is regular. Therefore the morphism $e_N \colon X(N) \to \operatorname{Spec} \mathbb{Z}[\boldsymbol{\mu}_N]$ is a local complete intersection (being a flat morphism of finite type between regular schemes, EGA IV 19.3.2). Therefore the sheaf of relative differentials extends to an invertible sheaf on $X(N)_{/\mathbb{Z}}$, namely the relative dualising sheaf (sheaf of regular differentials), and one can then show that $(1/N)KS_N$ extends to an *isomorphism* of invertible sheaves on all of $X(N)_{/\mathbb{Z}}$

$$\frac{1}{N} KS_N \colon \boldsymbol{\omega}^2 \to \Omega^{\mathrm{reg}}_{X(N)/\mathbb{Z}[\boldsymbol{\mu}_N]}(\log \text{cusps}).$$

This is not needed in what follows.

Because $Y(N)^{\mathrm{ord}}$ is smooth over $\mathbb{Z}[\boldsymbol{\mu}_N]$, one has

$$\Omega^2_{Y(N)^{\mathrm{ord}}/\mathbb{Z}} = \Omega^1_{Y(N)^{\mathrm{ord}}/\mathbb{Z}[\boldsymbol{\mu}_N]} \otimes \Omega^1_{\mathbb{Z}[\boldsymbol{\mu}_N]/\mathbb{Z}}$$

and $\Omega^1_{\mathbb{Z}[\boldsymbol{\mu}_N]/\mathbb{Z}}$ is killed by N and generated by $\operatorname{dlog}(\zeta_N)$.

Proposition 5.1.5. *Let z, $z' \in \mathcal{E}^{\mathrm{univ}}(Y(N)_{/\mathbb{Z}})$ be disjoint from* $\ker[\times DD']$. *In $\Omega^2_{Y(N)^{\mathrm{ord}}/\mathbb{Z}}$ the identity*

$$\operatorname{dlog}\{\vartheta_D(z), \vartheta_{D'}(z')\} = \frac{1}{N} KS_N\big({}_D\mathrm{Eis}(z) \cdot {}_{D'}\mathrm{Eis}(z')\big) \otimes \operatorname{dlog} e_N(z, z')$$

holds.

Proof. This can be checked on q-expansions. Suppose that on the completion of $Y(N)$ along a cusp we have fixed an isomorphism of $\mathcal{E}^{\mathrm{univ}}$ with the Tate curve $\mathrm{Tate}(q)$ over $\mathbb{Z}[\boldsymbol{\mu}_N]((q^{1/N}))$, and that z, z' are the points $z = \zeta_N^{a_1} q^{a_2/N}$, $z' = \zeta_N^{b_1} q^{b_2/N}$. Applying the congruence 1.3.4 and the fact that $e_N(z, z') = \zeta_N^{a_2 b_1 - a_1 b_2}$ one get the desired result. (We have normalised the e_N-pairing as in [18, (2.8.5.3)].) □

We can now give Kato's description of the image of the Euler system $\{\xi_r\}$ under the dual exponential map (see §3.2 above)

$$\exp_p^*\colon H^1(\mathbb{Q}(\boldsymbol{\mu}_r), \mathbb{T}_{p,N}) \to \mathbb{Q}_p \otimes_{\mathbb{Q}} \mathbb{Q}(\boldsymbol{\mu}_r) \otimes_{\mathbb{Q}} \mathrm{Fil}^1 H^1_{\mathrm{dR}}(Y(N)/\mathbb{Q})$$

recalling that $\mathrm{Fil}^1 H^1_{\mathrm{dR}}(Y(N)/\mathbb{Q}) = H^0(X(N), \Omega^1_{X(N)/\mathbb{Q}}(\text{log cusps}))$.

Define the following differentials on the modular curve in terms of the weight 1 Eisenstein series:

$$\widetilde{\omega}_r = \frac{1}{Nr} KS_{Nr}\big({}_D\mathrm{Eis}(z_r) \cdot {}_{D'}\mathrm{Eis}(z'_r)\big) \in H^0(X(Nr), \Omega^1(\text{log cusps})).$$

$$\omega_r = \mathrm{tr}_{X(Nr)/X(N)\otimes\mathbb{Q}(\boldsymbol{\mu}_r)} \widetilde{\omega}_r \in H^0(X(N) \otimes \mathbb{Q}(\boldsymbol{\mu}_r), \Omega^1(\text{log cusps})) \qquad (5.1.6)$$

Theorem 5.1.7. *For every $r \in \mathcal{R}_p$,*

$$\exp_p^* \xi_r = \frac{e}{r}\, \omega_r$$

where $e \in \mathbb{Z}_p^*$ is as in (5.1.1).

