The Beilinson conjectures

CHRISTOPHER DENINGER AND ANTHONY J. SCHOLL*

Introduction

The Beilinson conjectures describe the leading coefficients of L-series of varieties over number fields up to rational factors in terms of generalized regulators. We begin with a short but almost selfcontained introduction to this circle of ideas. This is possible by using Bloch's description of Beilinson's motivic cohomology and regulator map in terms of higher Chow groups and generalized cycle maps. Here we follow [**Bl3**] rather closely. We will then sketch how much of the known evidence in favour of these conjectures — to the left of the central point — can be obtained in a uniform way. The basic construction is Beilinson's Eisenstein symbol which will be explained in some detail. Finally in an appendix a map is constructed from higher Chow theory to a suitable Ext-group in the category of mixed motives as defined by Deligne and Jannsen. This smooths the way towards an interpretation of Beilinson's conjectures in terms of a Deligne conjecture for critical mixed motives [**Sc2**]. It also explains how work of Harder [**Ha2**] and Anderson fits into the picture.

For further preliminary reading on the Beilinson conjectures, one should consult the Bourbaki seminar of Soulé [**So1**], the survey article by Ramakrishnan [**Ra2**] and the introductory article by Schneider [**Sch**]. For the full story see the book [**RSS**] and of course Beilinson's original paper [**Be1**]. Here one will also find the conjectures for the central and near-central points, which for brevity we have omitted here.

1. Motivic cohomology

Motivic cohomology is a kind of universal cohomology theory for algebraic varieties. There are two constructions both generalizing ideas from algebraic topology. The first one is due to Beilinson [**Be1**]. He defines motivic cohomology as a suitable graded piece of the γ-filtration on Quillen's algebraic K-groups tensored with $\mathbf{Q}$. This is analogous to the introduction of singular cohomology as a suitable graded piece of topological K-theory by Atiyah in [**At**] 3.2.7.

For smooth varieties there is a second more elementary construction which is due to Bloch [**Bl2,3,4**]. It is modeled on singular cohomology: instead of continuous maps from the n-simplex to a topological space one considers algebraic correspondences from the algebraic n-simplex $\Delta_n \cong \mathbf{A}^n$ to the variety. We proceed with the details:

Let k be a field and set for $n \geq 0$

$$\Delta_n = \operatorname{Spec} k[T_0, \ldots, T_n]/(\Sigma T_i - 1).$$

There are face maps

$$\partial_i : \Delta_n \hookrightarrow \Delta_{n+1} \quad \text{for} \quad 0 \leq i \leq n+1 \tag{1.1}$$

which in coordinates are given by

$$\partial_i(t_0, \ldots, t_n) = (t_0, \ldots, t_{i-1}, 0, t_i \ldots, t_n).$$

Let X be an equidimensional scheme over k. A face of $X \times \Delta_m$ is the image of some $X \times \Delta_{m'}$, $m' < m$ under a composition of face maps induced by (1.1)

$$\partial_i : X \times \Delta_n \hookrightarrow X \times \Delta_{n+1}.$$

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We denote by $z^q(X,n)$ the free abelian group generated by the irreducible codimension q subvarieties of $X\times\Delta_n$ meeting all faces properly. Here subvarieties $Y_1,Y_2\subset X\times\Delta_n$ of codimensions c_1,c_2 are said to meet properly if every irreducible component of $Y_1\cap Y_2$ has codimension $\geq c_1+c_2$ on X. Observe that $z^q(X,n)$ is a subgroup of correspondences from Δ_n to X. Using the differential

$$d=\sum_{i=0}^{n+1}(-1)^i\partial_i^*:z^q(X,n+1)\to z^q(X,n)$$

one obtains a complex of abelian groups $z^q(X,\cdot)$. If X is a smooth quasiprojective variety over k setting

$$\Gamma_X(q)^{\cdot}=z^q(X,2q-\cdot)$$

we define:

$$H^p_{\mathcal{M}}(X,\Lambda(q))=H^p(\Gamma_X(q)^{\cdot}\otimes\Lambda) \tag{1.2}$$

for any ring Λ. By one of the main results of Bloch these groups coincide for $\Lambda=\mathbf{Q}$ with the groups $H^p_{\mathcal{M}}(X,\mathbf{Q}(q))$ defined by Beilinson using algebraic K-theory. Using either definition the following formal properties can be proved:

(1.3) Theorem. *(1) $H^{\cdot}_{\mathcal{M}}(\quad,\mathbf{Q}(*))$ is a contravariant functor from the category of smooth quasiprojective varieties over k into the category of bigraded $\mathbf{Q}$-vector spaces. For proper maps $f:X\to Y$ of pure codimension $c=\dim Y-\dim X$ we also have covariant functoriality with a shift of degrees*

$$f_*:H^{\cdot}_{\mathcal{M}}(X,\mathbf{Q}(*))\to H^{\cdot+2c}_{\mathcal{M}}(Y,\mathbf{Q}(*+c)).$$

(2) There is a cup product which is contravariant functorial, associative and graded commutative with respect to $\cdot$.

(3) There are functorial isomorphisms compatible with the product structure

$$H^{2p}_{\mathcal{M}}(X,\mathbf{Q}(p))=CH^p(X)\otimes\mathbf{Q}\,.$$

(4) $H^1_{\mathcal{M}}(X,\mathbf{Q}(1))=\Gamma(X,\mathcal{O}^*)\otimes\mathbf{Q}$ functorially.

(5) Let $i:Y\hookrightarrow X$ be a closed immersion (of smooth varieties) of codimension c with open complement $j:U=X-Y\hookrightarrow X$. Then there is a functorial long exact localization sequence

$$\begin{aligned}\cdots\to H^{\cdot-2c}_{\mathcal{M}}(Y,\mathbf{Q}(*-c))&\xrightarrow{i_*}H^{\cdot}_{\mathcal{M}}(X,\mathbf{Q}(*))\xrightarrow{j^*}H^{\cdot}_{\mathcal{M}}(U,\mathbf{Q}(*))\\ &\to H^{\cdot+1-2c}_{\mathcal{M}}(Y,\mathbf{Q}(*-c))\to\cdots\,.\end{aligned}$$

(6) If $\pi:X'\to X$ is a finite galois covering with group G we have $\pi_*\pi^*=|G|$ id and $\pi^*\pi_*=\sum_{\sigma\in G}\sigma^*$. In particular

$$\pi^*:H^{\cdot}_{\mathcal{M}}(X,\mathbf{Q}(*))\xrightarrow{\sim}H^{\cdot}_{\mathcal{M}}(X',\mathbf{Q}(*))^G$$

is an isomorphism, i.e. $H^{\cdot}_{\mathcal{M}}(\quad,\mathbf{Q}(*))$ has galois descent.

For zero dimensional X over $\mathbf{Q}$ the motivic cohomology groups are known by the work of Borel [**Bo1,Bo2**] on algebraic K-theory of number fields. A proof of the following result which does not make use of algebraic K-theory seems to be out of reach.

(1.4) Theorem. *Let k be a number field, $X = \operatorname{Spec} k$. Then*

(1)
$$\begin{aligned} \dim_{\mathbf{Q}} H^1_{\mathcal{M}}(X, \mathbf{Q}(q) &= r_1 + r_2 \qquad \text{if } q > 1 \text{ is odd} \\ &= r_2 \qquad \text{if } q > 1 \text{ is even} \end{aligned}$$

where r_1, r_2 denote the numbers of real resp. complex places of k.

(2)
$$H^p_{\mathcal{M}}(X, \mathbf{Q}(q)) = 0 \quad \text{for} \quad p \neq 1.$$

Observe that for X as in the theorem we have:

$$H^1_{\mathcal{M}}(X, \mathbf{Q}(1)) = k^* \otimes_{\mathbf{Z}} \mathbf{Q}.$$

In view of the class number formula which involves a regulator formed with the units of k we see that for arithmetic purposes the groups $H^p_{\mathcal{M}}(X, \mathbf{Q}(q))$ may have to be replaced by smaller ones:

If X is a variety over $\mathbf{Q}$ we set:

(1.6)
$$H^p_{\mathcal{M}}(X, \mathbf{Q}(q))_{\mathbf{Z}} = H^p_{\mathcal{M}}(X, \mathbf{Q}(q)) \qquad \text{for } q > p.$$

(1.7)
$$H^p_{\mathcal{M}}(X, \mathbf{Q}(q))_{\mathbf{Z}} = \operatorname{Im}(H^p_{\mathcal{M}}(\mathcal{X}, \mathbf{Q}(q)) \to H^p_{\mathcal{M}}(X, \mathbf{Q}(q)) \qquad \text{for } q \leq p.$$

Here $\mathcal{X}$ is a proper regular model of X over $\operatorname{Spec} \mathbf{Z}$ which is supposed to exist. The groups $H^p_{\mathcal{M}}(\mathcal{X}, \mathbf{Q}(q))$ are either defined by the above construction which also works over $\operatorname{Spec} \mathbf{Z}$ or by using the K-theory of $\mathcal{X}$. The "motivic cohomology groups of an integral model" $H^p_{\mathcal{M}}(X, \mathbf{Q}(q))_{\mathbf{Z}}$ are independent of $\mathcal{X}$. It is a conjecture that (1.6) holds if the definition in (1.7) is extended to $q > p$.

2. Deligne cohomology and regulator map

The definition of Deligne cohomology which is about to follow may seem rather unmotivated at first. We refer to (2.9) below where a conceptual interpretation of these groups as Ext's in a category of mixed Hodge structures is described.

(2.1) For a subring Λ of $\mathbf{C}$ we set $\Lambda(q) = (2\pi i)^q \Lambda \subset \mathbf{C}$. Let X be a smooth projective variety over $\mathbf{C}$ and consider the following complex of sheaves on the analytic manifold X_{an}:

$$\mathbf{R}(q)_{\mathcal{D}} = (R(q) \to \mathcal{O} \to \cdots \to \Omega^{q-1})$$

in degrees 0 to q. We set

$$H^p_{\mathcal{D}}(X, \mathbf{R}(q)) = H^p(X_{\text{an}}, \mathbf{R}(q)_{\mathcal{D}}).$$

Apart from the Deligne cohomology groups we need the singular (Betti) cohomology groups of X_{an}

$$H^p_B(X, \Lambda(q)) = H^p_{\text{sing}}(X_{\text{an}}, \Lambda(q))$$

and the de Rham groups

$$H^p_{DR}(X) = H^p_{\text{Zar}}(X, \Omega^{\cdot}_{X/\mathbf{C}}) \cong H^p(X_{\text{an}}, \Omega^{\cdot}).$$

(2.2) If X is smooth projective over $\mathbf{R}$ there is an antiholomorphic involution F_∞ on $(X_{\mathbf{C}})_{\text{an}}$, the infinite Frobenius. We set

$$H^p_{\mathcal{D}}(X, \mathbf{R}(q)) = H^p_{\mathcal{D}}(X_{\mathbf{C}}, \mathbf{R}(q))^+$$

where the superscript + denotes the fixed module under

$$\overline{F}^*_\infty = F^*_\infty \circ (\text{complex conjugation on the coefficients}).$$

The groups $H^p_B(X,\Lambda(q))$ are defined similarly if $1/2 \in \Lambda$. Observe that under the comparison isomorphism

$$H^p_{DR}(X_{\mathbf{C}}/\mathbf{C}) \xrightarrow{\sim} H^p_B(X_{\mathbf{C}},\mathbf{C})$$

the de Rham conjugation corresponds to $\overline{F}^*_\infty$ and hence

$$H^p_{DR}(X) \xrightarrow{\sim} H^p_B(X,\mathbf{C}).$$

(2.3) Recall that if $u : \mathcal{A}^\cdot \to \mathcal{B}^\cdot$ is a morphism of complexes of sheaves the cone of u is the complex

$$\mathrm{Cone}(\mathcal{A}^\cdot \xrightarrow{u} \mathcal{B}^\cdot) = \mathcal{A}^\cdot[1] \oplus \mathcal{B}^\cdot$$

with the differentials

$$\begin{aligned} \mathcal{A}^{q+1} \oplus \mathcal{B}^q &\to \mathcal{A}^{q+2} \oplus \mathcal{B}^{q+1} \\ (a,b) &\mapsto (-d(a), u(a) + d(b)). \end{aligned}$$

There are quasi-isomorphisms on X_{an}

$$\mathrm{Cone}(\Omega^{\geq q} \oplus \mathbf{R}(q) \to \Omega^\cdot)[-1] \xrightarrow{\sim} \mathbf{R}(q)_{\mathcal{D}}$$

where $\Omega^{\geq q} = (0 \to \cdots \to 0 \to \Omega^q \to \Omega^{q+1} \to \cdots)$ and u is the difference of the obvious embeddings. For a smooth projective variety X over $\mathbf{R}$ or $\mathbf{C}$ we thus obtain a long exact sequence

$$\begin{aligned} \cdots \to H^p_{\mathcal{D}}(X,\mathbf{R}(q)) \to & F^q H^p_{DR}(X) \oplus H^p_B(X,\mathbf{R}(q)) \\ & \to H^p_{DR}(X) \to H^{p+1}_{\mathcal{D}}(X,\mathbf{R}(q)) \to \cdots . \end{aligned} \tag{2.3.1}$$

Recall here the definition of the Hodge filtration (for $X/\mathbf{C}$ say):

$$F^q H^p_{DR}(X) = \mathrm{Im}(H^p_{\mathrm{Zar}}(X,\Omega^{\geq q}_{X/\mathbf{C}}) \to H^p_{\mathrm{Zar}}(X,\Omega^\cdot_{X/\mathbf{C}}))$$

and observe that by GAGA and the degeneration of the Hodge spectral sequence we have

$$F^q H^p_{DR}(X) \cong H^p_{\mathrm{Zar}}(X,\Omega^{\geq q}_{X/\mathbf{C}}) \cong H^p(X_{\mathrm{an}},\Omega^{\geq q}).$$

Now assume that X is a variety over $\mathbf{R}$. Using Hodge theory we obtain for $q > \frac{p}{2}+1$ exact sequences

$$0 \to F^q H^p_{DR}(X) \to H^p_B(X,\mathbf{R}(q-1)) \to H^{p+1}_{\mathcal{D}}(X,\mathbf{R}(q)) \to 0 \tag{$\mathcal{B}$}$$

$$0 \to H^p_B(X,\mathbf{R}(q)) \to H^p_{DR}(X)/F^q \to H^{p+1}_{\mathcal{D}}(X,\mathbf{R}(q)) \to 0. \tag{$\mathcal{D}$}$$

For a smooth projective variety X over $\mathbf{Q}$ these define $\mathbf{Q}$-structures

$$\begin{aligned} \mathcal{B}_{p,q} &= \det(H^p_B(X_{\mathbf{R}},\mathbf{Q}(q-1))) \otimes \det(F^q H^p_{DR}(X/\mathbf{Q}))^\vee \\ \mathcal{D}_{p,q} &= \det(H^p_{DR}(X/\mathbf{Q})/F^q) \otimes \det(H^p_B(X_{\mathbf{R}},\mathbf{Q}(q))^\vee \end{aligned} \tag{2.3.2}$$

on $\det H^{p+1}_{\mathcal{D}}(X_{\mathbf{R}},\mathbf{R}(q))$. Here $\det W$ denotes the highest exterior power of a finite dimensional vector space and $^\vee$ is the dual.

(2.4) For smooth quasiprojective varieties X over $\mathbf{C}$ the above definition of Deligne cohomology leads to vector spaces which are in general infinite dimensional. A more sophisticated definition imposing growth conditions at infinity remedies this defect.

By resolution of singularities there exists an open immersion

$$j : X \hookrightarrow \overline{X}$$

of X into a smooth, projective variety $\overline{X}$ over $\mathbf{C}$ such that the complement $D = \overline{X} - X$ is a divisor with only normal crossings. Consider the natural maps of complexes of sheaves $\Omega^{\geq q}_{\overline{X}}\langle D\rangle \to j_*\Omega^{\cdot}_X$ on $\overline{X}_{\mathrm{an}}$ and $\mathbf{R}(q) \to \Omega^{\cdot}_X$ on X_{an}. Choose injective resolutions

$$\mathbf{R}(q) \xrightarrow{\sim} I^{\cdot} \quad \text{and} \quad \Omega^{\cdot}_X \xrightarrow{\sim} J^{\cdot}$$

and set

$$Rj_*\mathbf{R}(q) = j_*I^{\cdot} \quad \text{and} \quad Rj_*\Omega^{\cdot}_X = j_*J^{\cdot}.$$

We get induced maps on $\overline{X}_{\mathrm{an}}$

$$\Omega^{\geq q}_{\overline{X}}\langle D\rangle \to Rj_*\Omega^{\cdot}_X \quad \text{and} \quad Rj_*\mathbf{R}(q) \to Rj_*\Omega^{\cdot}_X$$

and using the difference of these we can form

$$\mathbf{R}(q)_{\mathcal{D}} = \mathrm{Cone}(\Omega^{\geq q}_{\overline{X}}\langle D\rangle \oplus Rj_*\mathbf{R}(q) \to Rj_*\Omega^{\cdot}_X)[-1].$$

The Deligne cohomology groups

$$H^p_{\mathcal{D}}(X, \mathbf{R}(q)) = H^p(\overline{X}_{\mathrm{an}}, \mathbf{R}(q)_{\mathcal{D}})$$

are independent of the choice of resolutions and compactification. As before we can define $\mathcal{D}$-cohomology of varieties over $\mathbf{R}$.

For X over $\mathbf{R}$ or $\mathbf{C}$ there is still a long exact sequence (2.3.1) where now $F^qH^p_{DR}(X)$ is the Deligne Hodge filtration on $H^p_{DR}(X)$. Observe that by the degeneration of the logarithmic Hodge spectral sequence

$$F^qH^p_{DR}(X) \cong H^p(\overline{X}_{\mathrm{an}}, \Omega^{\geq q}_{\overline{X}}\langle D\rangle).$$

Assertions (1), (2), (5), (6) of theorem (1.3) have their counterparts for Deligne cohomology. The analogue of assertion (4) is the formula

$$H^1_{\mathcal{D}}(X, \mathbf{R}(1)) = \{g \in H^0(X_{\mathrm{an}}, \mathcal{O}/\mathbf{R}(1)) \mid dg \in H^0(\overline{X}_{\mathrm{an}}, \Omega^1_{\overline{X}}\langle D\rangle)\} \tag{2.4.1}$$

which follows immediately from the definition. The typical element of this group should be thought of as an $\mathbf{R}$-linear combination of logarithms of regular invertible functions on X.

(2.5) In the proofs of the Beilinson conjectures a more explicit description of $\mathcal{D}$-cohomology in terms of $\mathcal{C}^\infty$-differential forms is used. Let $\mathcal{A}^{\cdot}$ be the de Rham complex of real valued $\mathcal{C}^\infty$-forms and let $\pi_k : \mathbf{C} \to \mathbf{R}(k)$, $\pi_k(z) = \frac{1}{2}(z + (-1)^k\bar{z})$ be the natural projection. There is a quasi isomorphism u

$$\begin{array}{c}\mathbf{R}(q)_{\mathcal{D}} = \mathrm{Cone}(\Omega^{\geq q}_{\overline{X}}\langle D\rangle \oplus Rj_*\mathbf{R}(q) \to Rj_*\Omega^{\cdot}_X)[-1] \\ \Big\downarrow u \\ \widetilde{\mathbf{R}(q)}_{\mathcal{D}} := \mathrm{Cone}(\Omega^{\geq q}_{\overline{X}}\langle D\rangle \to j_*\mathcal{A}^{\cdot}_X \otimes \mathbf{R}(q-1))[-1]\end{array}$$

on $\overline{X}_{\mathrm{an}}$ induced by the projection

$$\Omega^{\geq q}_{\overline{X}}\langle D\rangle \oplus Rj_*\mathbf{R}(q) \to \Omega^{\geq q}_{\overline{X}}\langle D\rangle$$

and by the composition:

$$Rj_*\Omega^{\cdot}_X = j_*J^{\cdot} \to j_*\mathcal{A}^{\cdot}_X \otimes \mathbf{C} \xrightarrow{\pi_{q-1}} j_*\mathcal{A}^{\cdot}_X \otimes \mathbf{R}(q-1).$$

In particular

$$H^p_{\mathcal{D}}(X,\mathbf{R}(q)) \cong H^p_{\mathcal{D}}(\overline{X}_{\mathrm{an}},\widetilde{\mathbf{R}(q)}_{\mathcal{D}}).$$

For $p=q$ we obtain by a straightforward computation

$$H^p_{\mathcal{D}}(X,\mathbf{R}(p)) \cong \frac{\left\{\begin{array}{c}\varphi \in H^0(X_{\mathrm{an}},\mathcal{A}^{p-1}\otimes \mathbf{R}(p-1)) | d\varphi = \pi_{p-1}(\omega), \\ \omega \in H^0(\overline{X}_{\mathrm{an}},\Omega^p_{\overline{X}}\langle D\rangle)\end{array}\right\}}{dH^0(X_{\mathrm{an}},\mathcal{A}^{p-2}\otimes \mathbf{R}(p-1))}. \tag{2.5.1}$$

In case $p=1$ we find

$$H^1_{\mathcal{D}}(X,\mathbf{R}(1)) \cong \{\varphi \in H^0(X_{\mathrm{an}},\mathcal{A}^0) \mid d\varphi = \pi_0(\omega),\, \omega \in H^0(\overline{X}_{\mathrm{an}},\Omega^1_{\overline{X}}\langle D\rangle)\}.$$

Under this isomorphism the section g of (2.4.1) is mapped to $\varphi=\pi_0(g)$ and $\omega = dg$.

Using (2.5.1) as an identification the cup product of classes $[\varphi_a]$, $[\varphi_b]$ in $H^a_{\mathcal{D}}(X,\mathbf{R}(a))$ resp. $H^b_{\mathcal{D}}(X,\mathbf{R}(b))$ with associated forms ω_a, ω_b is represented by

$$\varphi_a \cup \varphi_b := \varphi_a \wedge \pi_b \omega_b + (-1)^a \pi_a \omega_a \wedge \varphi_b.$$

One checks that $\omega_a \wedge \omega_b$ is associated to $\varphi_a \cup \varphi_b$.

We also note that in this description the boundary map in (2.4.1) is given by

$$\begin{aligned}\partial : H^p_{\mathcal{D}}(X,\mathbf{R}(p)) &\to F^p H^p_{DR}(X) \oplus H^p_B(X,\mathbf{R}(p))\\ \varphi &\mapsto (\omega,[\omega]).\end{aligned}$$

Observe that $\pi_{p-1}[\omega]=0$ and hence $[\omega]\in H^p_B(X,\mathbf{R}(p))$.

(2.6) The final ingredient in the formulation of the Beilinson conjectures is the regulator map. This is a co- and contravariant functorial homomorphism

$$r_{\mathcal{D}} : H^{\cdot}_{\mathcal{M}}(X,\mathbf{Q}(*)) \to H^{\cdot}_{\mathcal{D}}(X,\mathbf{R}(*))$$

for smooth quasiprojective varieties X over $\mathbf{R}$ or $\mathbf{C}$ which commutes with cup products. If motivic cohomology is described in terms of K-theory $r_{\mathcal{D}}$ is a generalized Chern character. In the description of $H^{\cdot}_{\mathcal{M}}$ given in section 1 $r_{\mathcal{D}}$ becomes a generalized cycle map (see (2.8)). There is a commutative diagram:

$$\begin{array}{ccc} H^1_{\mathcal{M}}(X,\mathbf{Q}(1)) & \xrightarrow{r_{\mathcal{D}}} & H^1_{\mathcal{D}}(X,\mathbf{R}(1)) \\ \| & \overset{\log}{\nearrow} & \downarrow \wr \pi_0 \\ \mathcal{O}^*(X)\otimes \mathbf{Q} & \underset{\log||}{\rightarrow} & \left\{\varphi \in \Gamma(X,\mathcal{A}^0) \,\middle|\, \begin{array}{l} d\varphi = \pi_0(\omega),\ \omega \text{ with log-sing. at} \\ \text{infinity}\end{array}\right\} \end{array} \tag{2.6.1}$$

If X is a smooth quasiprojective variety over $\mathbf{Q}$ the regulator map is defined by composition:

$$r_{\mathcal{D}} : H^{\cdot}_{\mathcal{M}}(X,\mathbf{Q}(*)) \xrightarrow{\mathrm{res}} H^{\cdot}_{\mathcal{M}}(X_{\mathbf{R}},\mathbf{Q}(*)) \xrightarrow{r_{\mathcal{D}}} H^{\cdot}_{\mathcal{D}}(X_{\mathbf{R}},\mathbf{R}(*)).$$

(2.7) The formal properties of motivic and Deligne cohomology and of the regulator map which we have mentioned up to now are sufficient for an understanding of the proofs of Beilinson's conjectures in the cases sketched in section 4. For Bloch's actual construction of the regulator map as a generalized cycle class map in (2.8) however more properties of Deligne cohomology are required. We list them briefly:

(2.7.1) There are relative $\mathcal{D}$-cohomology groups $H^p_{\mathcal{D},Y}(X,\mathbf{R}(q))$ for smooth, quasi-projective X over $\mathbf{R}$ and $\mathbf{C}$ and arbitrary closed subschemes Y of X. These fit into a co- and contravariant functorial long exact sequence

$$\to H^p_{\mathcal{D},Y}(X,\mathbf{R}(q)) \to H^p_{\mathcal{D}}(X,\mathbf{R}(q)) \to H^p_{\mathcal{D}}(X-Y,\mathbf{R}(q)) \to H^{p+1}_{\mathcal{D},Y}(X,\mathbf{R}(q)) \to \cdots$$

(2.7.2) If $Y \subset X$ has pure codimension q there is a contravariant functorial cycle class $[Y]$ in $H^{2q}_{\mathcal{D},Y}(X,\mathbf{R}(q))$. Moreover weak purity holds: in other words, $H^p_{\mathcal{D},Y}(X,\mathbf{R}(q)) = 0$ for $p < 2q$.