Proof. By 5.1.5 we have in $H^0(X(Nr)^{\mathrm{ord}}, \Omega^2_{X(Nr)/\mathbb{Z}}(\text{log cusps}))$ the identity

$$\operatorname{dlog} \widetilde{\sigma}_r = \widetilde{\omega}_r \otimes \operatorname{dlog} e_{Nr}(z_r, z'_r).$$

Now take $r = r_0 p^m$ and tensor with $\mathbb{Z}_p$. Then by (5.1.1)

$$\operatorname{dlog} \widetilde{\sigma}_r = r_0^{-1} e\, \widetilde{\omega}_r \otimes \operatorname{dlog} \zeta_{p^m} \in H^0(X(Nr)^{\mathrm{ord}} \otimes \mathbb{Z}_p, \Omega^2(\text{log cusps})).$$

Taking the trace to $X(N) \otimes \mathbb{Q}(\boldsymbol{\mu}_r)$ gives, using the compatibility (§2.1) of trace and transfer

$$\operatorname{dlog} \sigma_r = r_0^{-1} e\, \omega_r \otimes \operatorname{dlog} \zeta_{p^m}.$$

Let $\mathfrak{o}_n$ be the ring of integers of $\mathbb{Q}_p(\boldsymbol{\mu}_{p^n})$. By the explicit reciprocity law 3.2.3,

$$\exp_p^* \xi_r = \lim_{n\to\infty} \frac{r_0^{-1} e}{p^n} \operatorname{tr}_{Y(N)\otimes\mathfrak{o}_n/Y(N)\otimes\mathfrak{o}_m} \omega_{r_0 p^n}.$$

But since $\operatorname{tr}_{Y(Nr_0p^n)/Y(Nr)} \widetilde{\omega}_{r_0p^n} = p^{n-m}\widetilde{\omega}_r$ by 2.5.3, this gives the desired formula. □

5.2 Elliptic curves

Suppose that $E/\mathbb{Q}$ is a modular elliptic curve of conductor N_E, with a Weil parameterisation

$$\varphi_E \colon X_0(N_E) \to E.$$

Choose a prime p not dividing $2N_E$, and write $T_p(E) = H^1(\overline{E}, \mathbb{Z}_p)(1)$ — of course, this is the same as the Tate module of E, but it is better to think in terms of cohomology, especially if we were to work more generally with any weight 2 eigenform (with character). Let the L-series of E be

$$L(E,s) = \sum_{n\geq 1} a_n n^{-s}$$

(again, this is best thought of here as the L-series attached to the motive $h^1(E)$). Let N be any positive multiple of N_E with $(N,p)=1$. (The actual choice of N is to be made later.) Consider the composite morphism

$$\varphi_{E,N} \colon X(N) \to X_0(N_E) \xrightarrow{\varphi_E} E.$$

There are Galois-equivariant maps of restriction and direct image

$$\begin{tikzcd} H^1(X(N)\otimes_{\mathbb{Q}}\overline{\mathbb{Q}}, \mathbb{Z}_p(1)) \arrow[r, "\text{restriction}"] \arrow[d, "\varphi_{E,N*}"] & H^1(Y(N)\otimes_{\mathbb{Q}}\overline{\mathbb{Q}}, \mathbb{Z}_p(1)) = \mathbb{T}_{p,N} \\ H^1(E\otimes_{\mathbb{Q}}\overline{\mathbb{Q}}, \mathbb{Z}_p(1)) = T_p(E) & \end{tikzcd}$$

Now the Manin-Drinfeld theorem (or rather its proof) implies that there is an idempotent Π_N^{cusp} in the Hecke algebra (with rational coefficients) which induces for every p a left inverse to the map labelled "restriction". So for some positive integer h_E (independent of p) the composite map

$$h_E\,\varphi_{E,N*}\circ\Pi_N^{\mathrm{cusp}}\colon \mathbb{T}_{p,N}\to T_p(E)$$

is well-defined. Choose D, D' prime to $6Np$, and systems (z_r), (z'_r) as in the previous section.

Theorem 5.2.1. *Define for* $r\in\mathcal{R}_p$

$$\xi_r(E):=(h_E\,\varphi_{E,N*}\circ\Pi_N^{\mathrm{cusp}})\xi_r(Y(N))\in H^1(\mathbb{Q}(\boldsymbol{\mu}_r),T_p(E)).$$

Then the family $\{\xi_r(E)\}$ *is an Euler system for* $T_p(E)$*; that is,*

- *For every* $r\in\mathcal{R}_p$, $\mathrm{cor}_{\mathbb{Q}(\boldsymbol{\mu}_{rp})/\mathbb{Q}(\boldsymbol{\mu}_r)}\xi_{rp}(E)=\xi_r(E)$*;*
- *If* ℓ *is prime and* $(\ell,NDD'r)=1$ *then*

$$\mathrm{cor}_{\mathbb{Q}(\boldsymbol{\mu}_{\ell r})/\mathbb{Q}(\boldsymbol{\mu}_r)}\xi_{\ell r}(E)=(1-\ell^{-1}a_\ell\mathrm{Frob}_\ell+\ell^{-1}\mathrm{Frob}_\ell^2)\,\xi_r(E).$$

where $\mathrm{Frob}_\ell\in\mathrm{Gal}(\mathbb{Q}(\boldsymbol{\mu}_r)/\mathbb{Q})$ *is the geometric Frobenius.*

Remark. Actually, Rubin considers cohomology classes not over $\mathbb{Q}(\boldsymbol{\mu}_r)$ but rather over the subfield $\mathbb{Q}_{m-1}(\boldsymbol{\mu}_{r_0})$, where $r=r_0p^m$ and $\mathbb{Q}_{m-1}/\mathbb{Q}$ is the unique extension of degree p^{m-1} contained in the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$. To get an Euler system in the precise sense of [29, §2], one should therefore take the corestriction of $\xi_r(E)$ to $\mathbb{Q}_{m-1}(\boldsymbol{\mu}_{r_0})$. Note that his formula for the norm relation differs from that here, as we are using geometric Frobenius: as Nekovář has explained to us, the relation (ii) can be rewritten more conceptually as $\mathrm{cor}(\xi_{\ell r})=Q_\ell(\mathrm{Frob}_\ell)\xi_r$, where $Q_\ell(x)=\det(1-\mathrm{Frob}_\ell x\mid T_p(E)^*(1))$. Writing $P_\ell(x)=Q_\ell(\ell^{-1}x)$ one gets the same formula as in *loc. cit.*

Proof. The first statement follows directly from the corresponding statement 5.1.3(i) for $\xi_{rp}(Y(N))$. The second follows from 5.1.3(ii) together with the fact that $\langle\ell\rangle=1$ and $T_\ell=a_\ell$ on $T_p(E)$. □

On differentials, the projector Π_N^{cusp} is the identity on cusp forms and annihilates Eisenstein series. Put

$$\omega_r^{\mathrm{cusp}}=\Pi_N^{\mathrm{cusp}}(\omega_r)\in H^0(X(N)\otimes\mathbb{Q}(\boldsymbol{\mu}_r),\Omega^1).$$

Then Theorem 5.1.7 gives:

$$\exp_p^*\xi_r(E)=\frac{eh_E}{r}\varphi_{E,N*}(\omega_r^{\mathrm{cusp}}). \tag{5.2.2}$$

To compute this in terms of the L-function, use the Rankin-Selberg integral from §4. Fix a differential ω_E on $E/\mathbb{Q}$ such that $\varphi_E^*\omega_E$ is a newform on $X_0(N_E)$, which we write as $2\pi iF(\tau,g)d\tau$ for a weight 2 cusp form F whose Whittaker function satisfies:

- A_q is K_q-invariant if $q \nmid N_E$, and is $K_0(q^\nu)$-invariant if $\mathrm{ord}_q(N_E) = \nu > 0$;
- $A_q(1) = 1$ for all q.

This means that $A_q(q) = q^{-2} a_q$ for every $q \nmid N_E$, and that $L(E, s) = L(\pi, s)$ where π is the representation of G_f generated by F.