(2.7.3) Homotopy: $H^p_{\mathcal{D}}(X \times \mathbf{A}^1, \mathbf{R}(q)) = H^p_{\mathcal{D}}(X, \mathbf{R}(q))$.

(2.7.4) For $Y \subset X$ of pure codimension there are complexes of $\mathbf{R}$-vector spaces $D^{\cdot}_Y(X,q)$ which are contravariant functorial with respect to cartesian diagrams

$$\begin{array}{ccc} Y' & \hookrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \hookrightarrow & X \end{array}$$

and such that

$$H^p_{\mathcal{D},Y}(X,\mathbf{R}(q)) = H^p(D^{\cdot}_Y(X,q)) \quad \text{functorially.}$$

Remarks. (1) The relative $\mathcal{D}$-cohomology groups are defined by

$$H^p_{\mathcal{D},Y}(X,\mathbf{R}(q)) = H^p(X, \mathrm{Cone}(\mathbf{R}(q)_{\mathcal{D},X} \xrightarrow{\mathrm{res}} \mathbf{R}(q)_{\mathcal{D},X-Y})[-1])$$

where $\mathbf{R}(q)_{\mathcal{D},X}$ and $\mathbf{R}(q)_{\mathcal{D},X-Y}$ are the Deligne complexes on X and $X - Y$ computed with respect to compatible compactifications. The long exact sequence is then an immediate consequence.

(2) For the complexes $D^{\cdot}(X,q) = D^{\cdot}_{\phi}(X,q)$ we can choose:

$$D^{\cdot}(X,q) = \varinjlim s\check{C}(\mathcal{U}, \mathbf{R}(q)_{\mathcal{D}})$$

the limit over all coverings $\mathcal{U}$ of X_{an} of the associated simple complex to the Čech complex with coefficients in $\mathbf{R}(q)$. Moreover

$$D^{\cdot}_Y(X,q) := \mathrm{Cone}(D^{\cdot}(X,q) \xrightarrow{\mathrm{res}} D^{\cdot}(X-Y,q))[-1].$$

(2.8) We now proceed to the construction **[Bl3]** of the regulator map for smooth quasiprojective varieties X over $\mathbf{R}$ or $\mathbf{C}$. Consider the cohomological double complex

$$D^{\cdot}(X^*,q) = D^{\cdot}(X \times \Delta_{-*}, q)$$

non-zero for $\cdot \geq 0$, $* \leq 0$ with $*$-differential:

$$d = \sum_{i=0}^{-a} (-1)^i \partial_i^* : D^b(X^a,q) \to D^b(X^{a+1},q).$$

Similarly another double complex is defined

$$D^{\cdot}_{\mathrm{supp}}(X^*,q) = \varinjlim_{Z \in z^q(X,-*)} D^{\cdot}_{\mathrm{Supp}\, Z}(X_{-*},q).$$

For technical reasons we truncate these complexes (non-trivially) in large negative $*$-degree:

$$\mathbf{D}^{\cdot}_{(\mathrm{supp})}(X^*,q) = \tau_{*\geq -N} D^{\cdot}_{(\mathrm{supp})}(X^*,q)$$

where $N >> 0$ is an even integer.

Consider the spectral sequence

$$E_1^{a,b} = H^b(\mathbf{D}^{\cdot}(X^a,q)) \Rightarrow H^{a+b}(s\mathbf{D}^{\cdot}(X^*,q))$$

where s denotes the associated simple complex of a double complex. Because of the homotopy axiom

$$E_1^{a,b} = H^b_{\mathcal{D}}(X, \mathbf{R}(q)) \quad \text{for} \quad -N \le a \le 0,\ b \ge 0$$

and $E_1^{a,b} = 0$ for all other a, b. Moreover $d_1^{a,b} = 0$ except for a even, $-N \le a < 0$ and $b \ge 0$ in which case $d_1^{a,b} = \mathrm{id}$. Hence we obtain isomorphisms

$$H^p(s\mathbf{D}^{\cdot}(X^*, q)) = H^p_{\mathcal{D}}(X, \mathbf{R}(q)).$$

In the spectral sequence

$$E_1^{a,b} = H^b(\mathbf{D}^{\cdot}_{\mathrm{supp}}(X^a, q)) \Rightarrow E^{a+b} = H^{a+b}(s\mathbf{D}^{\cdot}_{\mathrm{supp}}(X^*, q))$$

we have

$$E_1^{a,b} = \varinjlim_{Z \in z^q(X, -a)} H^b_{\mathcal{D}, \mathrm{Supp}\, Z}(X^a, \mathbf{R}(q))$$

for $-N \le a \le 0$, $b \ge 0$ and $E_1^{a,b} = 0$ otherwise. The cycle map induces a natural map of complexes

$$\Gamma_X(q)^* \to E_1^{*-2q, 2q}$$

and hence for all p a map

$$H^p_{\mathcal{M}}(X, \mathbf{Z}(q)) \to E_2^{p-2q, 2q}.$$

Due to weak purity the groups $E_1^{a,b}$ are zero for $b < 2q$ and all $r \ge 1$. Hence there are natural maps

$$E_2^{a,2q} \twoheadrightarrow E_\infty^{a,2q} \hookrightarrow E^{a+2q}.$$

Choosing $-N < p - 2q$ the regulator map $r_{\mathcal{D}}$ is defined by composition:

$$\begin{tikzcd} H^p_{\mathcal{M}}(X, \mathbf{Z}(q)) \arrow[r] \arrow[d, "r_{\mathcal{D}}"'] & H^p(s\mathbf{D}^{\cdot}_{\mathrm{supp}}(X^*, q)) \arrow[d, "\mathrm{nat.}"] \\ H^p_{\mathcal{D}}(X, \mathbf{R}(q)) \arrow[r, equal] & H^p(s\mathbf{D}^{\cdot}(X^*, q)). \end{tikzcd}$$

It is independent of N. Similarly a regulator (or cycle) map into continuous étale cohomology **[Ja1]** can be constructed.

(2.9) We now sketch how the notions introduced fit into the philosophy of motives. More details will be given in the appendix.

Assume X is smooth, projective over $\mathbf{R}$. Let $MH_{\mathbf{R}}$ be the abelian category of $\mathbf{R}$-mixed Hodge structures with the action of a real Frobenius. According to Beilinson (**[Be3]**, see also **[Ca]**) there is a natural isomorphism for $p+1 < 2q$

$$H^{p+1}_{\mathcal{D}}(X, \mathbf{R}(q)) \xrightarrow{\sim} \mathrm{Ext}^1_{MH_{\mathbf{R}}}(\mathbf{R}(0), H^p_B(X)(q)).$$

One would like to give a similar interpretation to the motivic cohomology groups as Ext-groups in a suitable abelian category of "mixed motives". The ultimate definition of such a category remains to be found. However via realizations (ℓ-adic, Betti, ...) working definitions have been found for $MM_{\mathbf{Q}}$ and $MM_{\mathbf{Z}}$ the categories of mixed motives over $\mathbf{Q}$ resp. $\mathbf{Z}$, see **[De3,Ja2,Sc2]**. It is shown in the appendix that for smooth, projective varieties X over $\mathbf{Q}$ there are natural maps (conjecturally isomorphisms) for $p+1 < 2q$

$$H^{p+1}_{\mathcal{M}}(X, \mathbf{Q}(q)) \to \mathrm{Ext}^1_{MM_{\mathbf{Q}}}(\mathbf{Q}(0), H^p(X)(q))$$

and one hopes that the image of $H^{p+1}_{\mathcal{M}}(X, \mathbf{Q}(q))_{\mathbf{Z}}$ is precisely $\mathrm{Ext}^1_{MM_{\mathbf{Z}}}(\mathbf{Q}(0), H^p(X)(q))$. Moreover there is a commutative diagram

$$\begin{tikzcd} H^{p+1}_{\mathcal{M}}(X, \mathbf{Q}(q)) \arrow[r, "r_{\mathcal{D}}"] \arrow[d] & H^{p+1}_{\mathcal{D}}(X_{\mathbf{R}}, \mathbf{R}(q)) \arrow[d, "\wr"] \\ \mathrm{Ext}^1_{MM_{\mathbf{Q}}}(\mathbf{Q}(0), H^p(X)(q)) \arrow[r, "H_B"] & \mathrm{Ext}^1_{MH_{\mathbf{R}}}(\mathbf{R}(0), H^p_B(X_{\mathbf{R}})(q)) \end{tikzcd}$$

where H_B maps a mixed motive to the Betti realization over $\mathbf{R}$ endowed with its mixed $\mathbf{R}$-Hodge structure.

3. The conjectures

Recall the definition of the i-th L-series of a smooth projective variety X over $\mathbf{Q}$ by the following Euler product:

$$L(H^i(X),s)=\prod_p P_p(H^i(X),p^{-s}).$$

Here we have set

$$P_p(H^i(X),t)=\det\bigl(1-Fr_p t\bigm| H^i_{\acute{e}t}(X_{\overline{\mathbf{Q}}_p},\mathbf{Q}_\ell)^{I_p}\bigr)$$

where ℓ is a prime different from p, I_p is the inertia group in $G_{\mathbf{Q}_p}$ and Fr_p is the inverse of a Frobenius element in $G_{\mathbf{Q}_p}$. For primes p where X has good reduction the polynomial $P_p(H^i(X),t)$ has coefficients in $\mathbf{Q}$ independent of ℓ. The product of the $P_p(H^i(X),p^{-s})$ extended over the good primes converges absolutely in the usual topology for $\operatorname{Re} s>\frac{i}{2}+1$. Conjectures [**Se**]:

- The polynomials $P_p(H^i(X),t)$ lie in $\mathbf{Q}[t]$ for *all* p, and are independent of ℓ, and nonvanishing for $|t|<p^{-1-i/2}$.
- The Euler product has a meromorphic continuation to the whole plane.
- There is a functional equation relating $L(H^i(X),s)$ and $L(H^i(X),i+1-s)$ as in [**Se**].

Concerning the special values of these L-functions there is the following conjecture.

(3.1) Conjecture. *Assume $n>\frac{i}{2}+1$. Then:*
(3.1.1) $r_{\mathcal{D}}\otimes\mathbf{R}:H^{i+1}_{\mathcal{M}}(X,\mathbf{Q}(n))_{\mathbf{Z}}\otimes\mathbf{R}\to H^{i+1}_{\mathcal{D}}(X_{\mathbf{R}},\mathbf{R}(n))$ is an isomorphism.
(3.1.2) $r_{\mathcal{D}}(\det H^{i+1}_{\mathcal{M}}(X,\mathbf{Q}(n))_{\mathbf{Z}})=L(H^i(X),n)\mathcal{D}_{i,n}$ in $\det H^{i+1}_{\mathcal{D}}(X_{\mathbf{R}},\mathbf{R}(n))$ with $\mathcal{D}_{i,n}$ as defined in (2.3.2).