At this point it would be wise to recall that we have in §3.1 normalised the reciprocity law of local class field theory to take uniformisers to geometric Frobenius. This gives the classical isomorphism:

$$\begin{array}{ccccc} \hat{\mathbb{Z}}^* & \hookrightarrow & \mathbb{A}_{\mathbb{Q}}^* & \xrightarrow{\text{global CFT}} & \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}) \\ a & & \longmapsto & & (\zeta_n \mapsto \zeta_n^a) \end{array}$$

If λ is any idele class character of conductor M, with associated Dirichlet character $\lambda_{\mathrm{mod}\, M} \colon (\mathbb{Z}/M\mathbb{Z})^* \to \mathbb{C}^*$, we then have

$$L(\pi \otimes \lambda^{-1}, s) = L(E, \lambda_{\mathrm{mod}\, M}, s) := \sum_{(m,M)=1} a_m \lambda_{\mathrm{mod}\, M}(m) m^{-s}$$

We also define the incomplete and twisted L-series

$$L_N(E, \lambda_{\mathrm{mod}\, M}, s) := \sum_{(m,MN)=1} a_m \lambda_{\mathrm{mod}\, M}(m) m^{-s}$$
$$L_N(E, s; \alpha) := \sum_{(m,N)=1} e^{2\pi i m\alpha} a_m m^{-s}$$

as in §4 above. Now put

- δ_E = number of connected components of $E(\mathbb{R})$;
- Ω_E^+ = fundamental real period of ω_E;
- $\Omega_E^- = \delta_E \times$ fundamental imaginary period of ω_E

so that

$$\int_{E(\mathbb{C})} \omega_E \wedge \bar{\omega}_E = \Omega_E^+ \Omega_E^- \in i\mathbb{R}$$

The set of complex points $\mathrm{Spec}\, \mathbb{Q}(\boldsymbol{\mu}_r)(\mathbb{C})$ is the set of primitive r^{th} roots of unity $\{e^{2\pi i x/r}\}$ in $\mathbb{C}$, which we identify with $(\mathbb{Z}/r\mathbb{Z})^*$. Write $\iota_x \colon \mathbb{Q}(\boldsymbol{\mu}_r) \hookrightarrow \mathbb{C}$ for the corresponding embedding $\zeta_r \mapsto e^{2\pi i x/r}$. Suppose $\lambda \colon \mathbb{A}_f^*/\mathbb{Q}_{>0}^* \to \mathbb{C}^*$ is a character of conductor dividing r. Then we can compute

$$\sum_{x\in(\mathbb{Z}/r\mathbb{Z})^*} \int\limits_{E(\mathbb{C})} \lambda_{\operatorname{mod} r}(x)\cdot \iota_x\varphi_{E,N*}(\omega_r^{\mathrm{cusp}})\wedge\bar\omega_E$$

$$= \int\limits_{Y(N)(\mathbb{C})\times(\mathbb{Z}/r\mathbb{Z})^*} \lambda_{\operatorname{mod} r}(x)\cdot \iota_x\omega_r^{\mathrm{cusp}}\wedge\varphi^*_{E,N}\bar\omega_E$$

$$= \int\limits_{Y(N)(\mathbb{C})\times(\mathbb{Z}/r\mathbb{Z})^*} \lambda_{\operatorname{mod} r}(x)\cdot \iota_x\omega_r\wedge\varphi^*_{E,N}\bar\omega_E$$

(since cusp forms and Eisenstein series are orthogonal)

$$= \int\limits_{Y(Nr)(\mathbb{C})} (\lambda_{\operatorname{mod} Nr}\circ e_{Nr})\cdot\widetilde{\omega}_r\wedge\varphi^*_{E,Nr}\bar\omega_E \tag{5.2.3}$$

where the map $e_{Nr}\colon Y_{Nr}(\mathbb{C})\to\operatorname{Spec}\mathbb{Q}(\boldsymbol{\mu}_{Nr})(\mathbb{C})=(\mathbb{Z}/Nr\mathbb{Z})^*$ is that defined in (2.3.4).