If the above hypothesis on the L-function of $H^i(X)$ are satisfied assertion (3.1.2) is equivalent to:

$$r_{\mathcal{D}}(\det H^{i+1}_{\mathcal{M}}(X,\mathbf{Q}(n))_{\mathbf{Z}})=L(H^i(X),i+1-n)^*\mathcal{B}_{i,n} \tag{3.1.3}$$

in $\det H^{i+1}_{\mathcal{D}}(X_{\mathbf{R}},\mathbf{R}(n))$ where $L(H^i(X),k)^*$ denotes the leading coefficient at $s=k$ in the Taylor development of the L-series [**Ja3**].

The following result on the order of vanishing follows from a straightforward calculation and the expected functional equation[**Sch**]:

$$\begin{aligned}\operatorname{ord}_{s=i+1-n}L(H^i(X),s)&=\dim H^{i+1}_{\mathcal{D}}(X_{\mathbf{R}},\mathbf{R}(n))\\&=\dim H^{i+1}_{\mathcal{M}}(X,\mathbf{Q}(n))_{\mathbf{Z}}\quad\text{assuming (3.1.1).}\end{aligned}\tag{3.1.4}$$

Observe that the conjectures determine the special values of the L-series up to a non-vanishing rational number. Equation (3.1.3) is the original proposal by Beilinson. The version (3.1.2) is a reformulation due to Deligne. It requires less information about the L-series to make sense.

For the remaining values of n: the right central point $n=\frac{i}{2}+1$ and the central point $n=\frac{i+1}{2}$ the conjectures have to be modified ([**Be1**], Conjecture 3.7 *et seq.*). Since we don't deal with examples for these cases we skip the formulation. A uniform approach is possible in the framework of mixed motives: The Beilinson conjectures are seen to be equivalent to a Deligne conjecture for critical mixed motives [**Sc2**]. An integral refinement of the conjectures has been proposed by Bloch and Kato [**BlK**] using their philosophy of Tamagawa measures for motives. Essentially the only case where (3.1.1) is known is for X the spectrum of a number field. In this case the result is due to Borel with a different definition of the regulator map. For a comparison of the regulator maps see [**Be1,Rap**]. For a proof of Borel's result the K-theoretical approach to motivic cohomology is essential.

In a number of cases to be treated in section 5 and 6 the following weakened version of the conjectures can be proved.

(3.2) Conjecture. *Assume $n > \frac{i}{2}+1$. Then (3.1.1) and (3.1.2) (or (3.1.3)) hold with $H^{i+1}_{\mathcal{M}}(X,\mathbf{Q}(n))_{\mathbf{Z}}$ replaced by a suitable $\mathbf{Q}$-subspace.*

Thus motivic cohomology as we have defined it should at least be large enough so that a sensible regulator can be formed having the expected relation to the L-values.

(3.3) Generalization to Chow motives. For some well known L-series the above framework is too restrictive. For example the Dirichlet L-functions of algebraic number theory are not covered. This is remedied by extending the above notions and conjectures to the category of Chow motives, which should be thought of as generalised varieties. Fix a number field $T/\mathbf{Q}$ – the field of coefficients. Let $\mathcal{V}_k$ be the category of smooth projective varieties over a field k. Consider the category $\mathcal{C}_k(T)$ with objects TX for each object X in $\mathcal{V}_k$ and morphisms

$$\mathrm{Hom}(TX,TY) = CH^{\dim Y}(X\times_k Y)\otimes T.$$

For $a: TX_1\to TX_2$ and $b: TX_2\to TX_3$ composition is defined by intersecting cycles:

$$b\circ a = p_{13*}(p_{12}^*a\cdot p_{23}^*b)$$

where $p_{ij}: X_1\times X_2\times X_3\to X_i\times X_j$ are the projections. Sending X to TX and a morphism f to its graph $\tilde{f}$ defines a covariant functor from $\mathcal{V}_k\to\mathcal{C}_k(T)$. The category of effective Chow motives $\mathcal{M}_k^+(T)$ is obtained by adding images of projectors to $\mathcal{C}_k(T)$. Objects are pairs $M=(TX,p)$ where $p\in\mathrm{End}(TX)$, $p^2=p$ and morphisms are the obvious ones. Setting

$$H_?(M) = p^*(H_?(X)\otimes T)$$

the cohomologies and conjecture (3.1) factorize over $\mathcal{M}_{\mathbf{Q}}^+(T)$. They determine special values of $T\otimes\mathbf{C}$-valued L-series $L(H^i(M),s)$ up to numbers in T^*. See **[Be1,Ja3,Kl,Ma]** for more details.

Remarks. The category $\mathcal{M}_{\mathbf{Q}}^+(T)$ is not abelian. If instead of Chow theory one considers cycles modulo homological equivalence one obtains what is essentially Grothendieck's category of (effective) motives. Standard conjectures on algebraic cycles would imply that it is an abelian semisimple category. Nowadays these motives are called pure in contrast to more general "mixed" motives which should come e.g. from the H^i of singular varieties. The category of these mixed motives $MM_{\mathbf{Q}}$ was already alluded to in section (2.9). As yet there is no Grothendieck style definition for $MM_{\mathbf{Q}}$ using cycles but only a definition via realizations.

As an example of a Chow motive let us construct the motive M_χ of a Dirichlet character χ of a number field k: Via class field theory we may view χ as a one-dimensional representation of the absolute galois group G_k of k with values in a number field T:

$$\chi: G_k\to T^*.$$

We may assume that T is generated over $\mathbf{Q}$ by the values of χ. Choose a finite abelian extension F/k such that χ factorizes over $G=\mathrm{Gal}(F/k)$ and set

$$M_\chi = e_\chi(T\mathrm{Spec}(F))$$

in $\mathcal{M}_k^+(T)$ where e_χ is the idempotent:

$$e_\chi = \frac{1}{|G|}\sum_{\sigma\in G}\chi(\sigma)\tilde{\sigma}^{-1} \quad \text{in} \quad T[G].$$

Observe that M_χ is independent of the choice of F.

We end this section with a short discussion of known cases for the conjectures. (3.1) is known for $X=\mathrm{Spec}\,F$, $F/\mathbf{Q}$ a number field **[Bo2]** and for the motives M_χ attached to Dirichlet characters of $k=\mathbf{Q}$ or $k=K$ an imaginary quadratic field **[Be1,Den2]**. In section 5 and 6 we will deduce the evidence for the weak conjecture (3.2) from the theory of Beilinson's Eisenstein symbol map (section 4). The logical dependences in our approach are depicted in a diagram:

$$\begin{array}{ccc}
 & & \begin{array}{c}\text{modular curves}/\mathbf{Q};\\ \text{Shimura curves}/\mathbf{Q}\ [\mathbf{Ra1}]\end{array} \\
 & & \uparrow \\
 & & \text{modular curves } [\mathbf{Be1,Be2,SS1}] \\
 & & \uparrow \\
\text{Eisenstein symbol} & \rightarrow & \begin{array}{c}\text{modular forms of}\\ \text{weight} \geq 2\ [\mathbf{Sc2}]\end{array} \\
\downarrow & \searrow & \\
\begin{array}{c}\text{Dirichlet characters of imaginary}\\ \text{quadratic fields } [\mathbf{Den2}]\end{array} & & \begin{array}{c}\text{algebraic Hecke characters}\\ \text{of imaginary quadratic}\\ \text{fields } [\mathbf{Den1,Den2}]\end{array} \\
\downarrow & & \downarrow \\
\begin{array}{c}\text{Dirichlet characters}\\ \text{of } \mathbf{Q}\ [\mathbf{Be1,Den2,N}]\end{array} & & \begin{array}{c}\text{CM elliptic curves over}\\ \text{number fields of Shimura}\\ \text{type } [\mathbf{Bl1,Den1}]\end{array}
\end{array}$$

4. Kuga-Sato varieties and the Eisenstein symbol

The Eisenstein symbol is a certain "universal" construction of elements of motivic cohomology of an elliptic curve, or more generally self-products of an elliptic curve. It has its origins in the work of Bloch [**Bl1**] on K_2 of elliptic curves but was constructed in generality by Beilinson [**Be2**]. For a constant elliptic curve a slightly refined construction is made in [**Den1**].

4.1 We first introduce the modular and Kuga-Sato varieties. In what follows n will be an integer ≥ 3. Let M_n be the modular curve of level n, parameterising elliptic curves E together with level n structure $(\mathbf{Z}/n)^2 \xrightarrow{\sim} E[n]$. Thus the set of complex points $M_n(\mathbf{C})$ is the disjoint union of $\phi(n)$ copies of $\Gamma(n)\backslash\mathcal{H}$, the quotient of the upper half-plane by the principal congruence subgroup $\Gamma(n) \subset SL_2(\mathbf{Z})$.

The assumption $n \geq 3$ assures that there is a universal family of elliptic curves:

$$\pi : X_n \to M_n.$$

Write $\overline{M}_n$ for the usual compactification of M_n, and $M_n^\infty = \overline{M}_n - M_n$ for the cusps of $\overline{M}_n$ (a sum of copies of $\mathrm{Spec}\mathbf{Q}(\zeta_n)$). Then we can consider the minimal (regular) model of X_n over $\overline{M}_n$:

$$\bar{\pi} : \overline{X}_n \to \overline{M}_n$$

whose restriction to M_n is just π. For each cusp $s \in M_n^\infty$, the fibre $\bar{\pi}^{-1}(s)$ is a Néron polygon with n sides. Write $\hat{X}_n \subset \overline{X}_n$ for the connected component of the smooth part (Néron model) of $\overline{X}_n$. Then $\bar{\pi}^{-1}(s) \cap \hat{X}_n$ is (non-canonically) isomorphic to the multiplicative group $\mathbf{G}_m$.

(4.2) For $l \geq 0$ write X_n^l, $\overline{X}_n^l$, $\hat{X}_n^l$ for the l-fold fibre product of X_n (resp. $\overline{X}_n$, $\hat{X}_n$) over $\overline{M}_n$. The variety $\overline{X}_n^l$ has singularities for $l \geq 2$; we shall consider these in 5.2 below. Since X_n^l is a group scheme over M_n, in addition to the obvious projections

$$p_i : X_n^l \to X_n \quad \text{for } 1 \leq i \leq l$$

onto the factors, there is a further projection

$$p_0 = -p_1 - \cdots - p_l : X_n^l \to X_n.$$

These $(l+1)$ projections p_i allow us to regard X_n^l as a closed subscheme of X_n^{l+1}. This gives an action of the symmetric group $\mathbf{S}_{l+1}$ on X_n^l, permuting the projections $p_0, \ldots\ p_l$. The same construction works also for $\hat{X}_n^l$.

(4.3) From the localisation sequence (1.3.5) for the pair $(\hat{X}^l_n, X^l_n)$ we have:

$$H^{l+1}_{\mathcal{M}}(\hat{X}^l_n, \mathbf{Q}(l+1)) \to H^{l+1}_{\mathcal{M}}(X^l_n, \mathbf{Q}(l+1)) \to H^l_{\mathcal{M}}(M^\infty_n \times \mathbf{G}^l_m, \mathbf{Q}(l)) \to H^{l+2}_{\mathcal{M}}(\hat{X}^l_n, \mathbf{Q}(l+1)). \tag{4.3.1}$$

Consider the eigenspaces for the sign character sgn_{l+1} of $\mathbf{S}_{l+1}$. Under the involution $\sigma : x \mapsto x^{-1}$ of $\mathbf{G}_m$, the motivic cohomology (1.3.4) of $\mathbf{G}_m$ decomposes:—

$$H^1_{\mathcal{M}}(\mathbf{G}_m/k, \mathbf{Q}(1)) = k[x, x^{-1}]^* \otimes \mathbf{Q} = \underbrace{k^* \otimes \mathbf{Q}}_{\sigma=+1} \oplus \underbrace{\mathbf{Q}.x}_{\sigma=-1}.$$

Using this it is not hard to see that

$$H^l_{\mathcal{M}}(M^\infty_n \times \mathbf{G}^l_m, \mathbf{Q}(l))_{\mathrm{sgn}_{l+1}} = H^0_{\mathcal{M}}(M^\infty_n, \mathbf{Q}(0)) = \mathbf{Q}[M^\infty_n]. \tag{4.3.2}$$

Here $\mathbf{Q}[M^\infty_n]$ denotes the set of $\mathbf{Q}$-valued functions on the closed points of M^∞_n. By composing (4.3.1) and (4.3.2) we therefore obtain a "residue map" in motivic cohomology:

$$H^{l+1}_{\mathcal{M}}(X^l_n, \mathbf{Q}(l+1))_{\mathrm{sgn}_{l+1}} \xrightarrow{\mathrm{Res}^l_{\mathcal{M}}} \mathbf{Q}[M^\infty_n]. \tag{4.3.3}$$

Beilinson's key result is then:—

(4.4) Theorem (**[Be2]**, Theorem 3.1.7). $\mathrm{Res}^l_{\mathcal{M}}$ *is surjective for* $l \geq 1$.

(4.5) Remarks. This theorem can be viewed as a generalisation of the theorem of Manin and Drinfeld, which is the case $l = 0$. For then $X^0_n = M_n$ and (4.3.1) comes from the exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}^*(\overline{M}_n) & \longrightarrow & \mathcal{O}^*(M_n) & \xrightarrow{\mathrm{Div}} & \mathbf{Z}[M^\infty_n] & \xrightarrow{c} & \mathrm{Pic}\overline{M}_n \\ & & \| & & & & & & \\ & & \mathbf{Q}(\zeta_n)^* & & & & & & \end{array}$$

by tensorising with $\mathbf{Q}$. Here $\mathrm{Res}^0_{\mathcal{M}} = \mathrm{Div}$ is the divisor map, and c maps a divisor supported on the cusps to its class in $\mathrm{Pic}\overline{M}_n$. According to the Manin-Drinfeld theorem, the divisors of degree zero

$$\mathbf{Z}[M^\infty_n]^0 \overset{\mathrm{def}}{=} \ker\left\{\mathbf{Z}[M^\infty_n] \xrightarrow{\deg} \mathbf{Z}\right\}$$

are torsion in $\mathrm{Pic}\,\overline{M}_n$, or equivalently

$$\mathrm{Res}^0_{\mathcal{M}} : H^1_{\mathcal{M}}(M_n, \mathbf{Q}(1)) \twoheadrightarrow \mathbf{Q}[M^\infty_n]^0.$$

For $l > 0$ the picture should be even better. Firstly, there is no restriction to divisors of degree zero. Secondly, the general Beilinson conjectures would imply that $\mathrm{Res}^l_{\mathcal{M}}$ is actually an isomorphism. (To see this one examines carefully the exact sequence (4.3.1).) This makes Beilinson's proof of (4.4) philosophically reasonable—he constructs a totally explicit left inverse to $\mathrm{Res}^l_{\mathcal{M}}$, the *Eisenstein symbol* map

$$\mathcal{E}^l_{\mathcal{M}} : \mathbf{Q}[M^\infty_n] \to H^{l+1}_{\mathcal{M}}(X^l_n, \mathbf{Q}(l+1))_{\mathrm{sgn}_{l+1}}$$

whose construction we now describe.

(4.6) Let $U_n \subset X_n$ be the complement of the n^2 sections of order dividing n, and write

$$U^{l\prime}_n = \bigcap_{0 \leq i \leq l} p_i^{-1}(U_n) \subset X^l_n.$$

We first construct symbols on $U_n^{l\prime}$ as follows. Start with any invertible functions $g_0, \ldots\, g_l \in \mathcal{O}^*(U_n)$. Then

$$p_0^*(g_0) \cup \cdots \cup p_l^*(g_l) \in H^{l+1}_{\mathcal{M}}(U_n^{l\prime}, \mathbf{Q}(l+1)). \tag{4.6.1}$$

To get an element of $H^{l+1}_{\mathcal{M}}(X_n^l, \mathbf{Q}(l+1))$ we apply three projectors:

—$U_n^{l\prime}$ is stable under the symmetric group $\mathbf{S}_{l+1}$, and we take the sgn-eigenspace;

—The group of sections of finite order $(\mathbf{Z}/n)^{2l}$ acts on $U_n^{l\prime}$ by translations, and we project onto the subspace of invariants;

—For an integer $m \geq 1$, there is a multiplication map

$$[m^{-1}] : H^{\cdot}_{\mathcal{M}}(U_n^{l\prime}, \mathbf{Q}(*)) \to H^{\cdot}_{\mathcal{M}}(U_n^{l\prime}, \mathbf{Q}(*))$$

defined as follows: consider the diagram

$$\begin{tikzcd}
U_n^{l\prime} & U_{mn}^{l\prime} \arrow[l, "j"'] \arrow[d, "{[\times m]}"] \\
& U_n^{l\prime}
\end{tikzcd}$$

Here j denotes the inclusion map, and the multiplication $[\times m]$ is a Galois étale covering with group $(\mathbf{Z}/m)^{2l}$. By (1.3.6) we have

$$\begin{tikzcd}
H^{\cdot}_{\mathcal{M}}(U_n^{l\prime}, \mathbf{Q}(*)) \arrow[r, "j^*"] \arrow[dr, dashed, "{[m^{-1}]}"'] & H^{\cdot}_{\mathcal{M}}(U_{mn}^{l\prime}, \mathbf{Q}(*))_{(\mathbf{Z}/m)^{2l}} \\
& H^{\cdot}_{\mathcal{M}}(U_{mn}^{l\prime}, \mathbf{Q}(*)) \arrow[u, "\wr"', "{[\times m]^*}"]
\end{tikzcd}$$

whence there is a map $[m^{-1}]$ as indicated. Denote by a subscript l the maximal quotient of $H^{\cdot}_{\mathcal{M}}(U_n^{l\prime}, \mathbf{Q}(*))$ on which $[m^{-1}]$ is multiplication by m^{-l}, for every $m \geq 1$. (In fact it suffices to consider only one $m > 1$.)

(4.7) Theorem. *The restriction from X_n^l to $U_n^{l\prime}$ induces an isomorphism*

$$H^{\cdot}_{\mathcal{M}}(X_n^l, \mathbf{Q}(*))_{\mathrm{sgn}_{l+1}} \xrightarrow{\sim} H^{\cdot}_{\mathcal{M}}(U_n^{l\prime}, \mathbf{Q}(*))_{\mathrm{sgn}_{l+1}, (\mathbf{Z}/n)^{2l}, l}.$$

Applying this to the elements (4.6.1) projected to the right hand group gives a map

$$\bigotimes^{l+1} \mathcal{O}^*(U_n) \otimes \mathbf{Q} \to H^{l+1}_{\mathcal{M}}(X_n^l, \mathbf{Q}(l+1))_{\mathrm{sgn}_{l+1}}. \tag{4.7.1}$$

(4.8) Lemma. *The divisor map $\mathcal{O}^*(U_n) \otimes \mathbf{Q} \to \mathbf{Q}[(\mathbf{Z}/n)^2]^0$ is surjective.*

Proof. Let $s : M_n \to X_n$ be a section of order dividing n, and let $e : M_n \to X_n$ be the unit section. We have to show that $\mathcal{O}(s-e)$ is torsion in $\mathrm{Pic} X_n$. It certainly is torsion in the relative Picard group $\mathrm{Pic}(X_n/M_n)$, so for some $N \geq 1$ and some line bundle $\mathcal{L}$ on M_n we have $\mathcal{O}(s-e)^{\otimes N} \simeq \pi^*\mathcal{L}$. Hence $\mathcal{L} = e^*\pi^*\mathcal{L} \simeq e^*\mathcal{O}(s-e)^{\otimes N} = e^*\mathcal{O}(-e)^{\otimes N} = \mathcal{N}_e^{\otimes N}$ where $\mathcal{N}_e$ is the normal bundle of the unit section. Hence $\mathcal{L} \simeq \underline{\omega}_{X_n/M_n}^{\otimes(-N)}$, and $\underline{\omega}^{\otimes 12}$ is trivial (a nowhere-vanishing section being the discriminant Δ).

We now have a diagram:

$$\begin{tikzcd}
\bigotimes^{l+1} \mathbf{Q}[(\mathbf{Z}/n)^2]^0_{\mathrm{sgn}_{l+1}, (\mathbf{Z}/n)^{2l}} \arrow[dr, dashed] & \bigotimes^{l+1} \mathcal{O}^*(U_n) \arrow[l, "\mathrm{Div}"'] \arrow[d] \\
\mathbf{Q}[(\mathbf{Z}/n)^2]^0 \arrow[u, "\wr", "\vartheta"'] \arrow[r, "E^l_{\mathcal{M}}"'] & H^{l+1}_{\mathcal{M}}(X_n^l, \mathbf{Q}(l+1))_{\mathrm{sgn}_{l+1}}
\end{tikzcd}$$

It is not hard to show that the map (4.7.1) factors through the dotted arrow as shown. The isomorphism ϑ is given by

$$\beta \mapsto \beta \otimes \alpha \otimes \cdots \otimes \alpha, \qquad \alpha = n^2(0) - \sum_{x \in (\mathbf{Z}/n)^2} (x).$$

This defines a composite map $E^l_{\mathcal{M}}$ as indicated, which is "almost" the Eisenstein symbol.

(4.9) At this point we want to describe the composite of the map $E^l_{\mathcal{M}}$ just constructed with the regulator map. Let us restrict attention to the component of $M_n(\mathbf{C})$ containing the cusp at infinity, and write τ for the variable on the complex upper half-plane. The corresponding component of $X^l_n(\mathbf{C})$ then can be described as the quotient

$$\Gamma(n)\backslash \mathcal{H} \times \mathbf{C}^l / \mathbf{Z}^{2l}$$

where the actions of $\Gamma(n)$ and $\mathbf{Z}^{2l}$ are given by:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : (\tau, z_1, \ldots, z_l) \mapsto \left(\frac{a\tau+b}{c\tau+d}, \frac{z_1}{c\tau+d}, \ldots, \frac{z_l}{c\tau+d}\right)$$
$$(u_1, v_1, \ldots, u_l, v_l) : (\tau, z_1, \ldots, z_l) \mapsto (\tau, z_1 + u_1\tau + v_1, \ldots, z_l + u_l\tau + v_l).$$

Let $\beta \in \mathbf{Q}[(\mathbf{Z}/n)^2]^0$. In terms of the description (2.5.1) of Deligne cohomology by differential forms, $r_{\mathcal{D}} E^l_{\mathcal{M}}(\beta) \in H^{l+1}_{\mathcal{D}}(X^l_n/\mathbf{R}, \mathbf{R}(l+1))$ is represented by

$$\phi = \sum_{j=0}^{l} \sum_{c_1, c_2 \in \mathbf{Z}}{}' \frac{\psi_\beta(c_1, c_2)\mathrm{Im}(\tau)}{(\overline{c_1\tau + c_2})^{j+1}(c_1\tau+c_2)^{l-j+1}} (d\bar{z}_1 \wedge \cdots \qquad (4.9.1)$$
$$\cdots \wedge d\bar{z}_j \wedge dz_{j+1} \wedge \cdots \wedge dz_l)_{\mathrm{sgn}_l} + (d\tau,\ d\bar{\tau} \text{ term})$$

where for $c = (c_1, c_2) \in (\mathbf{Z}/n)^2$

$$\psi_\beta(c) = \sum_{d \in (\mathbf{Z}/n)^2} \beta(d) e^{2\pi i (c_1 d_2 - c_2 d_1)/n}.$$

(The omitted terms in (4.9.1) involving $d\tau$, $d\bar{\tau}$ vanish in the applications of §§5, 6.) See 4.12 below for remarks concerning the proof of this formula.

(4.10) To pass from $E^l_{\mathcal{M}}$ to $\mathcal{E}^l_{\mathcal{M}}$ we first recall that the set of closed points of M^∞_n is canonically isomorphic to

$$GL_2(\mathbf{Z}/n) / \begin{pmatrix} * & * \\ 0 & \pm 1 \end{pmatrix}.$$

The definition of the residue map (4.3.2) involves choosing for each $s \in M^\infty_n$ an isomorphism of the fibre of $\hat{X}$ at s with $\mathbf{G}_m$ (see 4.1 above), and the two such isomorphisms are interchanged by $-1 \in GL_2(\mathbf{Z}/n)$. If we replace $H^0_{\mathcal{M}}(M^\infty_n, \mathbf{Q}(0))$ by the (non-canonically) isomorphic space $V^{(-)^l}$ defined as

$$V^\pm = \left\{ f : GL_2(\mathbf{Z}/n) \to \mathbf{Q} \;\middle|\; f\left(g\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}\right) = f(g) = \pm f(-g) \right\}$$

then the map $\mathrm{Res}^l_{\mathcal{M}}$ becomes $GL_2(\mathbf{Z}/n)$-equivariant.