At this point we need to choose the parameters z_r, z'_r of the Euler system ξ_r in such a way that the expression (5.2.3) can be computed using Theorem 4.6.3. In fact it will be necessary to replace $\xi_r(E)$ by a certain linear combination of Euler systems. The choices to be made are best broken down into a number of steps:

Step 1: Fix a prime p with $p\not| N_E$, and $\varepsilon\in\{\pm1\}$. We will restrict to characters λ with $\lambda(-1)=\varepsilon$.

Step 2: If $\alpha=y/M\in\mathbb{Q}$, the value of the twisted Dirichlet series at $s=1$ is a period integral

$$L_{MN_E}(E,1;\alpha) = -\int_\alpha^{i\infty}\sum_{(n,MN_E)=1} a_nq^n\,2\pi i\,d\tau$$

and one knows that this is a rational multiple of a period along a closed path in $X(N)(\mathbb{C})$, for suitable N. Moreover the cusp form $\sum_{(n,N)=1}a_nq^n\,dq/q$ is obtained from the eigenform $\varphi^*\omega_E$ by applying a suitable Hecke operator. It follows that for any $\alpha\in\mathbb{Q}$,

$$L_{MN_E}(E,1;\alpha)-\varepsilon L_{MN_E}(E,1;-\alpha) \tag{5.2.4}$$

is a rational multiple of Ω_E^ε. Moreover, one can find α with denominator prime to any chosen integer for which (5.2.4) is nonzero, by [36].

We choose an $\alpha=y/M$ with $M>0$ and $(M,y)=(M,p)=1$, and for which (5.2.4) is non-zero. By what has been just said, there will be a finite collection of such y/M which will cover all possible choices of p. We then take

$$N = \prod_{q|MN_E} q^{\nu_q}, \quad \nu_q = \max(2, \mathrm{ord}_q(N_E), \mathrm{ord}_q(M) + 1).$$

Step 3: Fix auxiliary integers D, $D' > 1$ with $(DD', 6pN_E) = 1$ and $D \equiv D' \equiv 1 \pmod{M}$. Let $r = r_0 p^m \in \mathcal{R}_p$; thus $m \geq 1$ and $r_0 > 0$ is squarefree and coprime to $pDD'N$. In the notation of §4.6 we put $R = r$, $T = \{q|r\}$, $S = \{q|N\}$ and choose the ideles t, $t' \in \mathbb{A}_f^*$ to have local components

$$t_q = \begin{cases} 1 & \text{if } q \nmid Nr \\ (Nr)^{-1} & \text{if } q|Nr \end{cases}; \qquad t'_q = \begin{cases} 1 & \text{if } q \nmid Mr \\ -r^{-1}y|M|_q & \text{if } q|M \\ (Mr)^{-1} & \text{if } q|r \end{cases}$$

Then (4.6.1) holds, and $t \in (Nr)^{-1}\hat{\mathbb{Z}}^*$, $t' \in (Mr)^{-1}\hat{\mathbb{Z}}^*$. In §4.6 this data then determines functions ϕ, $\phi' \in \mathcal{S}(\mathbb{A}_f^2)$. Let $\delta \in \hat{\mathbb{Z}}^*$ be the finite unit idele

$$\delta_q = \begin{cases} D & \text{if } q|Nr; \\ 1 & \text{otherwise} \end{cases}$$

and set, by analogy with (1.3.2),

$${}_D\phi = D^2\phi - D[\delta]\phi$$

in the notation of (4.1.2). Likewise define δ' and ${}_{D'}\phi'$ in the obvious way. Since $(Nr, D) = 1$, if $\mathrm{cond}(\lambda)|r$ we have

$$\lambda(\delta) = \prod_{q|Nr} \lambda_q(D) = \lambda_{\mathrm{mod}\, r}(D). \tag{5.2.5}$$

Step 4: We have $\phi = \mathrm{char}[(t + \hat{\mathbb{Z}}) \times \hat{\mathbb{Z}}] = \mathrm{char}[(Nr)^{-1} + \hat{\mathbb{Z}} \times \hat{\mathbb{Z}}]$. Choose $z_r \in \mathcal{E}^{\mathrm{univ}}(Y(Nr))$ to be the point which in complex coordinates is

$$\frac{1}{Nr} \in (\frac{1}{Nr}\mathbb{Z} + \frac{\tau}{Nr}\mathbb{Z})/(\mathbb{Z} + \tau\mathbb{Z}) \simeq (\mathbb{Z}/Nr\mathbb{Z})^2.$$