(4.11) Now consider the family of maps

$$\lambda^l_n : \mathbf{Q}[(\mathbf{Z}/n)^2]^0 \to V^{(-)^l}$$
$$(\lambda^l_n \phi)(g) = \sum_{x,y=0}^{n-1} \phi(g.{}^t(x,y)) B_{l+2}\left(\frac{y}{n}\right)$$

(where B_{l+2} are Bernoulli polynomials). It is fairly elementary to prove that λ_n^l is surjective. (These maps are the finite level analogues of the horospherical map τ of **[Be2]**, paragraph following 3.1.6.) One now proves that (up to a non-zero constant factor) the diagram

(4.11.1)
$$\begin{tikzcd} & \mathbf{Q}[(\mathbf{Z}/n)^2]^0 \arrow[dl, "E^l_{\mathcal{M}}"'] \arrow[d, "\lambda^l_n"] \\ H^{l+1}_{\mathcal{M}}(X^l_n, \mathbf{Q}(l+1))_{\mathrm{sgn}_{l+1}} \arrow[r, "\mathrm{Res}^l_{\mathcal{M}}"'] & V^{(-)^l} \end{tikzcd}$$

is commutative, and that $E^l_{\mathcal{M}}$ factors through λ^l_n. Thus there is a map

$$\mathcal{E}^l_{\mathcal{M}}: V^{(-)^l} \to H^{l+1}_{\mathcal{M}}(X^l_n, \mathbf{Q}((l+1))_{\mathrm{sgn}_{l+1}}$$

satisfying

$$\mathcal{E}^l_{\mathcal{M}} \circ \lambda^l_n = E^l_{\mathcal{M}} \quad \text{and} \quad \mathrm{Res}^l_{\mathcal{M}} \circ \mathcal{E}^l_{\mathcal{M}} = \mathrm{id}.$$

This proves Theorem 4.4.

(4.12) We finally say some words about the commutativity of (4.11.1), on which the theorem rests. Beilinson's original proof uses the fact (from Borel's theorem) that the regulator map

$$r_{\mathcal{D}}: H^0_{\mathcal{M}}(M^{\infty}_n, \mathbf{Q}(0)) \to H^0_{\mathcal{D}}(M^{\infty}_n/\mathbf{R}, \mathbf{R}(0))$$

is injective. From this we see that one need only check the commutativity of the analogue of (4.11.1) in Deligne cohomology. To do this Beilinson explicitly calculates $r_{\mathcal{D}} \circ E^l_{\mathcal{M}}$, by integrating along the fibres of the projection $X^l_n(\mathbf{C}) \to M_n(\mathbf{C})$—see **[Be2]** §3.3 for details. (The resulting formula we gave as (4.9.1) above.)

An alternative proof **[SS2]** is by direct computation of $\mathrm{Res}^l_{\mathcal{M}} \circ E^l_{\mathcal{M}}$ using the Néron model of X^l_n. In this approach, the formula (4.9.1) is obtained as a *consequence* of the commutativity of (4.11.1). In fact, the analogue of $\mathrm{Res}^l_{\mathcal{M}}$ in Deligne cohomology is an *isomorphism*

$$H^{l+1}_{\mathcal{D}}(X^l_n/\mathbf{R}, \mathbf{R}(l+1))_{\mathrm{sgn}_{l+1}} \xrightarrow{\sim} H^0_{\mathcal{D}}(M^{\infty}_n/\mathbf{R}, \mathbf{R}(0))$$

(by consideration of the Hodge numbers) whose inverse is given by real analytic Eisenstein series.

5. L-functions of modular forms.

In this section we sketch how, mildly generalising the results of Beilinson, the Eisenstein symbol can be used to exhibit a relation between special values of L-functions of cusp forms of weight ≥ 2 and higher regulators.

5.1 Let $k \geq 0$ be an integer, and f a classical cusp form of weight $k+2$, which we assume to be a newform on some $\Gamma_0(N)$ with character χ_f. For simplicity we shall assume that the field generated by the Fourier coefficients of f is $\mathbf{Q}$.

As is well known **[De1]**, attached to f is a strictly compatible system of ℓ-adic representations $\{V_\ell(f)\}$, whose associated L-function is the Hecke L-series $L(f,s)$. Moreover $V_\ell(f)$ is a subspace of the parabolic cohomology

$$H^1_{\text{ét}}(\overline{M}_n \otimes \overline{\mathbf{Q}}, \phi_* \mathrm{Sym}^k R^1\pi_*\mathbf{Q}_\ell) \tag{5.1.1}$$

for suitable n. (Recall that ϕ denotes the inclusion $M_n \hookrightarrow \overline{M}_n$.) In Lemma 7 of **[De1]** a canonical resolution of singularities of $\overline{X}^k_n$ is constructed, which we denote by $\overline{\overline{X}}{}^k_n$, and it is shown that $V_\ell(f)$ is a constituent of $H^{k+1}_{\text{ét}}(\overline{\overline{X}}{}^k_n \otimes \overline{\mathbf{Q}}, \mathbf{Q}_\ell)$.

5.2 Theorem [Sc1]. *There exists a projector Π_f in the ring of algebraic correspondences on $\overline{\overline{X}}{}^k_n$ modulo homological equivalence such that for every prime ℓ*

$$V_\ell(f) = \Pi_f\left[H^{k+1}_{\text{ét}}(\overline{\overline{X}}{}^k_n \otimes \overline{\mathbf{Q}}, \mathbf{Q}_\ell)\right].$$

Remarks. (1) In fact Π_f annihilates H^i for $i \neq k+1$.

(2) The pair $V(f) = \left[\overline{\overline{X}}{}^k_n, \Pi_f\right]$ is a *motive* in the sense of Grothendieck (cf. 3.3 above); by the above remark and the theorem, the ℓ-adic representations of $V(f)$ are $\{V_\ell(f)\}$. The Betti realisation of $V(f)$ is given by the singular parabolic cohomology groups (Eichler-Shimura). It has Hodge type $(k+1,0)+(0,k+1)$ and the $(k+1,0)$ part is spanned by the differential form on $\overline{\overline{X}}{}^k_n$

$$\omega_f = 2\pi i f(\tau) d\tau \wedge dz_1 \wedge \cdots \wedge dz_k.$$

(3) A construction of $V(f)$ as a motive defined by absolute Hodge cycles was given by Jannsen (**[Ja2]**, §1; see also **[Scha]**, V.1.1).

(4) For the purposes of testing Beilinson's conjectures, one would like $V(f)$ to be a Chow motive (3.3). In general this seems rather difficult to establish. However, one can consider in place of $V_\ell(f)$ the whole parabolic cohomology group (5.1.1) of level n. There is then a Chow motive with this group for its ℓ-adic realisation. (See step (i) below.)

(5) One may also consider, for p prime to the level of f, the p-adic realisation $V_p(f)$, which is a crystalline representation of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ **[Fa,FM]**. A consequence of 5.2 is that the characteristic polynomial of Frobenius on the associated filtered module is the Hecke polynomial $t^2 - a_p t + \chi_f(p)p^{k+1}$.

(5.3) Sketch of the construction. For $k=0$ the theorem amounts to the decomposition of the Jacobian of $\overline{M}_n$ under the action of the Hecke algebra, and is classical. In this case the problem (4) does not arise. In the case $k>0$ there are two steps:

(i) The use of automorphisms: acting on X^k_n one has the following groups of automorphisms and characters:

—$(\mathbf{Z}/n)^{2k}$, the translations by sections of finite order;

—$\boldsymbol{\mu}_2^k$, inversions in the components of the fibres;

—$\mathbf{S}_k$, the symmetric group.

These generate a group Γ of automorphisms of X_n^k, and this extends to a group of automorphisms of $\overline{\overline{X}}{}_n^k$. There is a unique character of Γ which restricts to the trivial character on $(\mathbf{Z}/n)^{2k}$, the product character on $\boldsymbol{\mu}_2^k$, and the sign character of $\mathbf{S}_k$. This defines a projector Π in the group algebra $\mathbf{Q}[\mathrm{Aut}\,\overline{\overline{X}}{}_n^k]$. By explicit calculation of the cohomology of the boundary of $\overline{\overline{X}}{}_n^k$ one shows that Π cuts out the parabolic cohomology (5.1.1).

(ii) To pass to the individual $V(f)$'s one projects using an idempotent in the Hecke algebra (which is semisimple as an algebra of correspondences modulo homological equivalence).

(5.4) The integers $s=1,\dots,k+1$ are critical for $L(f,s)$. At these points the Beilinson conjectures reduce to the conjunction of Deligne's conjecture (already proved in **[De2]**) and the vanishing of $\Pi\big[H_{\mathcal{M}}^{k+2}(\overline{\overline{X}}{}_n^k,\mathbf{Q}(r))_{\mathbf{Z}}\big]$ for $1\le r\le k+1$, $r\ne k/2$ (for which there is at present no evidence). At $s=-l\le 0$ the L-function has a simple zero, and the conjectures predict a relation between $L'(f,-l)$ and a regulator coming from $H_{\mathcal{M}}^{k+2}(\overline{\overline{X}}{}_n^k,\mathbf{Q}(k+l+2))$.

The target for this regulator is the Deligne cohomology group

$$H_{\mathcal{D}}^{k+2}(\overline{\overline{X}}{}_n^k/\mathbf{R},\mathbf{R}(k+l+2))=H_B^{k+1}(\overline{\overline{X}}{}_n^k,\mathbf{R}(k+l+1))^+$$

and its Π_f-component is the space $(H_B(V(f))\otimes_{\mathbf{Q}}\mathbf{R}(k+l+1))^+$, which is one-dimensional.

(5.5) Theorem. *There is a subspace $\mathcal{P}_n\subset H_{\mathcal{M}}^{k+2}(\overline{\overline{X}}{}_n^k,\mathbf{Q}(k+l+2))$ such that*

$$\Pi_f[r_{\mathcal{D}}(\mathcal{P}_n)]=L'(f,-l).(H_B(V(f))\otimes\mathbf{Q}(k+l+1))^+\,.$$

(5.6) Remarks. (1) For $k=0$ (the case of modular curves) this was proved by Beilinson **[Be1,Be2,SS1]**. The main ideas for the general case can already be found there. The case $k=1$, $l=0$ was also considered by Ramakrishnan (unpublished). Full details for the general case will appear in **[Sc3]**.

(2) Recall that for the correct formalism of Beilinson's conjecture it is necessary to consider "motivic cohomology over $\mathbf{Z}$" (cf. 1.7 above). Although in general we cannot prove that $\mathcal{P}_n\subset H_{\mathcal{M}}^{k+2}(\overline{\overline{X}}{}_n^k,\mathbf{Q}(k+l+2))_{\mathbf{Z}}$, we have the following:

(i) Standard conjectures on the K-theory of varieties over finite fields would imply that $H_{\mathcal{M}}^{k+2}(\overline{\overline{X}}{}_n^k,\mathbf{Q}(k+l+2))_{\mathbf{Z}}=H_{\mathcal{M}}^{k+2}(\overline{\overline{X}}{}_n^k,\mathbf{Q}(k+l+2))$ except in the case $k=l=0$.

(ii) For curves these conjectures are known **[Ha1]**. Thus for $k=0$ the only obstruction to integrality occurs when $l=0$; in this case it is known (see **[SS1]**, §7) that $\mathcal{P}_n\subset H_{\mathcal{M}}^2(\overline{M}_n,\mathbf{Q}(2))_{\mathbf{Z}}$.

(iii) For $k>0$ one can at least show that $\mathcal{P}_n$ contains enough elements which are integral away from primes dividing n, using a modification of a trick of Soulé **[So1]**.

(5.7) Construction of $\mathcal{P}_n$. Consider the diagram

$$\begin{tikzcd} X_n^k \arrow[d, "r"] & X_n^{k+l} \arrow[l, "p"'] \arrow[r, "q"] & X_n^l \\ S=\mathrm{Spec}\,\mathbf{Q}(\zeta_n) & & \end{tikzcd}$$

where p, q are the projections onto the first k and last l factors of the fibre product, respectively. We define two subspaces

$$\mathcal{U}_n,\,\mathcal{V}_n\subset H_{\mathcal{M}}^{k+2}(X_n^k,\mathbf{Q}(k+l+2))(\Pi)$$

(where the projector Π is as in (5.3.1) above) as follows:

$$\mathcal{U}_n=p_*\left(q^*H_{\mathcal{M}}^{l+1}(X_n^l,\mathbf{Q}(l+1))\cup H_{\mathcal{M}}^{k+l+1}(X_n^{k+l},\mathbf{Q}(k+l+1))\right)$$
$$\mathcal{V}_n=r^*H_{\mathcal{M}}^1(S,\mathbf{Q}(l+1))\cup H_{\mathcal{M}}^{k+1}(X_n^k,\mathbf{Q}(k+1)).$$

(Note that the Eisenstein symbol and Borel's theorem give a plentiful supply of elements of $\mathcal{U}_n$ and $\mathcal{V}_n$.) Let σ be the restriction

$$H^{k+2}_{\mathcal{M}}(\overline{\overline{X}}{}^k_n, \mathbf{Q}(k+l+2))(\Pi) \xrightarrow{\sigma} H^{k+2}_{\mathcal{M}}(X^k_n, \mathbf{Q}(k+l+2))(\Pi)$$

(which is in fact an inclusion) and write

$$\mathcal{Q}_n = \sigma^{-1}(\mathcal{U}_n + \mathcal{V}_n).$$

We then define (cf. 4.10 above)

$$\mathcal{P}_n = \bigcup_{n|n'} \rho^l_{n,n'*}(\mathcal{Q}_{n'}).$$

(5.8) Calculation of the regulator. At this point we should observe that the assumption that the Fourier coefficients of f are rational simplifies the calculation somewhat; in particular, we need not distinguish between f and its complex conjugate. There is a nondegenerate pairing (Poincaré duality)

$$<,>: H_B(V(f)) \times H_B(V(f)) \to \mathbf{Q}(-k-1)$$

and one has to prove that

(5.8.1) $$< r_{\mathcal{D}}(\mathcal{P}_n), \omega_f > = L'(f,-l).c^+(V(f)(k+l+1)) \cdot \mathbf{Q}.$$

Here c^+ is Deligne's period **[De2]**. To calculate the left hand side we pull back to $X^{k+l}_{n'}$ for suitable n', and use the description (2.5) of the cup-product in Deligne cohomology. One obtains an integral of the form

(5.8.2) $$\frac{1}{(2\pi i)^{k+l}} \int_{X^{k+l}_{n'}} \mathcal{E}^{k+l}_{\mathcal{D}} \wedge \overline{q^* E_{l+2}} \wedge p^* \omega_f.$$

In this expression $\mathcal{E}^{k+l}_{\mathcal{D}}$ is the image of an Eisenstein symbol in Deligne cohomology, and E_{l+2} is a (variable) weight $k+2$ holomorphic Eisenstein series. This is a standard Rankin-Selberg integral and can be calculated explicitly. The Eisenstein series E_{l+2} is a linear combination of Eisenstein series E_χ, for various Dirichlet characters χ with $\chi(-1) = (-1)^l$, and the integral becomes a linear combination of terms which, up to a finite number of Euler factors, are of the form

$$L(f,k+l+2).L(f\otimes\chi,k+1).L(\chi.\chi_f,k+l+2)^{-1}.$$

At this stage one applies Shimura's algebraicity results on the twisted L-functions $L(f \otimes \chi, k+1)$ (which are critical values) and the functional equation for $L(f,s)$. In this way it can be shown that the left hand side of (5.8.1) is contained in the right hand side. The final step is to prove the equality—that is, to find suitable Eisenstein symbols for which the integral (5.8.2) is non-zero. For this one has to analyse the bad Euler factors carefully, and it is essential to work adelically. See **[Be2]**, §4 or **[SS1]**, §§2,4,5,6 for further details.

6. L-functions of algebraic Hecke characters

In this section we describe a construction involving the Eisenstein symbol which will give us elements in the motivic cohomology of motives attached to Hecke characters of imaginary quadratic number fields. The regulators of these elements have the expected relation to special values of Hecke L-series. As a corollary one obtains results on Beilinson's conjectures for CM elliptic curves of Shimura type and for Dirichlet characters of $\mathbf{Q}$ and of imaginary number field. Full details are contained in **[Den1,2]**.

(6.1) Consider an algebraic Hecke character $\epsilon : I_K/K^* \to \mathbf{C}^*$ of weight w of an imaginary quadratic field K. We wish to understand the special values $L(\epsilon, n)$ for $n > \frac{w}{2}+1$ of the corresponding L-series in terms of Beilinson's conjectures. In fact one can treat the L-values of all conjugates of ϵ simultaneously. Thus it is better to take a slightly different point of view and to look at the associated CM character

$$\phi : I_K \to T^*.$$

Here T/K is a number field and there exist integers a, b with $a+b=w$ such that

$$\phi(x) = x^a \bar{x}^b \quad \text{for all } x \text{ in } K^* \subset I_K.$$

From ϕ we obtain an L-series taking values in $T \otimes \mathbf{C} = \mathbf{C}^{\mathrm{Hom}(T,\mathbf{C})}$ by setting $L(\phi, s) = (L(\phi_\sigma, s))_\sigma$ where ϕ_σ is the Hecke character associated to ϕ via the embedding σ of T.

For critical n Beilinson's conjectures reduce to the Deligne conjecture, which for $L(\phi, n)$ is proved in **[GS1,2]** and in much greater generality in **[Bla]**.

For non-critical n we first have to find a Chow motive (3.3) with coefficients in T whose L-series equals $L(\phi, s)$. Note that if ϕ is a Dirichlet character χ of K—i.e. if $a = b = 0$—we can take the motive M_χ constructed in (3.3). For the general case one needs the theory of CM elliptic curves of Shimura type **[GS1]**. These are elliptic curves E with CM by $\mathcal{O}_K$ which are defined over an abelian extension F of K such that the extension $F(E_{\mathrm{tors}})/K$ is abelian as well. One checks that $e_0 = [E \times 0]$, $e_2 = [0 \times E]$ and $e_1 = 1 - e_0 - e_2$ are pairwise orthogonal projectors of the motive $\mathbf{Q}E$ in $\mathcal{M}_F^+(\mathbf{Q})$. The motive $h_1(E) = e_1(\mathbf{Q}E)$ in $\mathcal{M}_F^+(\mathbf{Q})$, viewed as a motive in $\mathcal{M}_K^+(\mathbf{Q})$, will be called M.

(6.2) Proposition. *For $w \geq 1$ and possibly after enlarging the field T there exists an elliptic curve as above such that $M^{\otimes w}$ contains a direct factor M_ϕ with $\mathrm{End}(M_\phi) = T$ and $L(H^w(M_\phi), s) = L(\phi, s)$.*

In the last equation M_ϕ is viewed as a motive in $\mathcal{M}_K^+(T)$ via **[De2]** 2.1.

Note that it is sufficient to treat Hecke characters of positive weight since multiplication of ϕ by the norm just results in a shift by one of s in the L-series. For the same reason we may assume that a, $b \geq 0$.

(6.3) Theorem. *Assume that $w \geq 1$, $n > \frac{w}{2}+1$ and in addition that n is non-critical for M_ϕ, i.e. $n > \mathrm{Max}(a,b)$. Then the L-series $L(\bar{\phi}, s)$ has a first order zero for $s = -l := w+1-n$ and there is an element ξ in $H^{w+1}_{\mathcal{M}}(M_\phi, \mathbf{Q}(n))$ such that*

$$r_{\mathcal{D}}(\xi) \equiv L'(\bar{\phi}, -l)\eta \quad \mathit{mod}\ T^*$$

in the free rank one $T \otimes \mathbf{R}$-module

$$H^{w+1}_{\mathcal{D}}(M_{\phi\mathbf{R}}, \mathbf{R}(n)) = H^w_B(M_{\phi\mathbf{R}}, \mathbf{R}(n-1)).$$

Here η is a T-generator of $H^w_B(M_{\phi\mathbf{R}}, \mathbf{Q}(n-1))$.

Remarks. (1) In general the conjectures involve the motivic cohomology of an integral model. However since $E \to \mathrm{Spec} K$ has potential good reduction one can show that

$$H^{w+1}_{\mathcal{M}}(M_\phi, \mathbf{Q}(n)) = H^{w+1}_{\mathcal{M}}(M_\phi, \mathbf{Q}(n))_{\mathbf{Z}}$$

for $n \neq \frac{w}{2}+1$, using **[So1]** 3.1.3, Corollary 2.

(2) In **[Den1]** a refined version of (6.3) is proved for $w=1$ where one considers motivic cohomology with almost integral coefficients. This was possible by a careful reexamination of the entire (slightly modified) construction of Beilinson's Eisenstein symbol specialised to a constant elliptic curve.

(6.4) Construction of ξ and calculation of $r_{\mathcal{D}}(\xi)$. For simplicity we shall assume that $l \geq 0$. For the finitely many negative l in the theorem a slightly different construction is required. Set $k = w+2l > 0$ and fix some integer $N \geq 1$. For a choice of a square root of the discriminant d_K of K consider the map

$$\delta = (\mathrm{id}, \sqrt{d_K}) : E \to E^2 = E \times_F E$$

and let $\mathrm{pr} : E^{l+w} = E^l \times_F E^w \to E^w$ be the projection. Choose a Galois extension F' of F such that the N-torsion points of $E' = E \otimes_F F'$ are rational over F'. The choice of a level N structure $\alpha : (\mathbf{Z}/N)^2 \xrightarrow{\sim} E'_N$ on E' determines a commutative diagram

$$\begin{tikzcd} E' \arrow[r, "i_\alpha"] \arrow[d] & X_N \arrow[d] \\ \mathrm{Spec} F' \arrow[r] & M_N \end{tikzcd}$$

Using (1.3)(6) we find a canonical map $\mathcal{E}_{\mathcal{M}}$ independent of α which makes the following diagram commute:

$$\begin{tikzcd} \mathbf{Q}[(\mathbf{Z}/N)^2]^0 \arrow[r, "E^k_{\mathcal{M}}"] \arrow[d, equal] & H^{k+1}_{\mathcal{M}}(X^k_N, \mathbf{Q}(k+1))_{\mathrm{sgn}_{k+1}} \arrow[d, "i^*_\alpha"] \\ \mathbf{Q}[E'_N]^0 & H^{k+1}_{\mathcal{M}}(E'^k, \mathbf{Q}(k+1))_{\mathrm{sgn}_{k+1}} \\ \mathbf{Q}[E_N]^0 \arrow[u] \arrow[r, "\mathcal{E}_{\mathcal{M}}"] & H^{k+1}_{\mathcal{M}}(E^k, \mathbf{Q}(k+1))_{\mathrm{sgn}_{k+1}} \arrow[u] \end{tikzcd}$$

Now consider the following composition $\mathcal{K}_{\mathcal{M}}$ of maps:

$$\begin{tikzcd} H^{k+1}_{\mathcal{M}}(E^k, \mathbf{Q}(k+1))_{\mathrm{sgn}_{k+1}} \arrow[r, "(\delta^l \times \mathrm{id})^*"] \arrow[dddd, "\mathcal{K}_{\mathcal{M}}"] & H^{k+1}_{\mathcal{M}}(E^{l+w}, \mathbf{Q}(k+1)) \arrow[d, "\mathrm{pr}_*"] \\ & H^{w+1}_{\mathcal{M}}(E^w, \mathbf{Q}(n)) \arrow[d] \\ & H^{w+1}_{\mathcal{M}}(h_1(E)^{\otimes w}, \mathbf{Q}(n)) \arrow[d] \\ & H^{w+1}_{\mathcal{M}}(M^{\otimes w}, \mathbf{Q}(n)) \arrow[d, phantom] \\ H^{w+1}_{\mathcal{M}}(M_\phi, \mathbf{Q}(n)) & H^{w+1}_{\mathcal{M}}(M^{\otimes w}, \mathbf{Q}(n)) \arrow[l] \end{tikzcd}$$

For $l < 0$ the map $\mathcal{K}_{\mathcal{M}}$ is defined differently **[Den2]** §2. The required element ξ is obtained in the form $\xi = \mathcal{K}_{\mathcal{M}}\mathcal{E}_{\mathcal{M}}(\beta)$ for suitable N and divisor β in $\mathbf{Q}[E_N]^0$. To prove that it has the right properties we must first of all calculate explicitly the analogous maps $\mathcal{E}_{\mathcal{D}}$ and $\mathcal{K}_{\mathcal{D}}$ in Deligne cohomology. For $\mathcal{K}_{\mathcal{D}}$ this is easy. For $\mathcal{E}_{\mathcal{D}}$ we can use formula (4.9.1) for $E^k_{\mathcal{D}}$ specialised to the value of τ corresponding to our elliptic curve E. Note that in order to derive (4.9.1) Beilinson makes essential use of the compactification $\overline{M}_N$ of M_N—see **[Be2]**, §3.3. In **[Den1]** a different method for the calculation of $\mathcal{E}_{\mathcal{D}}$ is described which only uses analysis on E itself.