For different r the points z_r are compatible: $\ell z_{\ell r} = z_r$. We then can use (1.3.3) to write the Eisenstein series in terms of the complex parameterisation as

$${}_D\mathrm{Eis}(z_r) = E_1({}_D\phi)\, du. \tag{5.2.6}$$

Step 5: The function ϕ' has local components

$$\phi'_q = \begin{cases} \mathrm{char}[\mathbb{Z}_q \times \mathbb{Z}_q] & \text{if } q \not| Nr; \\ \mathrm{char}[\mathbb{Z}_q \times (1/Mr + \mathbb{Z}_q)] & \text{if } q|r; \\ \mathrm{char}\begin{bmatrix} t'_q + \mathbb{Z}_q \\ \mathbb{Z}_q^* \end{bmatrix} - q^{-1}\mathrm{char}\begin{bmatrix} t'_q + q^{-1}\mathbb{Z}_q \\ \mathbb{Z}_q^* \end{bmatrix} & \text{if } q|N. \end{cases}$$

The last expression can be rewritten as

$$\mathrm{char}\begin{bmatrix} t'_q + \mathbb{Z}_q \\ \mathbb{Z}_q \end{bmatrix} - [q]\mathrm{char}\begin{bmatrix} q^{-1}t'_q + q^{-1}\mathbb{Z}_q \\ \mathbb{Z}_q \end{bmatrix} \\ - q^{-1}\mathrm{char}\begin{bmatrix} t'_q + q^{-1}\mathbb{Z}_q \\ \mathbb{Z}_q \end{bmatrix} + q^{-1}[q]\mathrm{char}\begin{bmatrix} q^{-1}t'_q + q^{-2}\mathbb{Z}_q \\ \mathbb{Z}_q \end{bmatrix}.$$

Now by (4.3.1) there exist a finite set of points $z'_{r,j} \in \mathcal{E}^{\mathrm{univ}}(Y(Nr))$ and constants $b_j \in N^{-1}\mathbb{Z}$ which are independent of r, such that

$$\sum_j b_j \cdot {}_{D'}\mathrm{Eis}(z'_{r,j}) = E_1({}_{D'}\phi')\, du$$

and $\ell z'_{\ell r,j} = z'_{r,j}$. Moreover the differences $z'_{r,j} - z'_{r,i}$ will be N-torsion, and in complex coordinates $Nz'_{r,j}$ will be the point

$$(-Nt' \bmod \hat{\mathbb{Z}})\tau \in (\frac{1}{r}\mathbb{Z} + \frac{\tau}{r}\mathbb{Z})/(\mathbb{Z}+\tau\mathbb{Z}) \simeq (\mathbb{Z}/r\mathbb{Z})^2.$$

It follows that

$$e_{Nr}(z_r, z'_{r,j}) = \zeta_{p^m}^{-(Mr_0)^{-1}} \times (\text{prime-to } p \text{ root of } 1)$$

and thus that the constant e of (5.1.1) equals $(-M^{-1}) \in \mathbb{Z}_p^*$.

Step 6: Put $\widetilde{\sigma}_{r,j} = \{\vartheta_D(z_r), \vartheta_{D'}(z'_{r,j})\}$, and let $\xi_{r,j}(E)$ be the associated Euler system for $T_p(E)$. The required Euler system is then

$$c_r = \sum_j b_j \xi_{r,j}(E) \in H^1(\mathbb{Q}(\boldsymbol{\mu}_r), T_p(E)).$$

We can now compute the dual exponential of c_r. Put $\widetilde{\omega}_{r,j}$ for the differential on $Y(Nr)$ constructed from $(z_r, z'_{r,j})$. The Kodaira-Spencer map takes $(dt/t)^{\otimes 2}$ to dq/q, and therefore $du^{\otimes 2}$ to $(2\pi i)^{-1}d\tau$. Therefore

$$\sum_j b_j \cdot \widetilde{\omega}_{r,j} = \frac{(2\pi i)^{-1}}{Nr} E_1({}_D\phi) E_1({}_{D'}\phi')\, d\tau.$$