Looking at (4.9.1) we see that $\mathcal{E}_{\mathcal{D}}(\beta)$ is a certain linear combination of Eisenstein-Kronecker series. Hence it comes as no surprise that for suitable β the element $\mathcal{K}_{\mathcal{D}}\mathcal{E}_{\mathcal{D}}(\beta)$ is related to $L'(\bar{\phi}, -l)$ as specified in the theorem.

(6.5) Corollary. *(1) Let E/F be a CM elliptic curve of Shimura type as above. Then for $n \geq 2$ the weak Beilinson conjecture (3.2) holds for $L(H^1(E), n)$.*

(2) Assume that F is Galois over $\mathbf{Q}$ and let F^+ be a real subfield of F, i.e. $F^+ = F^\sigma \cap \mathbf{R}$ for some embedding σ of F into $\mathbf{C}$. Then for any elliptic curve E^+/F^+ whose base change to F is of Shimura type the analogue of (1) holds.

Remark. (2) generalises the case of CM elliptic curves over $\mathbf{Q}$ at $n=2$ treated by Bloch **[Bl1]** and Beilinson **[Be1]**; see also **[DW]**.

(6.6) Dirichlet characters. Given a character

$$\chi : G_K \to T^*$$

we can attach to it the motive M_χ of (3.3) and the twist $M_\phi(1)$ of a motive M_ϕ as in (6.2) for $\phi = \chi N_{K\otimes\mathbf{R}/\mathbf{R}}$. Possibly after extension of scalars both motives have the same L-function and should in fact be equal. The Beilinson conjectures for $M_\phi(1)$ follow from the theorem. For M_χ one can prove them directly using the map

$$\mathcal{K}_\mathcal{M}\mathcal{E}_\mathcal{M} : \mathbf{Q}[E_N]^0 \to H^1_\mathcal{M}(M_\chi, \mathbf{Q}(l+1)), \quad l > 0$$

where E is a CM elliptic curve of Shimura type over an abelian extension of K trivialising χ and $\mathcal{K}_\mathcal{M}$ is defined by composition:

$$\begin{array}{ccc} H^{2l+1}_\mathcal{M}(E^{2l}, \mathbf{Q}(2l+1))_{\mathrm{sgn}_{l+1}} & \xrightarrow{(\delta^l)^*} & H^{2l+1}_\mathcal{M}(E^l, \mathbf{Q}(2l+1)) \\ \Big\downarrow \mathcal{K}_\mathcal{M} & & \Big\downarrow \mathrm{pr}_* \\ H^1_\mathcal{M}(M_\chi, \mathbf{Q}(l+1)) & \xleftarrow{e_\chi} & H^1_\mathcal{M}(\mathrm{Spec} F, \mathbf{Q}(l+1)). \end{array}$$

By a very simple argument **[Den2]** (3.6) one can use the theory over K to prove the Beilinson conjectures for Dirichlet characters of $\mathbf{Q}$ as well. The complete results are these:

(6.7) Theorem. *For $k = \mathbf{Q}$ or K consider a character*

$$\chi : G_k \to T^*$$

and let $L(\chi, s) = \big(L(\sigma\chi, s)\big)_\sigma$ be its $T \otimes \mathbf{C}$-valued L-series. For $l > 0$ the map

$$r_\mathcal{D} \otimes \mathbf{R} : H^1_\mathcal{M}(M_\chi, \mathbf{Q}(l+1))_\mathbf{Z} \otimes \mathbf{R} \to H^1_\mathcal{D}(M_{\chi\mathbf{R}}, \mathbf{R}(l+1))$$

is an isomorphism of free $T \otimes \mathbf{R}$-modules. For $k = \mathbf{Q}$ and $\chi(c) = (-1)^l$ or $k = K$ their rank equals one. In this case we have

$$c_{M_\chi} \equiv L'(\chi, -l) \mod T^*$$

where $c_{M_\chi} \in (T \otimes \mathbf{R})^*/T^*$ denotes the regulator.

Remarks. (1) That $r_\mathcal{D} \otimes \mathbf{R}$ is an isomorphism follows from the work of Borel **[Bo1]** and Beilinson **[Be1]**, app. to §2; see also **[Rap]**.

(2) For $k = \mathbf{Q}$ a different proof of the theorem is given in **[Be1]** §7, see also **[N,E]**.

Appendix: motivic cohomology and extensions

In this appendix, we outline without proof the construction of extensions of motives attached to elements of motivic cohomology. Details should appear in a future paper by the second author. The underlying idea is certainly not new, aand is implicit in the constructions of **[Bl2]**. To motivate it, we consider first the case of ordinary Chow theory (i.e., $H^{2q}_{\mathcal{M}}(X,\mathbf{Q}(q))$). The corresponding extensions appear first in a paper of Deligne (**[De2]**, 4.3). So let X be smooth and projective over $\mathbf{Q}$, and let y be a cycle of codimension q, homologous to zero. Write Y for the support of y. Then there is an exact sequence of mixed motives (in the sense of **[Ja2]**, Chap.1):

$$0\to h^{2q-1}(X)\to h^{2q-1}(X-Y)\to h^{2q}_Y(X)\xrightarrow{\gamma} h^{2q}(X)\cdots.$$

The cycle class gives a map $cl(y):\mathbf{Q}(-q)\to h^{2q}_Y(X)$, and by hypothesis $\gamma\circ cl(y)=0$. Hence by pullback we obtain an extension

$$0\to h^{2q-1}(X)\to E_y\to \mathbf{Q}(-q)\to 0.$$

Theorem [Ja2]. *The class of the extension E_y depends only on the rational equivalence class of y. The following diagram commutes:*

$$\begin{array}{ccc}
\ker\{CH^q(X)\to H^{2q}(\overline{X},\mathbf{Q}_\ell(q))\} & \xrightarrow{y\mapsto E_y} & \mathrm{Ext}^1_{MM_{\mathbf{Q}}}(\mathbf{Q}(-q),h^{2q-1}(X)) \\
\Big\downarrow{\scriptstyle \text{cycle}} & & \Big\downarrow{\scriptstyle \ell\text{-adic realisation}} \\
\ker\{H^{2q}(X,\mathbf{Q}_\ell(q))\to H^{2q}(\overline{X},\mathbf{Q}_\ell(q))\} & \xrightarrow{H-S} & H^1(\overline{\mathbf{Q}}/\mathbf{Q},H^{2q-1}(\overline{X},\mathbf{Q}_\ell(q)))
\end{array}$$

Here H–S denotes the edge homomorphism in the Hochschild-Serre spectral sequence (in continuous étale cohomology **[Ja1]**)

$$E_2^{ab}=H^a(\overline{\mathbf{Q}}/\mathbf{Q},H^b(\overline{X},\mathbf{Q}_\ell)(q))\Rightarrow H^{a+b}(X,\mathbf{Q}_\ell(q)).$$

There is a similar statement for Deligne cohomology (cf. 2.9 above).

We now imitate this construction for higher cycles. In an attempt to make the notation tidier we write Δ^n_X for $\Delta_n\times X$, and $\partial\Delta^n_X$ for the union of the codimension one faces of Δ^n_X. By the normalisation theorem, any element of $H^{2q-n}_{\mathcal{M}}(X,\mathbf{Q}(q))=CH^q(X,n)\otimes\mathbf{Q}$ may be represented by a cycle $y\in z^q(X,n)$ with $\partial_i^*(y)=0$ for $0\le i\le n$. Choosing such a representative y, write $Y=\mathrm{supp}(y)$, $\partial Y=Y\cap\partial\Delta^n_X$, $U=\Delta^n_X-Y$, and $\partial U=U\cap\partial\Delta^n_X$. We consider the motive $h^{2q-1}(U,\partial U)$ which fits into a long exact sequence

$$h^{2q-2}(U)\to h^{2q-2}(\partial U)\to h^{2q-1}(U,\partial U)\to h^{2q-1}(U)\to h^{2q-1}(\partial U). \tag{A.1}$$

By purity we have $h^{2q-2}(U)=h^{2q-2}(\Delta^n_X)\equiv h^{2q-2}(X)$. It is also easy to deduce that $h^{2p-2}(\partial U)=h^{2p-2}(\partial\Delta^n_X)$ by considering the spectral sequence expressing the cohomology of $\partial\Delta^n_X$ in terms of that of its faces.

Lemma. *There is a decomposition*

$$h^i(\partial\Delta^n_X)\xrightarrow{\sim} h^i(X)\oplus h^{i-n+1}(X). \tag{A.2}$$

(In fact this decomposition is given by the 1- and sgn-eigenspaces for the action of the symmetric group of degree n.) Thus the sequence (A.1) becomes

$$0\to h^{2q-n-1}(X)\to h^{2q-1}(U,\partial U)\to h^{2q-1}(U)\to h^{2q-1}(\partial U).$$

This fits into a bigger diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & h^{2q-n-1}(X) \arrow[d] & & \\
 & 0 \arrow[d] & h^{2q-1}(U,\partial U) \arrow[r] \arrow[d] & \ker\beta \arrow[d] & \\
0 \arrow[r] & h^{2q-1}(\Delta^n_X) \arrow[r] \arrow[d] & h^{2q-1}(U) \arrow[r] \arrow[d] & h^{2q}_Y(\Delta^n_X)^0 \arrow[r] \arrow[d, "\beta"] & 0 \\
0 \arrow[r] & h^{2q-1}(\partial\Delta^n_X) \arrow[r] & h^{2q-1}(\partial U) \arrow[r] & h^{2q}_{\partial Y}(\partial\Delta^n_X)^0 \arrow[r] & 0
\end{tikzcd}$$

Here we have written

$$h^{2q}_Y(\Delta^n_X)^0 = \ker\{h^{2q}_Y(\Delta^n_X) \to h^{2q}(\Delta^n_X)\}$$
$$h^{2q}_{\partial Y}(\partial\Delta^n_X)^0 = \ker\{h^{2q}_{\partial Y}(\partial\Delta^n_X) \to h^{2q}(\partial\Delta^n_X)\}.$$

The cycle class of y gives a map $\mathbf{Q}(-q) \to \ker\beta$. From the snake lemma and (A.2) we have a long exact sequence

$$0 \to h^{2q-n-1}(X) \to h^{2q-1}(U,\partial U) \to \ker\beta \to h^{2q-n}(X). \tag{A.3}$$

Since $n>0$ the composite map $\mathbf{Q}(-q) \to \ker\beta \to h^{2q-n}(X)$ is zero (by weights), hence by pullback we obtain an extension

$$0 \to h^{2q-n-1}(X) \to E_y \to \mathbf{Q}(-q) \to 0.$$

Theorem. *The class of the extension E_y depends only on the class of y in $H^{2q-n}_{\mathcal{M}}(X,\mathbf{Q}(q))$. The following diagram commutes:*

$$\begin{tikzcd}
H^{2q-n}_{\mathcal{M}}(X,\mathbf{Q}(q)) \arrow[r, "y\mapsto E_y"] \arrow[d, "\text{cycle}"] & \mathit{Ext}^1_{MM_{\mathbf{Q}}}(\mathbf{Q}(-q), h^{2q-n-1}(X)) \arrow[d, "\ell\text{-adic realisation}"] \\
H^{2q-n}(X,\mathbf{Q}_\ell(q)) \arrow[r, "\text{Hochschild-Serre}"] & H^1(\overline{\mathbf{Q}}/\mathbf{Q}, H^{2q-n-1}(\overline{X},\mathbf{Q}_\ell(q)))
\end{tikzcd}$$

The analogous statement 2.9 for Deligne cohomology also holds.

Remark. In this construction, we have in the interest of clarity freely used "relative" and "local" motives $h^i_\bullet(-)$, $h^i(-,\bullet)$. Lest this trouble the reader, we point out that the extension E_y really belongs to the category $MM_{\mathbf{Q}}$ of mixed motives generated by $h^i(V)$ for quasi-projective varieties $V/\mathbf{Q}$ (see **[Ja2]**, Appendix C2). Indeed, the "relative" motive $h^{2q-1}(U,\partial U)$ can be constructed as part of the motive of a suitable singular variety (a mapping cylinder) and the motive $\ker\beta$ is simply the Tate twist of an Artin motive. Therefore the objects in the exact sequence (A.3) are all motives in $MM_{\mathbf{Q}}$. To construct the arrows we need only work in the various realisations, and there the relative and local cohomology groups are available.

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