We then get

$$\begin{aligned}
&\sum_{\gamma\in \mathrm{Gal}(\mathbb{Q}(\boldsymbol{\mu}_r)/\mathbb{Q})=(\mathbb{Z}/r\mathbb{Z})^*} \lambda_{\mathrm{mod}\, r}(\gamma)\exp_p^* c_r^\gamma \\
&\qquad = \left(\frac{-M^{-1}h_E}{r\Omega_E^+\Omega_E^-}\sum_j b_j \int_{Y(Nr)(\mathbb{C})} (\lambda_{\mathrm{mod}\, Nr}\circ e_{Nr})\cdot \widetilde{\omega}_{r,j}\wedge \varphi^*_{E,Nr}\bar{\omega}_E\right)\omega_E \\
&\qquad = -\frac{h_E \# GL_2(\mathbb{Z}/Nr\mathbb{Z})}{MNr^2\Omega_E^+\Omega_E^-}\langle E_1({}_D\phi)E_1({}_{D'}\phi')\otimes\lambda, F\rangle\omega_E.
\end{aligned}$$

By Theorem 4.6.3, taking $k = l = 1$ and $s = 0$,

$$\begin{aligned}
\langle E_1(\phi)E_1(\phi')\otimes\lambda, F\rangle &= C' r^{-2}\# GL_2(\mathbb{Z}/r\mathbb{Z})^{-1}\prod_{q\mid r}\lambda_q(MNt_qt'_q)^{-1} \\
&\quad \times L_{Nr}(E,\lambda_{\mathrm{mod}\, r},1)\big(L_N(E,1;y/M)-\lambda(-1)L_N(E,1;-y/M)\big)
\end{aligned}$$

for some $C'\in\mathbb{Q}^*$, depending only on E, M and N. Moreover, using (5.2.5) and the hypothesis that $D\equiv D'\equiv 1 \pmod M$,

$$\begin{aligned}
\langle E_1({}_D\phi)\,E_1({}_{D'}\phi')\otimes\lambda, F\rangle &= C' r^2 \# GL_2(\mathbb{Z}/r\mathbb{Z})^{-1} \\
&\times \prod_{q\mid r}\lambda_q(MNt_qt'_q)^{-1}DD'(D-\lambda_{\mathrm{mod}\, r}(D)^{-1})(D'-\lambda_{\mathrm{mod}\, r}(D')^{-1}) \\
&\times L_{Nr}(E,\lambda_{\mathrm{mod}\, r},1)\big(L_N(E,1;y/M)-\lambda(-1)L_N(E,1;-y/M)\big).
\end{aligned}$$

Now for $q|r$ we have $t_qt'_q = (MNr^2)^{-1}$, so $\prod_{q|r}\lambda_q(MNt_qt'_q) = 1$ since $\mathrm{cond}(\lambda)|r$. Combining everything one gets the final result:

Theorem 5.2.7. *Let $E/\mathbb{Q}$ be a modular elliptic curve of conductor N_E. Fix a non-zero 1-form $\omega_E\in\Omega^1(E/\mathbb{Q})$, with real and imaginary periods Ω_E^+, Ω_E^-. Let p be a prime not dividing N_E. Then there is an integer M prime to p, and for every pair of integers D, $D'>1$ with $(DD',6pN_E)=1$ and $D\equiv D'\equiv 1$ (mod M) an Euler system:*

$$c_r = c_r(E,p,D,D')\in H^1(\mathbb{Q}(\boldsymbol{\mu}_r),T_p(E)),\qquad r = r_0p^m,\ r_0 \text{ squarefree and prime to } pMN_E,\ m\ge 1$$

such that for each r and each character λ: $\mathrm{Gal}(\mathbb{Q}(\boldsymbol{\mu}_r)/\mathbb{Q})\simeq(\mathbb{Z}/r\mathbb{Z})^*\to\mathbb{C}^*$ with $\lambda(-1)=\pm1$

$$\sum_{\gamma\in\mathrm{Gal}(\mathbb{Q}(\boldsymbol{\mu}_r)/\mathbb{Q})}\lambda(\gamma)\exp_p^* c_r^\gamma = C_E^{\pm}\,DD'(D-\lambda(D)^{-1})(D'-\lambda(D')^{-1})\,\frac{L_{rMN_E}(E,\lambda,1)}{\Omega_E^{\pm}}\,\omega_E$$

for some constant $C_E^{\pm}$, depending only on E.

In the special case $r = p^m$ this is (with minor modifications of notation) Theorem 7.1 of [29].

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